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PROPERTIES OF FUNCTIONS RELATED TO HADAMARD TYPE INEQUALITY AND APPLICATIONS

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Abstract. The aim of this paper is first to generalize Hadamard's inequality. Further, Schur m-power convex of the associated continuous function of two variables by utilize the Hadamard type inequalities are obtained. As applications, a inequality related special mean is established. And we also improve Jordan's inequality.

1. Introduction

Convex functions and Schur convex functions have been found to play an important role in the theory of special functions and mathematical statistics (see [4,10,11,14,17]).

Let $\mathbb{R}_{++} = (0,\infty)$.

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is convex function on I, if

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y), \qquad \forall x,y \in I, t \in [0,1]. \tag{1.1}$$

f is said to be concave if $-f$ is convex.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function. The well-known Hadamard's inequality states as follows

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}, \quad a,b \in I \textit{ with } a<b. \tag{1.2}$$

For many recent results related to this classic result, see books [4,10,11,14,17] and the papers [5,8,15,16,21].

In [6], S. S. Dragomir established the following theorem which a refinement of the first inequality of (1.2).

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THEOREM 1.1. *([6]). Let $f:[a,b]\to\mathbb{R}$ be a convex function, and*

$$G(t)=\frac{1}{2(b-a)}\int_a^b [f(ta+(1-t)x)+f(tb+(1-t)x)]dx,\quad t\in[0,1].$$

Then G is convex on [0,1], and for all $t\in[0,1]$, we have

$$\frac{1}{b-a}\int_a^b f(x)dx=G(0)\leqslant G(t)\leqslant G(1)=\frac{f(a)+f(b)}{2}. \tag{1.3}$$

In [12, 13], M. Merkle discussed the Schur convexity of the associated function of two variables $F(x,y)=(f(y)-f(x))/(y-x)$ by using the Hadamard's inequality.

In [7], N. Elezović and J. Pečarić researched the Schur convexity on the upper and the lower limit of the integral for the mean of the convex function and established the following important result by utilize the Hadamard's inequality.

THEOREM 1.2. *([7]). Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a continuous function on I. Then*

$$\Phi(x,y):=\begin{cases}\dfrac{1}{y-x}\displaystyle\int_x^y f(t)dt, & x,y\in I,\ x\neq y\\ f(x), & x=y\in I.\end{cases}$$

is Schur convex (Schur concave) on I^2 if and only if f is convex (concave)on I.

For many results related to the Schur convex related to Hadamard's inequality, see the papers [2,3,19,20].

Recently, Yang [26-28] generalized the notion of Schur convexity to Schur m-power convexity, which contains the Schur convexity, Schur geometrical convexity, Schur harmonic convexity. Moreover, he discussed Schur m-power convexity of Stolarsky means [26], Gini means [27] and Daróczy means [28]. Wang and Yang proved that generalized Hamy symmetric function [22] and a class of multiplicatively functions [23] are Schur m-power convex.

The aim of this paper is first establish a generalization of Hadamard's inequality. Further, the Schur m-power convex of the associated continuous function of two variables by utilize the Hadamard type inequalities are obtained. As applications, a relevant double inequality which is a extension of the known inequality is established. And several refinements and new inequalities for Jordan's inequality are obtained.

2. Main results

Our main results are presented as follows.

THEOREM 2.1. *Let $f:I\subset\mathbb{R}_{++}\to\mathbb{R}$ be a continuous function on I. If f is convex and increasing, and a parameter $m\leqslant 1$ (or if f is convex and decreasing, and $m>1$), then*

$$\frac{1}{y-x}\int_x^y f(t)dt\leqslant\frac{x^{1-m}f(x)+y^{1-m}f(y)}{x^{1-m}+y^{1-m}}. \tag{2.1}$$

If f is concave and decreasing, and $m \leqslant 1$ (or if f is concave and increasing, and $m > 1$), then (2.1) is reversed.

REMARK 1. From the proof of the theorem 2.1, it can be seen that the monotonic conditions of functions in the theorem 2.1 for $m = 1$ can be omitted.

