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AN EXTENSION OF THE DAVIS–GUT LAW AND LAI LAW

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Abstract. Let $\{X, X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables with $EX = 0$ and $EX^2 = 1$ and the partial sums $S_n = \sum_{k=1}^n X_k$, $n \geqslant 1$. Assume that $f(x)$ and $g(x)$ are positive functions defined on $[0,\infty)$. In this short note, under some suitable conditions, we establish the following results

$$\sum_{n=1}^{\infty} f(n) P\{|S_n| > \beta\sqrt{n} g(n)\} < \infty \quad or \quad = \infty$$

according as

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \exp\{-\frac{\beta^2}{2} g^2(n)(1+\alpha(n))\} < \infty \quad or \quad = \infty$$

where $\alpha(n) = EX^2 I\{|X| > \sqrt{n} g(n)\}/EX^2 I\{|X| \leqslant \sqrt{n} g(n)\}$, $\beta > 0$. The results extend and generalize the known Davis-Gut Law and Lai Law.

1. Introduction and main results

Throughout this paper, let $\{X, X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables with $EX = 0$, $EX^2 = 1$ and denote $S_n = \sum_{k=1}^n X_k$, $n \geqslant 1$. Let $\mathscr{N}$ be the standard normal random variable. C denotes a positive constant, possibly varying from place to place, the notion $a_n \sim b_n$ stands for $\lim_{n\to\infty} a_n/b_n = 1$.

The following theorem, which was related to the classical Hartman-Wintner law of the iterated logarithm, is labeled as the Davis-Gut Law.

THEOREM A. *Let $\{X, X_n, n \geqslant 1\}$ be a sequence of i. i. d. random variables with the partial sums $S_n = \sum_{k=1}^n X_k$, $n \geqslant 1$. Then the following three statements are equivalent*

$$EX = 0 \ \ and \ \ EX^2 = 1, \tag{1.1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{|S_n| > (1+\varepsilon)\sqrt{2n\log\log n}\} \begin{cases} < \infty, \ if \ \ \varepsilon > 0, \\ = \infty, \ if \ \ \varepsilon < 0. \end{cases} \tag{1.2}$$

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$$\sum_{n=1}^{\infty}\frac{\log\log n}{n}P\{|S_n|>(1+\varepsilon)\sqrt{2n\log\log n}\}\begin{cases}<\infty, & if\ \ \varepsilon>0,\\ =\infty, & if\ \ \varepsilon<0.\end{cases} \tag{1.3}$$

(1.1)$\Rightarrow$ (1.2) can be deduced by Theorem 4 of Davis [3] which was remedied by Corollary 2.3 of Li et al. [7], (1.2)$\Rightarrow$ (1.1) can be got by Theorem 6.2 of Gut [4]. (1.1)$\Leftrightarrow$ (1.3) was proved by Li [6]. For analogue results, we refer to Li and Rosalsky [8], Liu et al. [9].

The next theorem, which was related to the law of the single logarithm, is labeled as the Lai Law.

THEOREM B. *Let* $\{X, X_n, n \geqslant 1\}$ *be a sequence of i. i. d. random variables with the partial sums* $S_n = \sum_{k=1}^{n} X_k$, $n \geqslant 1$. *Then the following two statements are equivalent*

$$EX=0,\ \ EX^2=1\ \ and\ \ E(X^2/\log|X|)^{r+1}<\infty,\ \ r>0, \tag{1.4}$$

$$\sum_{n=1}^{\infty}n^{r-1}P\{|S_n|>(1+\varepsilon)\sqrt{2rn\log n}\}\begin{cases}<\infty, & if\ \ \varepsilon>0,\\ =\infty, & if\ \ \varepsilon<0.\end{cases} \tag{1.5}$$

Lai [5] established the equivalence for $\varepsilon > 0$, Chen and Wang [2] extended it to $\varepsilon < 0$. Chen and Qi [1] discussed the similar results of (1.5) for $r = 0$. Liu and Meng [11] established the analogue results of Davis-Gut Law and Lai Law for finitely inhomogeneous walks.

