SOME IMPROVED INEQUALITIES IN AN INNER PRODUCT SPACE

Nicuşor Minculete and Radu Pălтănea

Abstract. The aim of the paper is to present some improvements of an extended Cauchy-Schwarz inequality for four vectors in a real inner product space and derive certain consequences.

1. Introduction

In [5], Harvey generalizes the well-known Cauchy-Schwarz inequality,

$$\|a\|^2\|b\|^2 \geqslant \left(a^T b\right)^2, \ a,b \in \mathbb{R}^n, \tag{1}$$

to an inequality involving four vectors, namely:

$$\|a\|^2\|b\|^2 + \|c\|^2\|d\|^2 \geqslant 2a^T c b^T d + \left(a^T b\right)^2 + \left(c^T d\right)^2 - \left(a^T d\right)^2 - \left(b^T c\right)^2, \tag{2}$$

for $a,b,c,d \in \mathbb{R}^n$, where $\|\cdot\|$ denotes the Euclidean norm in the space $\mathbb{R}^n$.

In [2], Choi proved a stronger result than in [5], namely:

$$\|a\|^2\|b\|^2 + \|c\|^2\|d\|^2 \geqslant \left(a^T b\right)^2 + \left(c^T d\right)^2 + 2|a^T c b^T d - a^T d b^T c|, \tag{3}$$

for all $a,b,c,d \in \mathbb{R}^n$. The equality holds if and only if $a_i b_j - a_j b_i = c_i d_j - c_j d_i$, for all $1 \leqslant i,j \leqslant n$, where a_i, b_i, c_i, d_i are the components of vectors a,b,c,d, respectively. We also find in [2] a similar inequality of (3) for complex vectors.

Next, we show a refinement of inequalities (2) and (3) in a real inner product space.

2. Main results

Let X be a real inner product space over the field of real numbers $\mathbb{R}$. The inner product $\langle \cdot, \cdot \rangle$ induces an associated norm, given by $\|x\| = \sqrt{\langle x,x \rangle}$, for all $x \in X$ and thus X is a normed space.

We introduce the following notation for $a,b,c,d \in X$:

$$\Gamma(a,b;c,d) = \langle a,c \rangle \langle b,d \rangle - \langle a,d \rangle \langle b,c \rangle. \tag{4}$$

Below we present several properties of $\Gamma(a,b;c,d)$.

LEMMA 1. *For any* $a,b,c,d,e,f,u,v \in X$ *we have*

a) $\Gamma(a,b;a,b) = \|a\|^2\|b\|^2 - \langle a,b\rangle^2 \geqslant 0;$

b) $\Gamma(c,d;a,b) = \Gamma(a,b;c,d);$

c) $\Gamma(a,b;c,c) = 0;$

d) $\Gamma(a,b;d,c) = -\Gamma(a,b;c,d);$

e) $\Gamma(a,b;b,c) = \langle a,b\rangle\langle b,c\rangle - \langle a,c\rangle\|b\|^2;$

f) Γ *is linear in each argument;*

g) *If* $e \perp a$, $e \perp u$ *and* $u \perp a$, *then* $\Gamma(a,u;e,f) = 0$, $\Gamma(a,u;e,a) = 0;$

h) $\langle a,c\rangle\Gamma(a,b;b,v) - \langle a,v\rangle\Gamma(a,b;b,c) = \langle a,b\rangle\Gamma(a,b;c,v);$

i) $\Gamma(a,b;a,c)\Gamma(a,b;b,v) - \Gamma(a,b;a,v)\Gamma(a,b;b,c) = \Gamma(a,b;a,b)\Gamma(a,b;c,v);$

j) $\langle a,c\rangle\Gamma(a,b;d,a) + \langle a,d\rangle\Gamma(a,b;a,c) = -\|a\|^2\Gamma(a,b;c,d).$

Proof. By simple calculations, we obtain the statements. $\square$

THEOREM 1. *Let* X *be an inner product space. For any elements* $a,b,c,d \in X$, *there are* $u,e,f \in X$, $u \perp a$, $e \perp \mathrm{Sp}\{a,u\}$, $f \perp \mathrm{Sp}\{a,u\}$ *and* $\alpha,\beta,p,q \in \mathbb{R}$, *such that*

$$\Gamma(a,b;a,b) \cdot \Gamma(c,d;c,d) - [\Gamma(a,b;c,d)]^2 \tag{5}$$
$$= \Gamma(a,b;a,b)\Big[\Gamma(a,\alpha f - pe;a,\alpha f - pe) + \Gamma(u,\beta f - qe;u,\beta f - qe) + \Gamma(e,f;e,f)\Big]$$