THEOREM 2.2. *Let $f : I \subset \mathbb{R}_{++} \to \mathbb{R}$ be a continuous function on I. If f is convex and increasing, and a parameter $m \leqslant 1$ (or if f is convex and decreasing, and $m > 1$), then*

$$\Phi(x,y) := \begin{cases} \frac{1}{y-x} \int_x^y f(t)dt, & x,y \in I, x \neq y \\ f(x), & x = y \in I \end{cases}$$

is Schur m-power convex on I^2. If f is concave and decreasing, and $m \leqslant 1$ (or if f is concave and increasing, and $m > 1$), then $\Phi(x,y)$ is Schur m-power concave on I^2.

REMARK 2. From the proof of the theorem 2.2, it can be seen that the monotonic conditions of functions in the theorem 2.1 for $m = 1$ can be omitted.

Take $m = 1, 0, -1$, we get the following corollaries.

COROLLARY 2.1. *Let $f : I \subset \mathbb{R}_{++} \to \mathbb{R}$ be a continuous function on I. If f is convex and monotonicity, then $\Phi(x,y)$ is Schur convex. If f is concave and monotonicity, then $\Phi(x,y)$ is Schur concave.*

REMARK 3. The Corollary 2.1 is the result in [7, P854, Theorem 1]. So, our results generalize this conclusion.

COROLLARY 2.2. *Let $f : I \subset \mathbb{R}_{++} \to \mathbb{R}$ be a continuous function on I. If f is convex and increasing, then $\Phi(x,y)$ is Schur geometrically convex and Schur harmonically convex. If f is concave and decreasing, then $\Phi(x,y)$ is Schur geometrically concave and Schur harmonically concave.*

THEOREM 2.3. *Let $f : I \subset \mathbb{R}_{++} \to \mathbb{R}$ be a continuous function, and*

$$G(t) = \frac{1}{2(y-x)} \int_x^y [f(tx + (1-t)u) + f(ty + (1-t)u)]du, \quad t \in [0,1].$$

For any $t \in [0,1]$, we define a function of x,y as follows:

$$P(x,y) := \begin{cases} G(t), & x,y \in I,\ x \neq y \\ f(x), & x = y \in I. \end{cases}$$

(i) for $m \geqslant 1$ and $\frac{x^{1-m}}{x^{1-m}+y^{1-m}} \leqslant t \leqslant 1$, if f is convex (concave) and decreasing on I, then $P(x,y)$ is Schur m-power convex (Schur m-power concave) on I^2;
(ii) for $m < 1$ and $0 \leqslant t \leqslant \frac{x^{1-m}}{x^{1-m}+y^{1-m}}$, if f is concave (convex) and increasing on I, then $P(x,y)$ is Schur m-power concave (Schur m-power convex) on I^2.

REMARK 4. From the proof of the theorem 2.3, it can be seen that the monotonic conditions of functions in the theorem 2.3 for $m = 1$ can be omitted.

COROLLARY 2.3. *For* $\frac{1}{2} \leqslant t \leqslant 1$*, if* f *is convex (concave) on* I*, then* $P(x,y)$ *is Schur convex (Schur concave) on* I^2*; for* $0 \leqslant t \leqslant \frac{1}{2}$*, if is concave (convex) on* I*, then* $P(x,y)$ *is Schur concave (Schur convex) on* I^2*.*

REMARK 5. The Corollary 2.3 is the result in [3, P1138,Corollary 1]. So, our results partially generalize this conclusion.

COROLLARY 2.4. *For* $\frac{x}{x+y} \leqslant t \leqslant 1$*, if* f *is convex and decreasing on* I*, then* $P(x,y)$ *is Schur geometrically convex on* I^2*; for* $0 \leqslant t \leqslant \frac{x}{x+y}$*, if is concave and increasing on* I*, then* $P(x,y)$ *is Schur geometrically concave on* I^2*.*

COROLLARY 2.5. *For* $\frac{x^2}{x^2+y^2} \leqslant t \leqslant 1$*, if* f *is convex and decreasing on* I*, then* $P(x,y)$ *is Schur harmonically convex on* I^2*; for* $0 \leqslant t \leqslant \frac{x^2}{x^2+y^2}$*, if is concave and increasing on* I*, then* $P(x,y)$ *is Schur harmonically concave on* I^2*.*

3. Definitions and Lemmas

We first recall several definitions as follows.

DEFINITION 3.1. [11] Suppose that $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ are two n-tuples real numbers.