The gap for $\varepsilon = 0$ of Theorem A was solved by Liu and Guo [10], they obtained

THEOREM C. *Let* $\{X, X_n, n \geqslant 1\}$ *be a sequence of i. i. d. random variables with* $EX = 0$, $EX^2 = 1$ *and the partial sums* $S_n = \sum_{k=1}^{n} X_k$, $n \geqslant 1$. *Then*

$$\sum_{n=1}^{\infty}\frac{1}{n}P\{|S_n|>\sqrt{2n\log\log n}\}<\infty\quad or\quad =\infty \tag{1.6}$$

according as

$$\sum_{n=1}^{\infty}\frac{1}{n\sqrt{\log\log n}}\cdot\frac{1}{(\log n)^{1+\alpha(n)}}<\infty\quad or\quad =\infty \tag{1.7}$$

where $\alpha(n) = EX^2I\{|X| > \sqrt{n\log\log n}\}/EX^2I\{|X| \leqslant \sqrt{n\log\log n}\}$.

$$\sum_{n=1}^{\infty}\frac{\log\log n}{n}P\{|S_n|>\sqrt{2n\log\log n}\}<\infty\quad or\quad =\infty \tag{1.8}$$

according as

$$\sum_{n=1}^{\infty}\frac{\sqrt{\log\log n}}{n(\log n)^{1+\alpha(n)}}<\infty\quad or\quad =\infty \tag{1.9}$$

where $\alpha(n) = EX^2I\{|X| > \sqrt{n\log\log n}\}/EX^2I\{|X| \leqslant \sqrt{n\log\log n}\}$.

In this paper, we want to extend and generalize the above Theorems to some functions. The main result of this note is the following.

THEOREM 1. (Main) *Let* $\{X, X_n, n \geqslant 1\}$ *be a sequence of i. i. d. random variables with* $EX = 0$, $EX^2 = 1$ *and the partial sums* $S_n = \sum_{k=1}^n X_k$, $n \geqslant 1$. *Let* $f(x)$, $g(x)$ *be positive functions defined on* $[0, \infty)$, *suppose that the following conditions hold*

(C1) $g(x)$ *is strictly increasing to* ∞,

(C2) $\sum_{k=1}^n kf(k) \leqslant Cn^2 f(n)$,

(C3) $\sum_{k=n}^\infty k^{-1/2} f(k) g^{-3}(k) \leqslant Cn^{1/2} f(n) g^{-3}(n)$,

(C4) $nf(n) = O(g^2(n))$, *or*

(C4)* $\limsup_{n\to\infty} \frac{nf(n)}{g^2(n)} = \infty$, $\limsup_{n\to\infty} \max_{ng^2(n) \leqslant x^2 \leqslant (n+1)g^2(n+1)} \frac{g^4(x^2)}{g^4(n)} \cdot \frac{f(n)}{f(\frac{x^2}{g^2(x^2)})} \leqslant C$, $E[\frac{X^4}{g^4(X^2)} f(\frac{X^2}{g^2(X^2)})] < \infty$. *Then*

$$\sum_{n=1}^{\infty} f(n) P\{|S_n| > \beta \sqrt{n} g(n)\} < \infty \quad or \quad = \infty \tag{1.10}$$

according as

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \exp\{-\frac{\beta^2}{2} g^2(n)(1+\alpha(n))\} < \infty \quad or \quad = \infty \tag{1.11}$$

where $\alpha(n) = EX^2 I\{|X| > \sqrt{n} g(n)\} / EX^2 I\{|X| \leqslant \sqrt{n} g(n)\}$, $\beta > 0$.

REMARK 1. There are many functions satisfying the assumptions of $f(x)$ and $g(x)$, such as $f(x) = \frac{l(x)}{x^\alpha}$, $g(x) = x^\gamma h(x)$, with some suitable conditions of α, $\gamma \geqslant 0$ and $l(x)$ and $h(x)$ are slowly varying at infinity. We list some examples in the following Corollaries.

COROLLARY 1. *Let* $f(n) = \frac{(\log\log n)^b}{n}$, $g(n) = \sqrt{\log\log n}$ *with* $EX^2 (\log\log|X|)^{\max\{0, b-1\}} < \infty$, $b \in R$, *then*

$$\sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n} P\{|S_n| > \beta \sqrt{n \log\log n}\} \begin{cases} < \infty, \ if \ \ \beta > \sqrt{2}, \\ = \infty, \ if \ \ \beta < \sqrt{2}. \end{cases}$$

And for $\beta = \sqrt{2}$, *we have*

$$\sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n} P\{|S_n| > \sqrt{2n \log\log n}\} < \infty \quad or \quad = \infty \tag{1.12}$$

according as

$$\sum_{n=1}^{\infty} \frac{(\log\log n)^{b-1/2}}{n(\log n)^{1+\alpha(n)}} < \infty \quad or \quad = \infty \tag{1.13}$$

where $\alpha(n) = EX^2 I\{|X| > \sqrt{n \log\log n}\} / EX^2 I\{|X| \leqslant \sqrt{n \log\log n}\}$.

