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ADDITIVE REFINEMENTS AND REVERSES OF YOUNG'S OPERATOR INEQUALITY WITH APPLICATIONS

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Abstract. In this paper we obtain some new additive refinements and reverses of Young's operator inequality. Applications related to the Hölder-McCarthy inequality for positive operators and for trace class operators on Hilbert spaces are given as well.

1. Introduction

Throughout this paper $A,\ B$ are positive operators on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. We use the following notations for operators

$$A\nabla_\nu B := (1-\nu)A+\nu B,$$

the *weighted operator arithmetic mean* and

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},$$

the *weighted operator geometric mean.* When $\nu=\frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

The famous *Young inequality* for scalars says that if $a,\ b>0$ and $\nu\in[0,1]$, then

$$a^{1-\nu}b^\nu \leqslant (1-\nu)a+\nu b \tag{1.1}$$

with equality if and only if $a=b$. The inequality (1.1) is also called ν-*weighted arithmetic-geometric mean inequality*.

We recall that *Specht's ratio* is defined by [13]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty), \\ 1 \text{ if } h=1. \end{cases} \tag{1.2}$$

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It is well known that $\lim_{h\to 1} S(h) = 1,\ S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0,\ h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right) a^{1-\nu} b^{\nu} \leqslant (1-\nu)a + \nu b \leqslant S\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{1.3}$$

where $a,\ b > 0,\ \nu \in [0,1],\ r = \min\{1-\nu, \nu\}$.

The second inequality in (1.3) is due to Tominaga [14] while the first one is due to Furuichi [6].

The operator version is as follows [6], [14] :

THEOREM 1. *For two positive operators $A,\ B$ and positive real numbers $m,\ m'$, $M,\ M'$ satisfying either of the following conditions:*

(i) $0 < mI \leqslant A \leqslant m'I < M'I \leqslant B \leqslant MI$,

(ii) $0 < mI \leqslant B \leqslant m'I < M'I \leqslant A \leqslant MI$,

we have

$$S\left((h')^r\right) A\sharp_\nu B \leqslant A\nabla_\nu B \leqslant S(h) A\sharp_\nu B, \tag{1.4}$$

where $h := \frac{M}{m},\ h' := \frac{M'}{m'}$ *and* $\nu \in [0,1]$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geqslant 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds.

$$K^r\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \leqslant (1-\nu)a + \nu b \leqslant K^R\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \tag{1.6}$$

where $a,\ b > 0,\ \nu \in [0,1],\ r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [15] while the second by Liao et al. [11].

The operator version is as follows [15], [11]:

THEOREM 2. *For two positive operators $A,\ B$ and positive real numbers $m,\ m'$, $M,\ M'$ satisfying either of the following conditions:*

(i) $0 < mI \leqslant A \leqslant m'I < M'I \leqslant B \leqslant MI$,

(ii) $0 < mI \leqslant B \leqslant m'I < M'I \leqslant A \leqslant MI$,

we have

$$K^r(h') A\sharp_\nu B \leqslant A\nabla_\nu B \leqslant K^R(h) A\sharp_\nu B, \tag{1.7}$$

where $h := \frac{M}{m},\ h' := \frac{M'}{m'},\ \nu \in [0,1],\ r = \min\{1-\nu, \nu\}$ *and* $R = \max\{1-\nu, \nu\}$.

Kittaneh and Manasrah [8], [9] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \leqslant (1-\nu)a+\nu b-a^{1-\nu}b^{\nu} \leqslant R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{1.8}$$

where $a,\ b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. The case $\nu=\frac{1}{2}$ reduces (1.8) to an identity.

For some operator versions of (1.8) see [8] and [9]. Other recent results for operators may be found in [1]-[5].

Motivated by the above results we establish in this paper some new additive refinements and reverses of Young's operator inequality. Applications related to the Hölder-McCarthy inequality for positive operators and for trace class operators on Hilbert spaces are given as well.

2. Additive Reverses

We consider the function $f_\nu:[0,\infty)\to[0,\infty)$ defined for $\nu\in(0,1)$ by

$$f_\nu(x)=1-\nu+\nu x-x^{\nu}. \tag{2.1}$$

The following lemma holds.