(1) $\mathbf{y}$ majorizes $\mathbf{x}$ (in symbols $\mathbf{x} \prec \mathbf{y}$), if $\sum_{i=1}^k x_{[i]} \leqslant \sum_{i=1}^k y_{[i]}, (k = 1, 2, \cdots, n-1)$ and $\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}$, where $x_{[1]} \geqslant x_{[2]} \geqslant \cdots \geqslant x_{[n]}$, $y_{[1]} \geqslant y_{[2]} \geqslant \cdots \geqslant y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.

(2) A real-valued function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is said to be Schur convex on Ω if

$$\mathbf{x} \prec \mathbf{y} \quad on \ \Omega \Rightarrow \qquad f(\mathbf{x}) \leqslant f(\mathbf{y}).$$

f is a Schur concave function on Ω if and only if $-f$ is a Schur convex function.

DEFINITION 3.2. [30] Suppose $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ are two n-tuples real numbers. Let $\Omega \subset \mathbb{R}^n_{++}$. A function $f : \Omega \to \mathbb{R}_{++}$ is called Schur geometrically convex if

$$\ln \mathbf{x} \prec \ln \mathbf{y} \quad on \ \Omega \Rightarrow \qquad \varphi(\mathbf{x}) \leqslant \varphi(\mathbf{y}).$$

f is Schur geometrically concave if $-f$ is Schur geometrically convex.

The following Theorem is basic and plays an important role in the theory of the Schur geometrically convex function.

LEMMA 3.1. ([30]) *Let $\varphi(\boldsymbol{x}) = \varphi(x_1, x_2, \cdots, x_n)$ be symmetric and continuous on $\Omega \subset \mathbb{R}^n_{++}$ and differentiable in Ω^0. Then $\varphi : \Omega \to \mathbb{R}_{++}$ is Schur geometrically convex (Schur geometrically concave) if and only if*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \geqslant 0 (\leqslant 0). \tag{3.1}$$

DEFINITION 3.3. [1, 25] Let $\Omega \in \mathbb{R}^n$.

(1) A set Ω is called harmonically convex if $\frac{\mathbf{xy}}{\lambda \mathbf{x} + (1-\lambda \mathbf{y})} \in \Omega$ for every $\mathbf{x}, \mathbf{y} \in \Omega$ and $\lambda \in [0, 1]$, where $\mathbf{xy} = \sum_{i=1}^n x_i y_i$ and $\frac{1}{\mathbf{x}} = \left(\frac{1}{x_1}, \cdots, \frac{1}{x_n}\right)$.

(2) A function $\varphi : \Omega \to \mathbb{R}_{++}$ is called Schur harmonically convex on Ω if $\frac{1}{\mathbf{x}} \prec \frac{1}{\mathbf{y}}$ implies $\varphi(\mathbf{x}) \leqslant \varphi(\mathbf{y})$. f is Schur harmonically concave if $-f$ is Schur harmonically convex.

LEMMA 3.2. ([1,25]) *Let $\Omega \in \mathbb{R}^n_{++}$ be a symmetric and harmonically convex set with inner points and let $\varphi : \Omega \to \mathbb{R}_{++}$ be a continuously symmetric function which is differentiable in Ω^0. then φ is Schur harmonically convex (Schur harmonically concave) on Ω if and only of*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi}{\partial x_1} - x_2^2 \frac{\partial \varphi}{\partial x_2}\right) \geqslant 0 (\leqslant 0). \tag{3.2}$$

Schur convex, Schur geometrically convex and Schur harmonically convex were introduced by Schur [11], Zhang [30] and Chu [1], respectively, and played a key role in analytic inequalities. Moreover, the theory of convex functions and Schur convex functions is one of the most important research fields in modern analysis and geometry.

Recently, Yang present the Schur f-convexity in [26] as follows.

DEFINITION 3.4. [26-28] Let $\Omega \subseteq \mathbb{R}^n$ be a set with nonempty interior and f be a strictly monotone function defined on Ω. Let

$$f(\mathbf{x}) = (f(x_1), f(x_2), \cdots, f(x_n)) \text{ and } f(\mathbf{y}) = (f(y_1), f(y_2), \cdots, f(y_n)).$$

Then function $\varphi : \Omega \to \mathbb{R}$ is said to be Schur f-convex on Ω if $f(\mathbf{x}) \prec f(\mathbf{y})$ on Ω implies $\psi(\mathbf{x}) \leqslant \psi(\mathbf{y})$.