COROLLARY 2. *Let* $f(n)=\frac{(\log n)^b}{n},\ g(n)=\sqrt{\log\log n}$ *with* $EX^2=1$ *for* $b\leqslant 0$ *and* $EX^2(\log|X|)^b(\log\log|X|)^{-1}<\infty$ *for* $b>0$, *then*

$$\sum_{n=1}^{\infty}\frac{(\log n)^b}{n}P\{|S_n|>\beta\sqrt{n\log\log n}\}\begin{cases}<\infty, & if\ \ \beta^2>2(1+b),\\ =\infty, & if\ \ \beta^2<2(1+b).\end{cases}$$

And for $\beta^2=2(1+b)>0$, *we have*

$$\sum_{n=1}^{\infty}\frac{(\log n)^b}{n}P\{|S_n|>\sqrt{2(1+b)n\log\log n}\}<\infty \quad or \quad =\infty \tag{1.14}$$

according as

$$\sum_{n=1}^{\infty}\frac{1}{n(\log\log n)^{1/2}}\frac{1}{(\log n)^{1+(1+b)\alpha(n)}}<\infty \quad or \quad =\infty \tag{1.15}$$

where $\alpha(n)=EX^2I\{|X|>\sqrt{n\log\log n}\}/EX^2I\{|X|\leqslant\sqrt{n\log\log n}\}$.

COROLLARY 3. *Let* $f(n)=\frac{(\log\log n)^b}{n},\ g(n)=\log^s n$ *with* $EX^2=1,\ s>0$, *then we have*

$$\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n}P\{|S_n|>\beta\sqrt{n}\log^s n\}<\infty.$$

COROLLARY 4. *Let* $f(n)=n^{r-1},\ g(n)=\sqrt{\log n}$ *with* $E[\frac{X^2}{\log|X|}]^{r+1}<\infty,\ 0<r<1/2$, *then*

$$\sum_{n=1}^{\infty}n^{r-1}P\{|S_n|>\beta\sqrt{n\log n}\}\begin{cases}<\infty, & if\ \ \beta>\sqrt{2r},\\ =\infty, & if\ \ \beta<\sqrt{2r}.\end{cases}$$

And for $\beta=\sqrt{2r}$, *we have*

$$\sum_{n=1}^{\infty}n^{r-1}P\{|S_n|>\sqrt{2rn\log n}\}<\infty \quad or \quad =\infty \tag{1.16}$$

according as

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+r\alpha(n)}(\log n)^{1/2}}<\infty \quad or \quad =\infty. \tag{1.17}$$

where $\alpha(n)=EX^2I\{|X|>\sqrt{n\log n}\}/EX^2I\{|X|\leqslant\sqrt{n\log n}\}$.

COROLLARY 5. *Let* $f(n)=n^{-\alpha},\ g(n)=n^{\gamma},\ 0<\alpha\leqslant 1,\gamma>0,\ \alpha+2\gamma\geqslant 1$, $EX^2=1$, *then*

$$\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}P\{|S_n|>\beta\sqrt{n}n^{\gamma}\}<\infty.$$

Especially, for $0<p<r<2$,

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P\{|S_n| > \beta n^{\frac{1}{p}}\} < \infty.$$

REMARK 2. Obviously, Corollaries 1-5 extend the known results (Theorems A-C), and we also give the gap $\varepsilon = 0$ in Theorem B.

The following examples shows that it is possible that the series converge or diverge in (1.13), (1.15) and (1.17) under some different conditions.