Proof. There is $\lambda \in \mathbb{R}$ and $u \in X$, such that $b = \lambda a + u$ and $u \perp a$. Also there is $\mu \in \mathbb{R}$ and $v \in X$, such that $d = \mu c + v$ and $v \perp c$. We obtain

$$\Gamma(a,b;a,b) = \Gamma(a,\lambda a + u;a,\lambda a + u) = \Gamma(a,u;a,u),$$
$$\Gamma(c,d;c,d) = \Gamma(c,\mu c + v;c,\mu c + v) = \Gamma(c,v;c,v).$$

On the other hand, using Lemma 1 - b), c), f) we obtain

$$\Gamma(a,b;c,d) = \lambda\mu\Gamma(a,a;c,c) + \lambda\Gamma(a,a;c,v) + \mu\Gamma(a,u;c,c) + \Gamma(a,u;c,v)$$
$$= \Gamma(a,u;c,v).$$

There are $\alpha,\beta,p,q \in \mathbb{R}$ and $e,f \in X$, such that

$$c = \alpha a + \beta u + e, \text{ and } e \perp \mathrm{Sp}\{a,u\};$$
$$v = pa + qu + f, \text{ and } f \perp \mathrm{Sp}\{a,u\}.$$

But using relations b)-g) from Lemma 1, we have

$$\begin{aligned}
\Gamma(a,u;c,v) &= \alpha\Gamma(a,u;a,v)+\beta\Gamma(a,u;u,v)+\Gamma(a,u,e,v)\\
&= \alpha p\Gamma(a,u;a,a)+\alpha q\Gamma(a,u;a,u)+\alpha\Gamma(a,u;a,f)+p\beta\Gamma(a,u;u,a)\\
&\quad +\beta q\Gamma(a,u;u,u)+\beta\Gamma(a,u;u,f)+p\Gamma(a,u;e,a)+q\Gamma(a,u;e,u)\\
&\quad +\Gamma(a,u;e,f)\\
&= (\alpha q-\beta p)\Gamma(a,u;a,u).
\end{aligned}$$

Also

$$\begin{aligned}
\Gamma(c,v;c,v) &= \Gamma(\alpha a+\beta u+e,v;\alpha a+\beta u+e,v)\\
&= \alpha^2\Gamma(a,v;a,v)+2\alpha\beta\Gamma(a,v;u,v)+\beta^2\Gamma(u,v;u,v)\\
&\quad +2\alpha\Gamma(a,v;e,v)+2\beta\Gamma(u,v;e,v)+\Gamma(e,v;e,v). \tag{6}
\end{aligned}$$

Then we get

$$\begin{aligned}
\Gamma(a,v;a,v) &= \Gamma(a,qu+f;a,qu+f)=q^2\Gamma(a,u;a,u)+\Gamma(a,f;a,f),\\
\Gamma(a,v;u,v) &= \Gamma(a,qu+f;u,pa+f)=-2pq\Gamma(a,u;a,u),\\
\Gamma(u,v;u,v) &= \Gamma(u,pa+f;u,pa+f)=p^2\Gamma(a,u;a,u)+\Gamma(u,f;u,f),\\
\Gamma(a,v;e,v) &= \Gamma(a,qu+f;e,v)=\Gamma(a,f;e,v)=p\Gamma(a,f;e,a),\\
\Gamma(u,v;e,v) &= \Gamma(u,pa+f;e,v)=\Gamma(u,f;e,v)=q\Gamma(u,f;e,u),\\
\Gamma(e,v;e,v) &= p^2\Gamma(a,e;a,e)+q^2\Gamma(e,u;e,u)+\Gamma(e,f;e,f).
\end{aligned}$$

By replacing these relations in (6) we obtain

$$\begin{aligned}
\Gamma(c,v;c,v) &= (\alpha q-\beta p)^2\Gamma(a,u;a,u)\\
&\quad +\alpha^2\Gamma(a,f;a,f)+2\alpha p\Gamma(a,f;e,a)+p^2\Gamma(e,a;e,a)\\
&\quad +\beta^2\Gamma(u,f;u,f)+2\beta q\Gamma(u,f,e,u)+q^2\Gamma(e,u,e,u)+\Gamma(e,f;e,f)\\
&= (\alpha q-\beta p)^2\Gamma(a,u;a,u)+B,
\end{aligned}$$

where

$$B=\Gamma(a,\alpha f-pe;a,\alpha f-pe)+\Gamma(u,\beta f-qe;u,\beta f-qe)+\Gamma(e,f;e,f).$$