ψ is said to be Schur f-concave if $-\psi$ is Schur f-convex.

Take $f(x) = x$, $\ln x$, x^{-1} in Definition 3.4, it yields the Schur convexity, Schur geometrical convexity and Schur harmonic convexity. It is clear that the Schur f-convexity is a generalization of the Schur convexity mentioned above. In general, we have:

DEFINITION 3.5. [26-28] Let $f : \mathbb{R}_{++} \to \mathbb{R}$ be defined by $f(x) = (x^m - 1)/m$ if $m \neq 0$ and $f(x) = \ln x$ if $m = 0$. Then function $\psi : \Omega \subseteq \mathbb{R}^n_{++} \to \mathbb{R}$ is said to be Schur m-power convex on Ω if $f(\mathbf{x}) \prec f(\mathbf{y})$ on Ω implies $\psi(\mathbf{x}) \leqslant \psi(\mathbf{y})$.

ψ is said to be Schur m-power concave if $-\psi$ is Schur m-power convex.

LEMMA 3.3. ([26–28]) *Let $\psi : \Omega \subseteq \mathbb{R}^n_{++} \to \mathbb{R}$ be continuous on Ω and differentiable in Ω^0. Then ψ is schur m-power convex (Schur m-power concave) on Ω if and only if ψ is symmetric on Ω and*

$$\begin{cases} \dfrac{x_1^m - x_2^m}{m}\left(x_1^{1-m}\dfrac{\partial\varphi}{\partial x_1} - x_2^{1-m}\dfrac{\partial\varphi}{\partial x_2}\right) \geqslant 0 \ \ (\leqslant 0),\ \text{if } m \neq 0, \\ (\ln x_1 - \ln x_2)\left(x_1\dfrac{\partial\varphi}{\partial x_1} - x_2\dfrac{\partial\varphi}{\partial x_2}\right) \geqslant 0 \ \ (\leqslant 0),\ \text{if } m = 0, \end{cases} \tag{3.3}$$

hold for any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \Omega^0$ with $x_1 \neq x_2$, where $\Omega \subseteq \mathbb{R}^n_{++}$ is a symmetric set with nonempty interior Ω^0.

The following lemma is clearly due to the monotonicity property of the function x^p on $\mathbb{R}_{++}$.

LEMMA 3.4. [22,23] *Then the two discrimination inequalities in Lemma 3.3 are equivalent to*

$$(x_1 - x_2)\left(x_1^{1-m}\frac{\partial\varphi}{\partial x_1} - x_2^{1-m}\frac{\partial\varphi}{\partial x_2}\right) \geqslant (\leqslant) 0. \tag{3.4}$$

LEMMA 3.5. *Let $f(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2}$, $x \in (0, \pi]$. Then $f(x)$ is convex and decreasing on $(0, \frac{\pi}{2}]$, and $f(x)$ is convex and increasing on $[\frac{\pi}{2}, \pi]$.*

Proof. Since

$$f'(x) = \frac{2\sin x - 2x\cos x - x^2 \sin x}{x^3},$$

$$f'' x = \frac{-x^3\cos x + 3x^2 \sin x + 6x\cos x - 6\sin x}{x^4}.$$

Let $g(x) = 2\sin x - 2x\cos x - x^2 \sin x$, and $g(0) = 0$. Then $g'(x) = -x^2\cos x \leqslant 0$. And $g(x) \leqslant g(0) = 0$. Further, $f'(x) \leqslant 0$. So $f(x)$ is decreasing on $(0, \frac{\pi}{2}]$.

Let $h(x) = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x$, and $h(0) = 0$. Then $h'(x) = x^3\sin x \geqslant 0$, and $h(x) \geqslant h(0) = 0$. Further, $f''(x) \geqslant 0$. So $f(x)$ is convex on $(0, \frac{\pi}{2}]$.