EXAMPLE 1. Let X be a random variable such that $\sqrt{c/d}X$ has the following density function

$$f_z(x) = \frac{1}{c} \cdot \frac{\log\log\log|x|}{|x|^3 \log|x| (\log\log|x|)^2} I\{|x| > z\}, \ \ z > 0$$

where

$$c = c(z) = 2\int_z^{\infty} \frac{\log\log\log x}{x^3 \log x (\log\log x)^2} dx, \qquad d = d(z) = 2\int_z^{\infty} \frac{\log\log\log x}{x \log x (\log\log x)^2} dx.$$

By the same discussion in Liu and Guo [10], we know d/c is strictly monotone increasing in $(0,\infty)$ and $\lim_{z\to 0} d/c = 0$ and $\lim_{z\to\infty} d/c = \infty$, and

$$EX = 0, \ \ EX^2 = 1, \ \ EX^2 I\{|X| > \sqrt{n\log\log n}\} \sim \frac{d}{c} \cdot \frac{\log\log\log n}{\log\log n},$$

$$(\log n)^{EX^2 I\{|X|>\sqrt{n\log\log n}\}} \sim (\log\log n)^{\frac{d}{c}}, \qquad n \to \infty.$$

Therefore there exist unique constants z_1 and z_2 such that

$$\frac{d(z_1)}{c(z_1)} = \frac{1}{2} + b > 0, \qquad \frac{d(z_2)}{c(z_2)} = \frac{1}{2(1+b)} > 0.$$

So we can easily conclude the series (1.13) converges or diverges according as $z \in (z_1,\infty)$ or $z \in (0,z_1)$, the series (1.15) converges or diverges according as $z \in (z_2,\infty)$ or $z \in (0,z_2)$.

EXAMPLE 2. As the same argument as in Example 1, $\sqrt{c'/d'}Y$ has the following density function

$$f_z(y) = \frac{1}{c'} \cdot \frac{\log\log|y|}{|y|^3 (\log|y|)^2} I\{|y| > z\}, \ \ z > 0$$

where

$$c' = c'(z) = 2\int_z^{\infty} \frac{\log\log x}{x^3 (\log x)^2} dx, \qquad d' = d'(z) = 2\int_z^{\infty} \frac{\log\log x}{x (\log x)^2} dx.$$

It is easy to prove d'/c' is strictly monotone increasing in $(0,\infty)$ and $\lim_{z\to 0} d'/c' = 0$ and $\lim_{z\to\infty} d'/c' = \infty$, and

$$EY = 0,\ \ EY^2 = 1,\ \ EY^2 I\{|Y| > \sqrt{n\log n}\} \sim \frac{d'}{c'} \cdot \frac{\log\log n}{\log n},$$

$$n^{EX^2 I\{|X|>\sqrt{n\log n}\}} \sim (\log n)^{\frac{d'}{c'}}, \qquad n\to\infty.$$

Therefore there exists an unique constant z_3 such that

$$\frac{d'(z_3)}{c'(z_3)} = \frac{1}{2r}.$$

So we can easily conclude the series (1.17) converges or diverges according as $z \in (z_3,\infty)$ or $z \in (0,z_3)$.

2. Proof of Theorem 1

The following lemmas are useful for the proof of Theorem 1. The first lemma is the nonuniform estimates of the remainder term in the central limit theorem (see Nagaev [12] ,Theorem 3, p.215).

LEMMA 1. *Let $\{X, X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables with $EX = 0$, $EX^2 = \sigma^2$ and $E|X|^q < \infty$, $2 < q \leqslant 3$, then there a positive constant C, such that*

$$|P\{\frac{S_n}{\sigma\sqrt{n}} \leqslant x\} - P\{\mathcal{N} \leqslant x\}| \leqslant \frac{CE|X|^q}{n^{q/2-1}(1+|x|^q)}.$$

LEMMA 2. *Under the conditions of Theorem 1, we have*

$$\sum_{n=1}^{\infty} nf(n)P\{|X| > \sqrt{n}g(n)\} < \infty, \tag{2.1}$$

$$\sum_{n=1}^{\infty} \frac{f(n)}{\sqrt{n}g^3(n)} E|X|^3 I\{|X| \leqslant \sqrt{n}g(n)\} < \infty. \tag{2.2}$$

Proof. Obviously, note that (C2),

$$\sum_{n=1}^{\infty} nf(n)P\{|X| > \sqrt{n}g(n)\} = \sum_{n=1}^{\infty} nf(n) \sum_{k=n}^{\infty} P\{\sqrt{k}g(k) < |X| \leqslant \sqrt{k+1}g(k+1)\}$$

$$= \sum_{k=1}^{\infty} P\{\sqrt{k}g(k) < |X| \leqslant \sqrt{k+1}g(k+1)\} \sum_{n=n_0}^{k} nf(n)$$

$$\leqslant C\sum_{k=1}^{\infty} k^2 f(k)P\{\sqrt{k}g(k) < |X| \leqslant \sqrt{k+1}g(k+1)\}.$$