Therefore we have

$$\begin{aligned}
\Gamma(a,b;a,b)\Gamma(c,d;c,d) &= \Gamma(a,u;a,u)\Gamma(c,v;c,v)\\
&= (\alpha q-\beta p)^2\Gamma^2(a,u;a,u)+\Gamma(a,u;a,u)B\\
&= \Gamma^2(a,u;c,v)+\Gamma(a,b;a,b)B\\
&= \Gamma^2(a,b;c,d)+\Gamma(a,b;a,b)B.
\end{aligned}$$

We obtained (5). $\quad\square$

REMARK 1. Relation (5) from Theorem 1 can be written in an equivalent mode

$$\Gamma(a,b;a,b)\Gamma(c,d;c,d) - \Gamma^2(a,b;c,d)$$
$$= \Gamma(a,b;a,b)[\|a\|^2\|\alpha f - pe\|^2 + \|u\|^2\|\beta f - qe\|^2 + \|e\|^2\|f\|^2 - \langle e,f\rangle^2], \quad (7)$$

because $a \perp (\alpha f - pe)$ and $u \perp (\beta f - qe)$.

COROLLARY 1. *Let X be an inner product space. For any elements $a,b,c,d \in X$*

$$\Gamma(a,b;a,b) \cdot \Gamma(c,d;c,d) \geqslant \Gamma(a,b;c,d)]^2. \quad (8)$$

REMARK 2. Another proof of Corollary 1 can be given using the Gramian.

COROLLARY 2. *Let X be a real inner product space. For any $a,b,c,d \in X$ we have*

$$\|a\|^2\|b\|^2 - (\langle a,b\rangle)^2 + \|c\|^2\|d\|^2 - (\langle c,d\rangle)^2 \geqslant 2\left|\langle a,c\rangle\langle b,d\rangle - \langle a,d\rangle\langle c,b\rangle\right|. \quad (9)$$

Proof. We apply Theorem 2 and the inequality $(x+y)^2 \geqslant 4|xy|$, with $x = \|a\|^2\|b\|^2 - (\langle a,b\rangle)^2$ and $y = \|c\|^2\|d\|^2 - (\langle c,d\rangle)^2$. $\square$

REMARK 3. The inequality in Corollary 2 is the generalization for arbitary inner space of the result given [5], for the space $\mathbb{R}^n$.

We pass now to obtain an other improvement of inequality (8).

LEMMA 2. *For any $a,b,c,d \in X$ we have*

$$\Gamma(a,b;a,c)\Gamma(a,b;a,d)\|c\|^2 - \langle a,c\rangle\Gamma(a,b;a,b)\Gamma(a,c;c,d) - \langle c,d\rangle\Gamma^2(a,b,a,c)$$
$$= \|a\|^2\Big[\Gamma(a,b;bc)\Gamma(a,c;c,d) + \Gamma(a,b;c,a)\Gamma(b,c;c,d)\Big].$$

Proof. Using the definition of $\Gamma(a,b;c,d)$ we deduce

$$\Gamma(a,b;a,c)\Gamma(a,b;a,d) = \Big(\|a\|^2\langle b,c\rangle - \langle a,c\rangle\langle a,b\rangle\Big)\Big(\|a\|^2\langle b,d\rangle - \langle a,b\rangle\langle a,d\rangle\Big)$$
$$= \|a\|^4\langle b,c\rangle\langle b,d\rangle - \|a\|^2\langle b,c\rangle\langle a,b\rangle\langle a,d\rangle$$
$$- \|a\|^2\langle a,c\rangle\langle a,b\rangle\langle b,d\rangle + \langle a,b\rangle^2\langle a,c\rangle\langle a,d\rangle.$$

$$\Gamma(a,b;a,b)\Gamma(a,c;c,d) = \Big(\|a\|^2\|b\|^2 - \langle a,b\rangle^2\Big)\Big(\langle a,c\rangle\langle c,d\rangle - \|c\|^2\langle a,d\rangle\Big)$$
$$= \|a\|^2\|b\|^2\langle a,c\rangle\langle c,d\rangle - \|a\|^2\|b\|^2\|c\|^2\langle a,d\rangle$$
$$- \langle a,b\rangle^2\langle a,c\rangle\langle c,d\rangle + \|c\|^2\langle a,b\rangle^2\langle a,d\rangle.$$

$$\Gamma^2(a,b;b,c) = \left(\|a\|^2 \langle b,c \rangle - \langle a,b \rangle \langle a,c \rangle \right)^2$$
$$= \|a\|^4 \langle b,c \rangle^2 - 2\|a\|^2 \langle b,c \rangle \langle a,b \rangle \langle a,c \rangle + \langle a,b \rangle^2 \langle a,c \rangle^2.$$