Similarly, we get that $f(x)$ is convex and increasing on $[\frac{\pi}{2}, \pi]$. □

4. Main results and Proof

Proof of Theorem 2.1. Since $f(x)$ is convex. Then

$$\begin{aligned}\Delta &= (x^{1-m}+y^{1-m})\frac{1}{y-x}\int_x^y f(t)dt-(x^{1-m}f(x)+y^{1-m}f(y))\\ &\leqslant (x^{1-m}+y^{1-m})\frac{f(x)+f(y)}{2}-(x^{1-m}f(x)+y^{1-m}f(y))\\ &= \frac{1}{2}[y^{1-m}(f(x)-f(y))-x^{1-m}(f(x)-f(y))]\\ &= \frac{1}{2}(f(x)-f(y))(y^{1-m}-x^{1-m})\\ &= -\frac{1}{2(y-x)^2}[(y-x)(f(y)-f(x))][(y-x)(y^{1-m}-x^{1-m})]. \end{aligned} \tag{4.1}$$

If $m\leqslant 1$ and f is increasing (or $m>1$ and f is decreasing), further, according to the monotonicity property of the functions f and x^r and using (4.1), we get that $\Delta\leqslant 0$.

If $f(x)$ is concave. Then

$$\begin{aligned}\Delta &= (x^{1-m}+y^{1-m})\frac{1}{y-x}\int_x^y f(t)dt-(x^{1-m}f(x)+y^{1-m}f(y))\\ &\geqslant (x^{1-m}+y^{1-m})\frac{f(x)+f(y)}{2}-(x^{1-m}f(x)+y^{1-m}f(y))\\ &= \frac{1}{2}[y^{1-m}(f(x)-f(y))-x^{1-m}(f(x)-f(y))]\\ &= \frac{1}{2}(f(x)-f(y))(y^{1-m}-x^{1-m})\\ &= -\frac{1}{2(y-x)^2}[(y-x)(f(y)-f(x))][(y-x)(y^{1-m}-x^{1-m})]. \end{aligned} \tag{4.2}$$

If $m\leqslant 1$ and f is decreasing (or $m>1$ and f is increasing), further, according to the monotonicity property of the functions f and x^r and using (4.1), we get that $\Delta\geqslant 0$. So the proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. Let $x\neq y$. Then

$$\frac{\partial\Phi(x,y)}{\partial x}=\frac{1}{(y-x)^2}\int_x^y f(t)dt-\frac{f(x)}{y-x},$$

$$\frac{\partial\Phi(x,y)}{\partial y}=-\frac{1}{(y-x)^2}\int_x^y f(t)dt+\frac{f(y)}{y-x}.$$

By Lemma 3.3 and 3.4, one has

$$\begin{aligned}\Delta_1 &= (y-x)\left(y^{1-m}\frac{\partial\Phi(x,y)}{\partial y} - x^{1-m}\frac{\partial\Phi(x,y)}{\partial x}\right)\\ &= (y-x)\left[-\frac{1}{(y-x)^2}\int_x^y f(t)dt\cdot(x^{1-m}+y^{1-m}) + \frac{1}{(y-x)}(y^{1-m}f(y)+x^{1-m}f(x))\right]\\ &= -\frac{1}{(y-x)}\int_x^y f(t)dt\cdot(x^{1-m}+y^{1-m}) + (y^{1-m}f(y)+x^{1-m}f(x)).\end{aligned} \tag{4.3}$$

From (4.3) and applying Theorem 2.1, Theorem 2.2 is valid. □

Proof of Theorem 2.3. It is sufficient prove that (i) , the proof of (ii) is similar with (i) . We need only consider the case of $m > 1$ and $\frac{x^{1-m}}{x^{1-m}+y^{1-m}} \leqslant t \leqslant 1$. It is clear that $P(x,y)$ is symmetric. For $x \neq y$, let

$$P_1(x,y) = \int_x^y f(tx+(1-t)u)du$$

and

$$P_2(x,y) = \int_x^y f(ty+(1-t)u)du.$$

Then

$$P(x,y) = \frac{1}{2(y-x)}[P_1(x,y)+P_2(x,y)] = G(t).$$

By using the change of the variable $s = tx+(1-t)y$, then

$$\begin{aligned}P_1(x,y) &= \frac{1}{1-t}\int_x^{tx+(1-t)y} f(s)ds\\ &= \frac{1}{1-t}\left[\int_0^{tx+(1-t)y} f(s)ds - \int_0^x f(s)ds\right].\end{aligned}$$

$$\frac{\partial P_1(x,y)}{\partial x} = \frac{t}{1-t}f(tx+(1-t)y) - \frac{f(x)}{1-t}. \tag{4.4}$$

$$\frac{\partial P_1(x,y)}{\partial y} = f(tx+(1-t)y). \tag{4.5}$$

Notice that $P_2(x,y) = -P_1(y,x)$, form (4.4) and (4.5), we have

$$\frac{\partial P_2(x,y)}{\partial x} = -\frac{\partial P_1(x,y)}{\partial y} = -f(tx+(1-t)y), \tag{4.6}$$

$$\frac{\partial P_2(x,y)}{\partial x} = -\frac{\partial P_1(x,y)}{\partial y} = \frac{f(x)}{1-t} - \frac{t}{1-t}f(tx+(1-t)y). \tag{4.7}$$