If (C4) holds, we have

$$\sum_{n=1}^{\infty} nf(n)P\{|X|>\sqrt{n}g(n)\} \leqslant C\sum_{k=1}^{\infty} kg^2(k)P\{\sqrt{k}g(k)<|X|\leqslant \sqrt{k+1}g(k+1)\}$$
$$\leqslant CE|X|^2<\infty.$$

If (C4*) holds, we have

$$\sum_{n=1}^{\infty} nf(n)P\{|X|>\sqrt{n}g(n)\} \leqslant C\sum_{k=1}^{\infty} k^2f(k)P\{\sqrt{k}g(k)<|X|\leqslant \sqrt{k+1}g(k+1)\}$$
$$=C\sum_{k=1}^{\infty} E[\frac{X^4}{g^4(X^2)}f(\frac{X^2}{g^2(X^2)})]\cdot\frac{k^2g^4(k)}{X^4}\cdot\frac{g^4(X^2)}{g^4(k)}\frac{f(k)}{f(\frac{X^2}{g^2(X^2)})}$$
$$\cdot I\{\sqrt{k}g(k)<|X|\leqslant \sqrt{k+1}g(k+1)\}$$
$$\leqslant C\sum_{k=1}^{\infty} E[\frac{X^4}{g^4(X^2)}f(\frac{X^2}{g^2(X^2)})]I\{\sqrt{k}g(k)<|X|\leqslant \sqrt{k+1}g(k+1)\}$$
$$\leqslant CE[\frac{X^4}{g^4(X^2)}f(\frac{X^2}{g^2(X^2)})]<\infty.$$

Thus the proof of (2.1) is complete.

By the same argument as in (2.1), if (C3) and (C4) hold, we obtain

$$\sum_{n=1}^{\infty}\frac{f(n)}{\sqrt{n}g^3(n)}E|X|^3I\{|X|\leqslant\sqrt{n}g(n)\}$$
$$\leqslant C\sum_{k=1}^{\infty}E|X|^3I\{\sqrt{k}g(k)<|X|\leqslant \sqrt{k+1}g(k+1)\}\sum_{n=k}^{\infty}\frac{f(n)}{\sqrt{n}g^3(n)}$$
$$\leqslant C\sum_{k=1}^{\infty}\frac{\sqrt{k}f(k)}{g^3(k)}E|X|^3I\{\sqrt{k}g(k)<|X|\leqslant \sqrt{k+1}g(k+1)\}$$
$$\leqslant C\sum_{k=1}^{\infty}\frac{kf(k)}{g^2(k)}E|X|^2$$
$$\cdot I\{\sqrt{k}g(k)<|X|\leqslant \sqrt{k+1}g(k+1)\}$$
$$\leqslant C\sum_{k=1}^{\infty}E|X|^2I\{\sqrt{k}g(k)<|X|\leqslant \sqrt{k+1}g(k+1)\}\leqslant CE|X|^2<\infty,$$

if (C3) and (C4*) hold, we obtain

$$\sum_{n=1}^{\infty}\frac{f(n)}{\sqrt{n}g^3(n)}E|X|^3I\{|X|\leqslant\sqrt{n}g(n)\}$$
$$\leqslant C\sum_{k=1}^{\infty}\frac{\sqrt{k}f(k)}{g^3(k)}E|X|^3I\{\sqrt{k}g(k)<|X|\leqslant \sqrt{k+1}g(k+1)\}$$

$$\leqslant C\sum_{k=1}^{\infty} E[\frac{X^4}{g^4(X^2)} f(\frac{X^2}{g^2(X^2)})] \cdot \frac{\sqrt{k}g(k)}{|X|} \cdot \frac{g^4(X^2)}{g^4(k)} \frac{f(k)}{f(\frac{X^2}{g^2(X^2)})}$$
$$\cdot I\{\sqrt{k}g(k) < |X| \leqslant \sqrt{k+1}g(k+1)\}$$
$$\leqslant CE[\frac{X^4}{g^4(X^2)} f(\frac{X^2}{g^2(X^2)})] < \infty.$$