Therefore we obtain

$$\|c\|^2 \Gamma(a,b;a,c)\Gamma(a,b;a,d) - \langle a,c \rangle \Gamma(a,b;a,b)\Gamma(a,c;c,d) - \langle c,d \rangle \Gamma^2(a,b;b,c)$$
$$= \|a\|^2 \Big[\|c\|^2 \|a\|^2 \langle b,c \rangle \langle b,d \rangle - \|c\|^2 \langle b,c \rangle \langle a,d \rangle - \|c\|^2 \langle a,c \rangle \langle a,b \rangle \rangle b,d \rangle$$
$$- \|b\|^2 \langle a,c \rangle^2 \langle c,d \rangle + \|b\|^2 \|c\|^2 \langle a,d \rangle \langle a,c \rangle - \|a\|^2 \langle c,d \rangle \langle b,c \rangle^2$$
$$+ 2\langle b,c \rangle \langle a,b \rangle \langle a,c \rangle \langle c,d \rangle \Big]$$
$$= \|a\|^2 \Big[\left(\langle a,b \rangle \langle b,c \rangle - \|b\|^2 \langle a,c \rangle \right) \left(\langle a,c \rangle \langle a,d \rangle - \langle a,d \rangle \|c\|^2 \right)$$
$$+ \left(\langle a,c \rangle \langle b,a \rangle - \|a\|^2 \langle b,c \rangle \right) \left(\langle b,c \rangle \langle c,d \rangle - \|c\|^2 \langle b,d \rangle \right) \Big]$$
$$= \|a\|^2 \Big[\Gamma(a,b;bc)\Gamma(a,c;c,d) + \Gamma(a,b;c,a)\Gamma(b,c;c,d) \Big].$$

It follows the statement. $\square$

Using the above lemma we obtain:

THEOREM 2. *Let X be an inner product space. For any elements $a,b,c,d \in X$ with $\{a,b\}$ linearly independent and c nonzero vector, we have*

$$\Gamma(a,b;a,b) \cdot \Gamma(c,d;c,d) - \Gamma^2(a,b;c,d)$$
$$= \frac{1}{\|c\|^4 \Gamma^2(a,b;a,b)} \Big[\Gamma(a,b;b,c)\Gamma(a,c;c,d) + \Gamma(a,b;c,a)\Gamma(b,c;c,d) \Big]^2.$$

Proof. As in Theorem 1 is $\lambda \in \mathbb{R}$ and $u \in X$ such that $b = \lambda a + u$ and $u \perp a$. Also there is $\mu \in \mathbb{R}$ and $v \in \mathbb{R}$, such that $d = \mu c + v$ and $v \perp c$. We obtain $\Gamma(a,b;a,b) = \Gamma(a,u;a,u)$, $\Gamma(c,d;c,d) = \|c\|^2 \|v\|^2$ and $\Gamma(a,b;c,d) = \Gamma(a,u;c,v)$.

There are $\alpha, \beta, p, q \in \mathbb{R}$ and $e, f \in X$, such that

$$c = \alpha a + \beta u + e, \text{ and } e \perp \text{Sp}\{a,u\};$$
$$v = pa + qu + f, \text{ and } f \perp \text{Sp}\{a,u\}.$$

With these notations we have

$$\Gamma(a,u;c,v) = (\alpha q - \beta p)\|a\|^2 \|u\|^2.$$

On the other hand,

$$\|c\|^2 = \alpha^2 \|a\|^2 + \beta^2 \|u\|^2 + \|e\|^2,$$
$$\|d\|^2 = p^2 \|a\|^2 + q^2 \|u\|^2 + \|f\|^2.$$

But

$$\Gamma(c,d;c,d) = \|c\|^2\|v^2 \geqslant (\alpha^2\|a\|^2 + \beta^2\|u\|^2)(p^2\|a\|^2 + q^2\|u\|^2)$$
$$= (\alpha p\|a\|^2 + \beta q\|u\|u^2)^2 + (\alpha p - \beta q)^2\|a\|^2\|u\|^2$$

which implies the following inequality

$$\Gamma(a,b;a,b) \cdot \Gamma(c,d;c,d) \geqslant \Gamma^(a,u;c,v) + (\alpha p\|a\|^2 + \beta q\|u\|u^2)^2. \qquad (10)$$