LEMMA 1. *For any $x\in[m,M]\subset[0,\infty)$ we have*

$$\max_{x\in[m,M]} f_\nu(x)=\Delta_\nu(m,M):=\begin{cases} f_\nu(m) \text{ if } M<1,\\ \max\{f_\nu(m),f_\nu(M)\} \text{ if } m\leqslant 1\leqslant M,\\ f_\nu(M) \text{ if } 1<m \end{cases} \tag{2.2}$$

and

$$\min_{x\in[m,M]} f_\nu(x)=\delta_\nu(m,M):=\begin{cases} f_\nu(M) \text{ if } M<1,\\ 0 \text{ if } m\leqslant 1\leqslant M,\\ f_\nu(m) \text{ if } 1<m. \end{cases} \tag{2.3}$$

Proof. The function f_ν is differentiable and

$$f_\nu'(x)=\nu\left(1-x^{\nu-1}\right)=\nu\frac{x^{1-\nu}-1}{x^{1-\nu}},$$

which shows that the function f_ν is decreasing on $[0,1]$ and increasing on $[1,\infty)$, $f_\nu(0)=1-\nu,\ f_\nu(1)=0$ and the equation $f_\nu(x)=1-\nu$ for $x>0$ has the unique solution $x_\nu=\nu^{\frac{1}{\nu-1}}>1$.

Therefore, by considering the 3 possible situations for the location of the interval $[m,M]$ and the number 1 we get the desired bounds (2.2) and (2.3). □

REMARK 1. *We have the inequalities*

$$0 \leqslant f_\nu(x) \leqslant 1-\nu \text{ for any } x \in \left[0, \nu^{\frac{1}{\nu-1}}\right]$$

and

$$1-\nu \leqslant f_\nu(x) \text{ for any } x \in \left[\nu^{\frac{1}{\nu-1}}, \infty\right).$$

THEOREM 3. *Assume that A, B are positive invertible operators and the constants $M > m > 0$ are such that*

$$mA \leqslant B \leqslant MA. \tag{2.4}$$

Let $\nu \in [0,1]$, then we have the inequalities

$$\delta_\nu(m,M)A \leqslant A\nabla_\nu B - A\sharp_\nu B \leqslant \Delta_\nu(m,M)A, \tag{2.5}$$

where $\Delta_\nu(m,M)$ and $\delta_\nu(m,M)$ are defined by (2.2) and (2.3), respectively.

Proof. From Lemma 1 we have the double inequality

$$\delta_\nu(m,M) \leqslant 1-\nu+\nu x - x^\nu \leqslant \Delta_\nu(m,M) \tag{2.6}$$

for any $x \in [m,M]$.

If X is an operator such that $mI \leqslant X \leqslant MI$, then by (2.6) and the continuous functional calculus, we have

$$\delta_\nu(m,M)I \leqslant (1-\nu)I + \nu X - X^\nu \leqslant \Delta_\nu(m,M)I. \tag{2.7}$$

If the condition (2.4) holds, then by multiplying in both sides with $A^{-1/2}$ we get $mI \leqslant A^{-1/2}BA^{-1/2} \leqslant MI$ and by taking $X = A^{-1/2}BA^{-1/2}$ in (2.7) we get

$$\begin{aligned} \delta_\nu(m,M)I &\leqslant (1-\nu)I + \nu A^{-1/2}BA^{-1/2} - \left(A^{-1/2}BA^{-1/2}\right)^\nu \\ &\leqslant \Delta_\nu(m,M)I. \end{aligned} \tag{2.8}$$

Now, if we multiply (2.8) in both sides with $A^{1/2}$ we get the desired result (2.5). □

COROLLARY 1. *For two positive operators A, B and positive real numbers m, m', M, M' put $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$.*
If

(i) $0 < mI \leqslant A \leqslant m'I < M'I \leqslant B \leqslant MI$,

then

$$f_\nu\left(h'\right)A \leqslant A\nabla_\nu B - A\sharp_\nu B \leqslant f_\nu(h)A. \tag{2.9}$$