By Lemma 3.3 and 3.4, we get that

$$\begin{aligned}
\Delta_2 &= (y-x)\left(y^{1-m}\frac{\partial P(x,y)}{\partial y} - x^{1-m}\frac{\partial P(x,y)}{\partial x}\right) \\
&= \frac{1}{2}\left[\left(y^{1-m}\frac{\partial P_1(x,y)}{\partial y} - x^{1-m}\frac{\partial P_1(x,y)}{\partial x}\right) + \left(y^{1-m}\frac{\partial P_2(x,y)}{\partial y} - x^{1-m}\frac{\partial P_2(x,y)}{\partial x}\right)\right] \\
&\quad - \frac{P_1(x,y)+P_2(x,y)}{2(y-x)}(x^{1-m}+y^{1-m}) \\
&= \frac{1}{2}\left[\left(y^{1-m} - \frac{x^{1-m}t}{1-t}\right) f(tx+(1-t)y) + \left(x^{1-m} - \frac{y^{1-m}t}{1-t}\right) f(ty+(1-t)x)\right. \\
&\quad \left. + \frac{x^{1-m}f(x)+y^{1-m}f(y)}{1-t}\right] - G(t)(x^{1-m}+y^{1-m}) \\
&= \frac{1}{2}\left[\left(\frac{y^{1-m}-(x^{1-m}+y^{1-m})t}{1-t}\right) f(tx+(1-t)y)\right. \\
&\quad + \left(\frac{x^{1-m}-(x^{1-m}+y^{1-m})t}{1-t}\right) f(ty+(1-t)x) \\
&\quad \left. + \frac{x^{1-m}f(x)+y^{1-m}f(y)}{1-t}\right] - G(t)(x^{1-m}+y^{1-m}).
\end{aligned}$$

For $m \geqslant 1$ and $\frac{x^{1-m}}{x^{1-m}+y^{1-m}} \leqslant t \leqslant 1$, then $y^{1-m}-(x^{1-m}+y^{1-m})t \leqslant x^{1-m}-(x^{1-m}+y^{1-m})t \leqslant 0$. Since f is convex and decreasing, thus we get

$$\begin{aligned}
\Delta_2 &\geqslant \frac{1}{2}\left[\left(\frac{y^{1-m}-(x^{1-m}+y^{1-m})t}{1-t}\right)(tf(x)+(1-t)f(y))\right. \\
&\quad + \left(\frac{x^{1-m}-(x^{1-m}+y^{1-m})t}{1-t}\right)(tf(y)+(1-t)f(x)) \\
&\quad \left. + \frac{x^{1-m}f(x)+y^{1-m}f(y)}{1-t}\right] - G(t)(x^{1-m}+y^{1-m}) \\
&= (x^{1-m}f(x)+y^{1-m}f(y)) - G(t)(x^{1-m}+y^{1-m}) \\
&\geqslant (x^{1-m}f(x)+y^{1-m}f(y)) - \frac{1}{2}(f(x)+f(y))(x^{1-m}+y^{1-m}) \\
&= \frac{1}{2}(y^{1-m}-x^{1-m})(f(y)-f(x)) \geqslant 0,
\end{aligned}$$

where we use $G(t) \leqslant G(1)$ in (1.3).