From the proof of Theorem 1, we find

$$\alpha = \frac{\langle a,c\rangle}{\|a\|^2}, \; p = \frac{\langle a,v\rangle}{\|a\|^2}, \; \beta = \frac{\Gamma(a,b,a,c)}{\Gamma(a,b,a,b)}, \; q = \frac{\Gamma(a,b;a,v)}{\Gamma(a,b;a,b)}.$$

So, we can calculate the following

$$\alpha p\|a\|^2 + \beta q\|u\|u^2 = \frac{\langle a,c\rangle\langle a,v\rangle\Gamma(a,b;a,b) + \Gamma(a,b;a,c)\Gamma(a,b;a,v)}{\|a\|^2\Gamma(a,b;a,b)}. \qquad (11)$$

We know that $v = d - \frac{\langle c,d\rangle}{\|v\|^2}c$, which implies

$$\langle a,v\rangle \; = -\frac{\Gamma(a,c;c,d)}{\|c\|^2}$$

$$\Gamma(a,b;a,v) \; = \Gamma(a,b;a,d) - \frac{\langle c,d\rangle}{\|c\|^2}\cdot\Gamma(a,b;a,c).$$

Relation (11) becomes

$$\alpha p\|a\|^2 + \beta q\|u\|u^2 = \frac{1}{\|a\|^2\|c\|^2\Gamma(a,b;a,b)}\Big[\Gamma(a,b;a,c)\Gamma(a,b;a,d)\|c\|^2$$
$$-\langle a,c\rangle\Gamma(a,b;a,b)\Gamma(a,c;c,d) - \langle c,d\rangle\Gamma^2(a,b,a,c)\Big].$$

and using Lemma 2 we obtain

$$\Big(\alpha p\|a\|^2 + \beta q\|u\|u^2\Big)^2$$
$$= \frac{1}{\|c\|^4\Gamma^2(a,b;a,b)}\Big[\Gamma(a,b;b,c)\Gamma(a,c,c,d) + \Gamma(a,b;c,a)\Gamma(b,c,c,d)\Big]^2. \quad (12)$$

Combining relations (10), (12) and taking into account that $\Gamma(a,u;c.v) = \Gamma(a,b;c,d)$ we obtain the statement $\square$

The result given in Corollary 2 could be extended in the following mode.

COROLLARY 3. *Let X be a real inner product space. For any integer $n \geqslant 2$ and any elements $x_1, x_2, \ldots, x_n \in X$, $y_1, y_2, \ldots, y_n \in X$ there holds*

$$(n-1)\sum_{i=1}^n \big(\|x_i\|^2\|y_i\|^2 - (\langle x_i, y_i\rangle)^2\big) \geqslant 2\sum_{1\leqslant i<j\leqslant n}\Big|\langle x_i, x_j\rangle\langle y_i, y_j\rangle - \langle x_i, y_j\rangle\langle x_j, y_i\rangle\Big|. \quad (13)$$

Proof. We have

$$(n-1)\sum_{i=1}^{n}\left(\|x_i\|^2\|y_i\|^2-(\langle x_i,y_i\rangle)^2\right)$$

$$=\sum_{1\leqslant i<j\leqslant n}\left[\|x_i\|^2\|y_i\|^2-(\langle x_i,y_i\rangle)^2+\|x_j\|^2\|y_j\|^2-(\langle x_j,y_j\rangle)^2\right].$$

Therefore the inequality given in (13) reduces to the following inequalities

$$\|x_i\|^2\|y_i\|^2-(\langle x_i,y_i\rangle)^2+\|x_j\|^2\|y_j\|^2-(\langle x_j,y_j\rangle)^2$$
$$\geqslant 2\left|\langle x_i,x_j\rangle\langle y_i,y_j\rangle-\langle x_i,y_j\rangle\langle x_j,y_i\rangle\right|,$$

for all the pairs of integers (i,j), such that $1\leqslant i<j\leqslant n$. Then we can apply Corollary 1, for any pair (i,j), taking $a:=x_i$, $b:=y_i$, $c:=x_j$, $d:=y_j$. □

REMARK 4. If in Corollary 1 we take $a=x$, $c=y$, $b=d=e$, with $\|e\|=1$, then we have

$$\left(\|x\|^2-\langle x,e\rangle^2\right)\left(\|y\|^2-\langle y,e\rangle^2\right)\geqslant\left(\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right)^2.$$

Therefore, we deduce

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\leqslant\sqrt{\|x\|^2-\langle x,e\rangle^2}\sqrt{\|y\|^2-\langle y,e\rangle^2}$$

$$\leqslant\min\{\|x\|\sqrt{\|y\|^2-\langle y,e\rangle^2},\|y\|\sqrt{\|x\|^2-\langle x,e\rangle^2}\}.$$

This inequality is an improvement of the inequality

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\leqslant\min\{\|x\|\sqrt{\|y\|^2-\langle y,e\rangle^2},\|y\|\sqrt{\|x\|^2-\langle x,e\rangle^2}\},$$

given by Dragomir ([4], Theorem 2.1).