If

(ii) $0 < mI \leqslant B \leqslant m'I < M'I \leqslant A \leqslant MI$,

then

$$f_\nu\left(\left(h'\right)^{-1}\right)A \leqslant A\nabla_\nu B - A\sharp_\nu B \leqslant f_\nu\left(h^{-1}\right)A. \tag{2.10}$$

Proof. If *(i)* is valid, then we have

$$A < \frac{M'}{m'}A = h'A \leqslant B \leqslant hA = \frac{M}{m}A,$$

and by (2.5) we have for $1 < h' \leqslant h$

$$f_\nu\left(h'\right)A \leqslant A\nabla_\nu B - A\sharp_\nu B \leqslant f_\nu\left(h\right)A,$$

and the inequality (2.9) is proved.

If *(ii)* is valid, then we have

$$\frac{1}{h}A \leqslant B \leqslant \frac{1}{h'}A < A$$

and by (2.5) for $\frac{1}{h} \leqslant \frac{1}{h'} < 1$ we also have

$$f_\nu\left(\frac{1}{h'}\right)A \leqslant A\nabla_\nu B - A\sharp_\nu B \leqslant f_\nu\left(\frac{1}{h}\right)A,$$

and the inequality (2.10) is proved. □

We have the following simpler bounds:

COROLLARY 2. *With the assumptions of Theorem 3 we have*

$$r \times \begin{cases} \left(1-\sqrt{M}\right)^2 A \text{ if } M < 1, \\ 0 \text{ if } m \leqslant 1 \leqslant M, \\ \left(\sqrt{m}-1\right)^2 A \text{ if } 1 < m, \end{cases} \tag{2.11}$$

$$\leqslant A\nabla_\nu B - A\sharp_\nu B$$

$$\leqslant R \times \begin{cases} \left(1-\sqrt{m}\right)^2 A \text{ if } M < 1, \\ \max\left\{\left(1-\sqrt{m}\right)^2, \left(\sqrt{M}-1\right)^2\right\} A \text{ if } m \leqslant 1 \leqslant M, \\ \left(\sqrt{M}-1\right)^2 A \text{ if } 1 < m, \end{cases},$$

where $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ *and* $R = \max\{1-\nu, \nu\}$.

Proof. From the inequality (1.8) we have for $b = t$ and $a = 1$ that

$$r\left(\sqrt{t}-1\right)^2 \leqslant f_\nu(t) = 1-\nu+\nu t - t^\nu \leqslant R\left(\sqrt{t}-1\right)^2$$

for any $t \in [0,1]$.

Then we have

$$\Delta_\nu(m,M) \leqslant R \times \begin{cases} (1-\sqrt{m})^2 \text{ if } M < 1, \\ \max\left\{(1-\sqrt{m})^2, \left(\sqrt{M}-1\right)^2\right\} \text{ if } m \leqslant 1 \leqslant M, \\ \left(\sqrt{M}-1\right)^2 \text{ if } 1 < m \end{cases}$$

and

$$\delta_\nu(m,M) \geqslant r \times \begin{cases} \left(1-\sqrt{M}\right)^2 \text{ if } M < 1, \\ 0 \text{ if } m \leqslant 1 \leqslant M, \\ (\sqrt{m}-1)^2 \text{ if } 1 < m, \end{cases}$$

which by Theorem 3 proves the corollary. □

REMARK 2. *With the assumptions of Corollary 1, we have, in the case (i), that*

$$r\left(\sqrt{h'}-1\right)^2 A \leqslant A\nabla_\nu B - A\sharp_\nu B \leqslant R\left(\sqrt{h}-1\right)^2 A, \tag{2.12}$$

and in the case (ii), that

$$r\frac{\left(1-\sqrt{h'}\right)^2}{h'} A \leqslant A\nabla_\nu B - A\sharp_\nu B \leqslant R\frac{\left(1-\sqrt{h}\right)^2}{h} A. \tag{2.13}$$

The following bounds in terms of Specht's ratio can be stated as well:

COROLLARY 3. *With the assumptions of Theorem 3 we have*

$$\begin{cases} [S(M^r)-1]M^\nu A \text{ if } M < 1, \\ 0 \text{ if } m \leqslant 1 \leqslant M, \\ [S(m^r)-1]m^\nu A \text{ if } 1 < m, \end{cases} \tag{2.14}$$

$$\leqslant A\nabla_\nu B - A\sharp_\nu B$$

$$\leqslant \begin{cases} [S(m)-1]m^\nu A \text{ if } M < 1, \\ \max\{[S(m)-1]m^\nu, [S(M)-1]M^\nu\} A \text{ if } m \leqslant 1 \leqslant M, \\ [S(M)-1]M^\nu A \text{ if } 1 < m. \end{cases}$$

Proof. From the inequality (1.3) we have for $a=1$ and $b=t$ that

$$S(t^r)t^\nu \leqslant 1-\nu+\nu t \leqslant S(t)t^\nu, \tag{2.15}$$

where $t>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$.

By subtracting t^ν in the inequality (2.15) we get

$$(0\leqslant)\,[S(t^r)-1]t^\nu \leqslant f_\nu(t) \leqslant [S(t)-1]t^\nu, \tag{2.16}$$

for any $t>0,\ \nu\in[0,1]$.

Then we have

$$\Delta_\nu(m,M)\leqslant\begin{cases}[S(m)-1]m^\nu \text{ if } M<1,\\ \max\{[S(m)-1]m^\nu,[S(M)-1]M^\nu\} \text{ if } m\leqslant 1\leqslant M,\\ [S(M)-1]M^\nu \text{ if } 1<m\end{cases}$$

and

$$\delta_\nu(m,M)\geqslant\begin{cases}[S(M^r)-1]M^\nu \text{ if } M<1,\\ 0 \text{ if } m\leqslant 1\leqslant M,\\ [S(m^r)-1]m^\nu \text{ if } 1<m,\end{cases}$$

which by Theorem 3 proves the corollary. □

REMARK 3. *With the assumptions of Corollary 1, we have in the case (i), that*

$$\left[S\left((h')^r\right)-1\right](h')^\nu A \leqslant A\nabla_\nu B - A\sharp_\nu B \leqslant [S(h)-1]h^\nu A, \tag{2.17}$$

and in the case (ii), that

$$\frac{S((h')^r)-1}{(h')^r}A \leqslant A\nabla_\nu B - A\sharp_\nu B \leqslant \frac{S(h)-1}{h^\nu}A. \tag{2.18}$$

From the inequality (1.6) we have

$$(0<)\,[K^r(t)-1]t^\nu \leqslant 1-\nu+\nu t-t^\nu \leqslant \left[K^R(t)-1\right]t^\nu \tag{2.19}$$

where $t>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

We then have the following bounds in terms of Kantorovich's constant:

COROLLARY 4. *With the assumptions of Theorem 3 we have*

$$\begin{cases} [K^r(M)-1]M^\nu A \text{ if } M<1, \\ 0 \text{ if } m\leqslant 1\leqslant M, \\ [K^r(m)-1]m^\nu A \text{ if } 1<m, \end{cases} \tag{2.20}$$

$$\leqslant A\nabla_\nu B - A\sharp_\nu B$$

$$\leqslant \begin{cases} \left[K^R(m)-1\right]m^\nu A \text{ if } M<1, \\ \max\left\{\left[K^R(m)-1\right]m^\nu, \left[K^R(M)-1\right]M^\nu\right\}A \text{ if } m\leqslant 1\leqslant M, \\ \left[K^R(M)-1\right]M^\nu A \text{ if } 1<m. \end{cases}$$

REMARK 4. *With the assumptions of Corollary 1, we have in the case (i), that*

$$\left[K^r\left(h'\right)-1\right]\left(h'\right)^\nu A\leqslant A\nabla_\nu B - A\sharp_\nu B\leqslant \left[K^R(h)-1\right]h^\nu A, \tag{2.21}$$

and in the case (ii), that

$$\frac{K^r(h')-1}{(h')^r}A\leqslant A\nabla_\nu B - A\sharp_\nu B\leqslant \frac{K^R(h)-1}{h^\nu}A. \tag{2.22}$$