If f is concave and decreasing, thus we get

$$\begin{aligned}\Delta_2 &\leqslant \frac{1}{2}\left[\left(\frac{y^{1-m}-(x^{1-m}+y^{1-m})t}{1-t}\right)(tf(x)+(1-t)f(y))\right.\\&\quad+\left(\frac{x^{1-m}-(x^{1-m}+y^{1-m})t}{1-t}\right)(tf(y)+(1-t)f(x))\\&\quad\left.+\frac{x^{1-m}f(x)+y^{1-m}f(y)}{1-t}\right]-G(t)(x^{1-m}+y^{1-m})\\&=(x^{1-m}f(x)+y^{1-m}f(y))-G(t)(x^{1-m}+y^{1-m})\\&\leqslant (x^{1-m}f(x)+y^{1-m}f(y))-\frac{1}{y-x}\int_x^y f(t)dt(x^{1-m}+y^{1-m})\\&\leqslant 0.\end{aligned}$$

where we use $G(t)\geqslant G(0)$ in (1.3) and Theorem 2.1. The proof of Theorem 2.3 is completed. □

5. Applications

THEOREM 5.1. *For $a,b\in\mathbb{R}_{++}$, and $m\geqslant 1$. Then*

$$G^2(a,b)\leqslant \frac{a^m+b^m}{a^{m-1}+b^{m-1}}\cdot L(a,b), \tag{5.1}$$

where $G(a,b)=\sqrt{ab}, L(a,b)=\frac{b-a}{\ln b-\ln a}$.

Proof. Let $f(x)=\frac{1}{x}$, $x\in(0,\infty)$. Then for $a,b\in(0,\infty)$ and $b>a$, one has

$$\frac{1}{b-a}\int_a^b \frac{1}{x}dx=L^{-1}(a,b). \tag{5.2}$$

Since $f(x)$ is convex and decreasing, by Theorem 2.1, it follows that

$$\frac{1}{b-a}\int_a^b \frac{1}{x}dx\leqslant \frac{a^m+b^m}{ab(a^{m-1}+b^{m-1})}. \tag{5.3}$$

Thus, from (4.2) and (4.3), we get

$$L^{-1}(a,b)\leqslant \frac{a^m+b^m}{ab(a^{m-1}+b^{m-1})}.$$

So, (5.1) holds. □

The classical Jordan's inequality [14] states that for $a\in(0,\frac{\pi}{2}]$

$$\frac{2}{\pi}\leqslant \frac{\sin x}{x}\leqslant 1, \tag{5.4}$$

with equality holds if and only if $x=\frac{\pi}{2}$.

Some new developments on refinements, generalizations and applications of Jordan's inequality can be found in [9,18,24,29] and related references therein.

By applying Corollary 2.1 and 2.2, several refinements and new inequalities for Jordan's inequality are obtained, as follows.

THEOREM 5.2. *For* $a\in(0,\frac{\pi}{2}]$. *Then*

$$\frac{\sin a}{a}\geqslant\frac{2}{2a+\pi}\left(1-\frac{2}{2a+\pi}\cos a\right)+\frac{2}{\pi} \tag{5.5}$$

$$\frac{\sin a}{a}\leqslant\frac{2}{\pi}+(\frac{\pi}{2}-a)\left[\frac{\sin(\frac{a}{2}+\frac{\pi}{4})}{\left(\frac{a}{2}+\frac{\pi}{4}\right)^2}-\frac{\cos(\frac{a}{2}+\frac{\pi}{4})}{\left(\frac{a}{2}+\frac{\pi}{4}\right)}\right]. \tag{5.6}$$

Proof. By Lemma 3.5 and Corollary 2.1, then the function

$$\Phi(x,y)=\frac{1}{y-x}\int_x^y\left(\frac{\cos t}{t}-\frac{\sin t}{t^2}\right)dt$$

is Schur convex on $(0,\frac{\pi}{2}]$. Since

$$\left(\frac{a+\frac{\pi}{2}}{2},\frac{a+\frac{\pi}{2}}{2}\right)\prec\left(\frac{\pi}{2},a\right)\prec\left(a+\frac{\pi}{2},0\right),$$

then

$$\Phi\left(\frac{a+\frac{\pi}{2}}{2},\frac{a+\frac{\pi}{2}}{2}\right)\leqslant\Phi\left(\frac{\pi}{2},a\right)\leqslant\Phi\left(a+\frac{\pi}{2},0\right).$$