We present an application of Theorem 2 to S_n numbers. Recall that if $(X,\langle\cdot,\cdot\rangle)$ is a inner product space and $\{e_1,\ldots,e_n\}$ is an orthonormal system of vectors of X, then for any vectors $x,y\in X$, we define as in [6]:

$$S_n(x,y)=\langle x,y\rangle-\sum_{k=0}^{n}\langle x,e_k\rangle\langle y,e_k\rangle,$$

$$\hat{x}=x-\sum_{k=0}^{n}\langle x,e_k\rangle e_k.$$

COROLLARY 4. *Let* $\{e_1,\ldots,e_n\}$ *be an orthonormal system in the inner product space* X. *Let* $\ell_1,\ell_2\in X$, *linearly independent, such that* $\langle e_k,\ell_i\rangle=0$, *for* $k=1,n$ *and* $i=1,2$. *Then, for any vectors* $x,y\in X$ *we have*

$$S_n(x,x)S_n(y,y)-S_n^2(x,y)\geqslant\frac{\Gamma^2(x,y;\ell_1,\ell_2)}{\Gamma(\ell_1,\ell_2;\ell_1,\ell_2)}+T(\hat{x},\hat{y},\ell_1,\ell_2),\tag{14}$$

where

$$T(\hat{x},\hat{y},\ell_1,\ell_2) = \frac{\left[\Gamma(\hat{x},\hat{y};\hat{y},\ell_1)\Gamma(x,\ell_1;\ell_1,\ell_2) + \Gamma(\hat{x},\hat{y};\ell_1,\hat{x})\Gamma(y,\ell_1;\ell_1,\ell_2)\right]^2}{\|\ell_1\|^4\Gamma^2(\hat{x},\hat{y};\hat{x},\hat{y})\Gamma(\ell_1,\ell_2;\ell_1,\ell_2)}.$$

Proof.

With the above notation we obtain immediately

$$\langle\hat{x},\hat{y}\rangle = \langle\hat{x},y\rangle = \langle x,\hat{y}\rangle = S_n(x,y). \tag{15}$$

From relations (4) and (15) we obtain

$$\Gamma(\hat{x},\hat{y};\hat{x},\hat{y}) = S_n(x,x)S_n(y,y) - (S_n(x,y))^2, \; x,y \in X. \tag{16}$$

From the properties of vectors ℓ_1,ℓ_2 we obtain for any vectors $x,y \in X$:

$$\Gamma(\hat{x},\hat{y};\ell_1,\ell_2) = \Gamma(x,y;\ell_1,\ell_2). \tag{17}$$

We apply Theorem 2 with the choice $a := \hat{x}$, $:= \hat{y}$, $c := \ell_1$, $d := \ell_2$ and we get

$$\Gamma(\hat{x},\hat{y};\hat{x},\hat{y}) \geqslant \frac{\Gamma^2(\hat{x},\hat{y};\ell_1,\ell_2)}{\Gamma(\ell_1,\ell_2;\ell_1,\ell_2)} + T(\hat{x},\hat{y},\ell_1,\ell_2). \tag{18}$$

From relations (16), (18) and (17) we obtain relation (14).   □

REMARK 5. Relation given in (14) and is an improvment of the inequality given by Kechriniotis and Delibasis [6]:

$$|S_n(x,y)|^2 \leqslant S_n(x,x)S_n(y,y) - R(x,y,\ell_1,\ell_2), \tag{19}$$

where

$$R(x,y,\ell_1,\ell_2) = \frac{\left[\langle x,\ell_1\rangle\langle y,\ell_2\rangle - \langle x,\ell_2\rangle\langle y,\ell_1\rangle\right]^2}{\|\ell_1\|^2\|\ell_2\|^2 - |\langle\ell_1,\ell_2\rangle|^2},$$

since

$$\frac{\left[\langle x,\ell_1\rangle\langle y,\ell_2\rangle - \langle x,\ell_2\rangle\langle y,\ell_1\rangle\right]^2}{\|\ell_1\|^2\|\ell_2\|^2 - |\langle\ell_1,\ell_2\rangle|^2} = \frac{\Gamma^2(x,y;\ell_1,\ell_2)}{\Gamma(\ell_1,\ell_2;\ell_1,\ell_2)}. \tag{20}$$

An other variant of inequality (8) is given in the next theorem.