Thus the proof of (2.2) is complete. □

LEMMA 3. *Under the conditions of Theorem 1, we have*

$$\sum_{n=1}^{\infty} f(n)P\{|\mathcal{N}| > \frac{\beta g(n) \pm 1/g(n)}{\sigma_n}\} < \infty \quad or \quad = \infty \tag{2.3}$$

according as

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \exp\{-\frac{\beta^2}{2} g^2(n)(1+\alpha(n))\} < \infty \quad or \quad = \infty \tag{2.4}$$

where $\sigma_n^2 = E(XI\{|X| \leqslant \sqrt{n}g(n)\} - EXI\{|X| \leqslant \sqrt{n}g(n)\})^2$, $\alpha(n) = EX^2 I\{|X| > \sqrt{n}g(n)\}/EX^2 I\{|X| \leqslant \sqrt{n}g(n)\}$, $\beta > 0$.

Proof. It is easy to show that $EX = 0$ and

$$|\frac{g^2(n)}{\sigma_n^2} - \frac{g^2(n)}{EX^2 I\{|X| \leqslant \sqrt{n}g(n)\}}| = \frac{g^2(n)|EXI\{|X| > \sqrt{n}g(n)\}|^2}{\sigma_n^2 EX^2 I\{|X| \leqslant \sqrt{n}g(n)\}}$$
$$\leqslant \frac{1}{\sigma_n^2 EX^2 I\{|X| \leqslant \sqrt{n}g(n)\}} \frac{1}{n} \to 0.$$

Note that $P\{|\mathcal{N}| > x\} \sim \frac{2}{\sqrt{2\pi}} \frac{1}{x} \exp\{-\frac{x^2}{2}\}$ as $x \to \infty$, $\sigma_n \to 1$, $1 = EX^2 = EX^2 I\{|X| \leqslant \sqrt{n}g(n) + EX^2 I\{|X| > \sqrt{n}g(n)$, then we have

$$\sum_{n=1}^{\infty} f(n)P\{|\mathcal{N}| > \frac{\beta g(n) \pm 1/g(n)}{\sigma_n}\}$$
$$\sim C\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \exp\{-\frac{\beta^2 g^2(n) \pm 2\beta + 1/g^2(n)}{2\sigma_n^2}\}$$
$$\sim C\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \exp\{-\frac{\beta^2 g^2(n)}{2\sigma_n^2}\}$$
$$\sim C\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \exp\{-\frac{\beta^2 g^2(n)}{2EX^2 I\{|X| \leqslant \sqrt{n}g(n)\}}\}$$
$$\sim C\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \exp\{-\frac{\beta^2 g^2(n)}{2}(1+\alpha(n))\},$$

therefore the proof is completed. It is easy to see that $\sum_{n=1}^{\infty} f(n)P\{|\mathcal{N}| > \frac{\beta g(n)+1/g(n)}{\sigma_n}\}$ and $\sum_{n=1}^{\infty} f(n)P\{|\mathcal{N}| > \frac{\beta g(n)-1/g(n)}{\sigma_n}\}$ have the same convergence by the above proof. □

Proof of Theorem 1. Define $S_n' = \sum_{k=1}^{n} X_k I\{|X_k| \leqslant \sqrt{n}g(n)\}$, Obviously

$$|ES_n'| = n|EXI\{|X| > \sqrt{n}g(n)\}| \leqslant n(EX^2)^{1/2}P^{1/2}\{|X| > \sqrt{n}g(n)\}$$
$$\leqslant n(\frac{EX^2}{ng^2(n)})^{1/2} = \frac{\sqrt{n}}{g(n)}. \tag{2.5}$$

Note that

$$\{|S_n| > \beta\sqrt{n}g(n)\} \subset \{\max_{1\leqslant k\leqslant n}|X_k| > \sqrt{n}g(n)\} \cup \{|S_n'| > \beta\sqrt{n}g(n)\},$$
$$\{|S_n'| > \beta\sqrt{n}g(n)\} \subset \{\max_{1\leqslant k\leqslant n}|X_k| > \sqrt{n}g(n)\} \cup \{|S_n| > \beta\sqrt{n}g(n)\},$$

and by (2.5)

$$\{|S_n' - ES_n'| > \beta\sqrt{n}g(n) + \frac{\sqrt{n}}{g(n)}\} \subset \{|S_n'| > \beta\sqrt{n}g(n)\}$$
$$\subset \{|S_n' - ES_n'| > \beta\sqrt{n}g(n) - \frac{\sqrt{n}}{g(n)}\}.$$