That is

$$\frac{\cos\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\frac{a+\frac{\pi}{2}}{2}}-\frac{\sin\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^2}\leqslant\frac{1}{a-\frac{\pi}{2}}\left.\frac{\sin x}{x}\right|_{\frac{\pi}{2}}^{a}\leqslant\frac{1}{-\frac{\pi}{2}-a}\left.\frac{\sin x}{x}\right|_{a+\frac{\pi}{2}}^{0}, \tag{5.7}$$

where $\frac{\sin 0}{0}=\lim_{t\to0^+}\frac{\sin x}{x}$. So, from (5.7), we get (5.5) and (5.6). □

REMARK. On the one hand, obviously,

$$\frac{\sin a}{a}\geqslant\frac{2}{2a+\pi}\left(1-\frac{2}{2a+\pi}\cos a\right)+\frac{2}{\pi}\geqslant\frac{\pi}{2}.$$

Therefore (5.5) is stronger than the left hand of (5.4).

On the other hand, since the function $\frac{\cos t}{t}-\frac{\sin t}{t^2}$ is decreasing on $(0,\frac{\pi}{2}]$, then

$$\frac{\sin(\frac{a}{2}+\frac{\pi}{4})}{\left(\frac{a}{2}+\frac{\pi}{4}\right)^2}-\frac{\cos(\frac{a}{2}+\frac{\pi}{4})}{\left(\frac{a}{2}+\frac{\pi}{4}\right)}\leqslant\left(\frac{2}{\pi}\right)^2.$$

Hence, (5.6) may be written as

$$\frac{\sin a}{a} \leqslant \frac{2}{\pi} + (\frac{\pi}{2} - a)\left(\frac{2}{\pi}\right)^2.$$

Sample calculation, as $\frac{\pi(4-\pi)}{4} \leqslant a \leqslant \frac{\pi}{2}$, we have $\frac{2}{\pi} + (\frac{\pi}{2} - a)\left(\frac{2}{\pi}\right)^2 \leqslant 1$. Hence, (5.6) partially improves the right hand of (5.4).

THEOREM 5.3. *Let $t \in [0,1)$, $a,b \in \mathbb{R}_{++}$, and*

$$L_r(a,b;t) := \begin{cases} \left[\dfrac{(b^r - a^r) - (u^r - v^r)}{r(1-t)(b-a)}\right]^{\frac{1}{r-1}}, & a \neq b, \\ a, & a = b, \end{cases}$$

where $u = tb + (1-t)a$, $v = ta + (1-t)b$. Then
(i) if $1 < r < 2$, for $m < 1$ and $0 \leqslant t \leqslant \frac{a^{1-m}}{a^{1-m}+b^{1-m}}$, $L_r(a,b;t)$ is Schur m-power concave on $\mathbb{R}^2_{++}$;
(ii) if $r \leqslant 1$ and $r \neq 0$, for $m \geqslant 1$ and $\frac{a^{1-m}}{a^{1-m}+b^{1-m}} \leqslant t \leqslant 1$, $L_r(a,b;t)$ is Schur m-power concave on $\mathbb{R}^2_{++}$.

Proof. Take $f(x) = x^{r-1}, r \neq 0$. For $a \neq b$, form theorem 2.3, we have

$$G(t) = \frac{1}{2(b-a)}\int_a^b [(ta + (1-t)x)^{r-1} + (tb + (1-t)x)^{r-1}]dx$$
$$= \frac{(b^r - a^r) - (u^r - v^r)}{2r(1-t)(b-a)}.$$

(i) if $1 < r < 2$, $m < 1$ and $0 \leqslant t \leqslant \frac{a^{1-m}}{a^{1-m}+b^{1-m}}$, since $f(x) = x^{r-1}$ is concave and increasing on $\mathbb{R}^2_{++}$. Furthermore, the function $\varphi : t \to t^{\frac{1}{r-1}}$ is increasing on $\mathbb{R}_{++}$, so $L_r(a,b;t)$ is Schur m-power concave on $\mathbb{R}^2_{++}$.

(ii) $r \leqslant 1$ ($r \neq 0$), $m \geqslant 1$ and $\frac{a^{1-m}}{a^{1-m}+b^{1-m}} \leqslant t \leqslant 1$, since $f(x) = x^{r-1}$ is convex and decreasing on $\mathbb{R}^2_{++}$. Furthermore, the function $\varphi : t \to t^{\frac{1}{r-1}}$ is decreasing on $\mathbb{R}_{++}$, so $L_r(a,b;t)$ is Schur m-power concave on $\mathbb{R}^2_{++}$. □

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