THEOREM 3. *Let X be a real inner product space. Then for any vectors $a,b,c,d \in X$, the following inequality*

$$\left(\|a\|^2\|c\|^2 + \langle a,b\rangle\langle c,d\rangle\right)\left(\|b\|^2\|d\|^2 + \langle a,b\rangle\langle c,d\rangle\right)$$
$$\geqslant \left(\langle a,b\rangle\|c\|^2 + \langle c,d\rangle\|b\|^2\right)\left(\langle a,b\rangle\|d\|^2 + \langle c,d\rangle\|a\|^2\right)$$
$$+ \left(\langle a,c\rangle\langle b,d\rangle - \langle a,d\rangle\langle b,c\rangle\right)^2$$

holds.

Proof. We consider the matrices

$$A = \begin{pmatrix} \|a\|^2 & \langle a,b \rangle \\ \langle a,b \rangle & \|b\|^2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \|c\|^2 & \langle c,d \rangle \\ \langle c,d \rangle & \|d\|^2 \end{pmatrix}.$$

It follows

$$AB = \begin{pmatrix} \|a\|^2 \|c\|^2 + \langle a,b \rangle \langle c,d \rangle & \|a\|^2 \langle c,d \rangle + \langle a,b \rangle \|d\|^2 \\ \langle a,b \rangle \|c\|^2 + \langle c,d \rangle \|b\|^2 & \langle a,b \rangle \langle c,d \rangle + \|b\|^2 \|d\|^2 \end{pmatrix}.$$

But, we have the property

$$\det AB = \det A \cdot \det B.$$

So, we deduce

$$\Big(\|a\|^2 \|c\|^2 + \langle a,b \rangle \langle c,d \rangle \Big) \Big(\|b\|^2 \|d\|^2 + \langle a,b \rangle \langle c,d \rangle \Big)$$
$$- \Big(\langle a,b \rangle \|c\|^2 + \langle c,d \rangle \|b\|^2 \Big) \Big(\langle a,b \rangle \|d\|^2 + \langle c,d \rangle \|a\|^2 \Big)$$
$$= \Big(\|a\|^2 \|b\|^2 - \langle a,b \rangle^2 \Big) \Big(\|c\|^2 \|d\|^2 - \langle c,d \rangle^2 \Big).$$

Using inequality (5) we obtain the desired inequality. □

For nonzero vectors x and y in X we define *the angular distance* $\alpha[x,y]$ between x and y by

$$\alpha[x,y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|, \tag{21}$$

see ([2]).

Therefore, from [1], using the relation:

$$\frac{1}{2} \|x\| \cdot \|y\| \cdot \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 = \|x\| \cdot \|y\| - \langle x,y \rangle, \tag{22}$$

we obtain that

$$\|x\| \cdot \|y\| - \langle x,y \rangle = \frac{1}{2} \|x\| \cdot \|y\| \cdot (\alpha[x,y])^2. \tag{23}$$

In what follows we obtain an other inequality in conection with relation (2).

THEOREM 4. *Let X be a real inner product space. Then for any nonzero vectors $a,b,c,d \in X$, with $\langle a,b \rangle, \langle c,d \rangle \geqslant 0$, the following inequality*

$$\|a\|^2 \|b\|^2 - \langle a,b \rangle^2 + \|c\|^2 \|d\|^2 - \langle c,d \rangle^2 \geqslant \frac{1}{2} \cdot \|a\| \cdot \|b\| \cdot \|c\| \cdot \|d\| \cdot (\alpha[a,b] \cdot \alpha[c,d])^2 \tag{24}$$

holds.