Hence

$$P\{|S_n' - ES_n'| > \beta\sqrt{n}g(n) + \frac{\sqrt{n}}{g(n)}\} - P\{\max_{1\leqslant k\leqslant n}|X_k| > \sqrt{n}g(n)\} \leqslant P\{|S_n| > \beta\sqrt{n}g(n)\}$$
$$\leqslant P\{|S_n' - ES_n'| > \beta\sqrt{n}g(n) - \frac{\sqrt{n}}{g(n)}\} + P\{\max_{1\leqslant k\leqslant n}|X_k| > \sqrt{n}g(n)\}.$$

By the fact $|a| - |a-b| \leqslant |b| \leqslant |a| + |a-b|$, then

$$P\{|\mathcal{N}| > \frac{\beta\sqrt{n}g(n) + \frac{\sqrt{n}}{g(n)}}{\sqrt{Var(S_n')}}\} - |P\{|S_n' - ES_n'| > \beta\sqrt{n}g(n) + \frac{\sqrt{n}}{g(n)}\}$$
$$-P\{|\mathcal{N}| > \frac{\beta\sqrt{n}g(n) + \frac{\sqrt{n}}{g(n)}}{\sqrt{Var(S_n')}}\}| - P\{\max_{1\leqslant k\leqslant n}|X_k| > \sqrt{n}g(n)\}$$
$$\leqslant P\{|S_n| > \beta\sqrt{n}g(n)\}$$
$$\leqslant P\{\max_{1\leqslant k\leqslant n}|X_k| > \sqrt{n}g(n)\} + |P\{|S_n' - ES_n'| > \beta\sqrt{n}g(n) - \frac{\sqrt{n}}{g(n)}\}$$
$$-P\{|\mathcal{N}| > \frac{\beta\sqrt{n}g(n) - \frac{\sqrt{n}}{g(n)}}{\sqrt{Var(S_n')}}\}| + P\{|\mathcal{N}| > \frac{\beta\sqrt{n}g(n) - \frac{\sqrt{n}}{g(n)}}{\sqrt{Var(S_n')}}\}.$$

By (2.1), we have

$$\sum_{n=1}^{\infty} f(n)P\{\max_{1\leqslant k\leqslant n}|X_k| > \sqrt{n}g(n)\} \leqslant \sum_{n=1}^{\infty} nf(n)P\{|X| > \sqrt{n}g(n)\} < \infty.$$

By Lemma 1 and (2.2), $Var(S'_n) = n\sigma_n^2$, $\sigma_n \to 1$, $\frac{\beta g(n) \pm \frac{1}{g(n)}}{\sigma_n} \to \beta g(n)$, we know

$$\sum_{n=1}^{\infty} f(n)|P\{|S'_n - ES'_n| > \beta\sqrt{n}g(n) \pm \frac{\sqrt{n}}{g(n)}\} - P\{|\mathscr{N}| > \frac{\beta\sqrt{n}g(n) \pm \frac{\sqrt{n}}{g(n)}}{\sqrt{Var(S'_n)}}\}|$$
$$= \sum_{n=1}^{\infty} f(n)|P\{|S'_n - ES'_n| > \beta\sqrt{n}g(n) \pm \frac{\sqrt{n}}{g(n)}\} - P\{|\mathscr{N}| > \frac{\beta g(n) \pm \frac{1}{g(n)}}{\sigma_n}\}|$$
$$\leqslant C\sum_{n=1}^{\infty} \frac{f(n)}{\sqrt{n}g^3(n)} E|X|^3 I\{|X| \leqslant \sqrt{n}g(n)\} < \infty.$$

Therefore

$$\sum_{n=1}^{\infty} f(n)P\{|S_n| > \beta\sqrt{n}g(n)\} < \infty \quad or \quad = \infty \tag{2.6}$$

according as

$$\sum_{n=1}^{\infty} f(n)P\{|\mathscr{N}| > \frac{\beta g(n) \pm \frac{1}{g(n)}}{\sigma_n}\} < \infty \quad or \quad = \infty, \tag{2.7}$$

then by Lemma 3, the proof of Theorem 1 is complete. □

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