Proof. Using the following identity $\|a\|^2\|b\|^2 - \langle a,b\rangle^2 = \left\|\, \|b\|a - \frac{\langle a,b\rangle}{\|b\|}b\,\right\|^2$ in a real inner product space, we have:

$$\|a\|^2\|b\|^2 - \langle a,b\rangle^2 + \|c\|^2\|d\|^2 - \langle c,d\rangle^2 = \left\|\, \|b\|a - \frac{\langle a,b\rangle}{\|b\|}b\,\right\|^2 + \left\|\, \|d\|c - \frac{\langle c,d\rangle}{\|d\|}d\,\right\|^2$$

$$\geqslant 2\left\|\, \|b\|a - \frac{\langle a,b\rangle}{\|b\|}b\,\right\| \cdot \left\|\, \|d\|c - \frac{\langle c,d\rangle}{\|d\|}d\,\right\|$$

$$\geqslant 2\Big(\|a\|\cdot\|b\| - |\langle a,b\rangle|\Big)\Big(\|c\|\cdot\|d\| - |\langle c,d\rangle|\Big).$$

Therefore, using (22), we deduce

$$\|a\|^2\|b\|^2 - \langle a,b\rangle^2 + \|c\|^2\|d\|^2 - \langle c,d\rangle^2$$

$$\geqslant 2\cdot\frac{1}{2}\|a\|\cdot\|b\|\cdot\left\|\frac{a}{\|a\|} - \frac{b}{\|b\|}\right\|^2 \cdot \frac{1}{2}\|c\|\cdot\|d\|\cdot\left\|\frac{c}{\|c\|} - \frac{d}{\|d\|}\right\|^2.$$

Combining this inequality with relation (21) we get the statement. □

REMARK 6. In certain cases inequality (24) is stronger than inequality (3), while in other cases the second one is stronger. For instance, if $X = \mathbb{R}^2$, $a = (1,1)$, $b = (1,0)$, $c = (1,-1)$, $d = (-1,1)$, then we deduce

$$\frac{1}{2}\cdot\|a\|\cdot\|b\|\cdot\|c\|\cdot\|d\|\cdot(\alpha[a,b]\cdot\alpha[c,d])^2 > 0 = 2|\langle a,c\rangle\langle b,d\rangle - \langle a,d\rangle\langle b,c\rangle|.$$

On the other hand, for $X = \mathbb{R}^2$, $a = \left(1,\frac{1}{2}\right)$, $b = (1,1)$, $c = \left(\frac{1}{2},1\right)$, $d = (1,0)$, then

$$2|\langle a,c\rangle\langle b,d\rangle - \langle a,d\rangle\langle b,c\rangle| = 1$$

$$> \frac{(\sqrt{10}-3)(\sqrt{5}-1)}{2} = \frac{1}{2}\cdot\|a\|\cdot\|b\|\cdot\|c\|\cdot\|d\|\cdot(\alpha[a,b]\cdot\alpha[c,d])^2.$$

REMARK 7. As in Corollary 2, if $\langle x_i,y_i\rangle \geqslant 0$, $\forall\, i = \overline{1,n}$, we obtain the following generalization of inequality (24):

$$(n-1)\sum_{i=1}^{n}\big(\|x_i\|^2\|y_i\|^2 - (\langle x_i,y_i\rangle)^2\big) \geqslant \sum_{1\leqslant i<j\leqslant n}\|x_i\|\cdot\|x_j\|\cdot\|y_i\|\cdot\|y_j\|\cdot(\alpha[x_i,x_j]\cdot\alpha[y_i,y_j])^2.$$

REFERENCES

[1] J. M. ALDAZ, *Strengthened Cauchy-Schwarz and Hölder inequalities*, J. Inequal. Pure Appl. Math. **10**, 4, (2009), 116 pp.

[2] D. CHOI, *A generalization of the Cauchy-Schwarz inequality*, J. Math. Inequal. **10**, 4 (2016), 1009–1012.

[3] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40**, 3 (1936), 396–414.

[4] S.S. Dragomir, *Some Grüss type inequalities in inner product spaces*, J. Inequal. Pure Appl. Math. **4**, 2 (2003), 42 pp.

[5] N. Harvey, *A generalization of the Cauchy-Schwarz inequality involving four vectors*, J. Math. Inequal. **9**, 2 (2015), 489–491.

[6] A. I. Kechriniotis and K .K. Delibasis, *On generalization of Grüss inequality in inner product spaces and applications*, J. Inequal. Appl. **2010**, 1 (2010), ID 167091.

(Received August 8, 2017)

Nicuşor Minculete
Transilvania University of Braşov
Faculty of Mathematics and Informatics
Iuliu Maniu Street no. 50, 500091, Braşov, Romania
e-mail: minculeten@yahoo.com

Radu Păltănea
Transilvania University of Braşov
Faculty of Mathematics and Informatics
Iuliu Maniu Street no. 50, 500091, Braşov, Romania
e-mail: radupaltanea@yahoo.com

Journal of Mathematical Inequalities
www.ele-math.com
jmi@ele-math.com