Let $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Assume that the positive invertible operators $A,\ B$ satisfy the condition

$$mA^p\leqslant B^q\leqslant MA^p. \tag{2.23}$$

Then by replacing A with A^p, B with B^q and $\nu=\frac{1}{q}$ in (2.5) we have

$$\delta_{\frac{1}{q}}(m,M)A^p\leqslant \frac{1}{p}A^p+\frac{1}{q}B^q-A^p\sharp_{\frac{1}{q}}B^q\leqslant \Delta_{\frac{1}{q}}(m,M)A^p, \tag{2.24}$$

where $\Delta_{\frac{1}{q}}(m,M)$ and $\delta_{\frac{1}{q}}(m,M)$ are defined by (2.2) and (2.3) respectively.

If the positive invertible operators $A,\ B$ satisfy the condition (2.23), then from

(2.11) we get for

$$r_{p,q}\times\begin{cases}\left(1-\sqrt{M}\right)^2A^p \text{ if } M<1,\\ 0 \text{ if } m\leqslant 1\leqslant M,\\ \left(\sqrt{m}-1\right)^2A^p \text{ if } 1<m\end{cases} \tag{2.25}$$

$$\leqslant \frac{1}{p}A^p+\frac{1}{q}B^q-A^p\sharp_{\frac{1}{q}}B^q$$

$$\leqslant R_{p,q}\times\begin{cases}\left(1-\sqrt{m}\right)^2A^p \text{ if } M<1,\\ \max\left\{\left(1-\sqrt{m}\right)^2,\left(\sqrt{M}-1\right)^2\right\}A^p \text{ if } m\leqslant 1\leqslant M,\\ \left(\sqrt{M}-1\right)^2A^p \text{ if } 1<m,\end{cases},$$

from (2.14) we get

$$\begin{cases}\left[S\left(M^{r_{p,q}}\right)-1\right]M^{\nu}A^p \text{ if } M<1,\\ 0 \text{ if } m\leqslant 1\leqslant M,\\ \left[S\left(m^{r_{p,q}}\right)-1\right]m^{\nu}A^p \text{ if } 1<m\end{cases} \tag{2.26}$$

$$\leqslant \frac{1}{p}A^p+\frac{1}{q}B^q-A^p\sharp_{\frac{1}{q}}B^q$$

$$\leqslant \begin{cases}\left[S\left(m\right)-1\right]m^{\nu}A^p \text{ if } M<1,\\ \max\left\{\left[S\left(m\right)-1\right]m^{\nu},\left[S\left(M\right)-1\right]M^{\nu}\right\}A^p \text{ if } m\leqslant 1\leqslant M,\\ \left[S\left(M\right)-1\right]M^{\nu}A^p \text{ if } 1<m,\end{cases}$$

while from (2.20) we get

$$\begin{cases} [K^{r_{p,q}}(M)-1]M^{\nu}A^{p} \text{ if } M<1, \\ 0 \text{ if } m\leqslant 1\leqslant M, \\ [K^{r_{p,q}}(m)-1]m^{\nu}A^{p} \text{ if } 1<m \end{cases} \tag{2.27}$$

$$\leqslant \frac{1}{p}A^{p}+\frac{1}{q}B^{q}-A^{p}\sharp_{\frac{1}{q}}B^{q}$$

$$\leqslant \begin{cases} \left[K^{R_{p,q}}(m)-1\right]m^{\nu}A^{p} \text{ if } M<1, \\ \max\left\{\left[K^{R_{p,q}}(m)-1\right]m^{\nu},\left[K^{R_{p,q}}(M)-1\right]M^{\nu}\right\}A^{p} \text{ if } m\leqslant 1\leqslant M, \\ \left[K^{R_{p,q}}(M)-1\right]M^{\nu}A^{p} \text{ if } 1<m, \end{cases}$$

where $r_{p,q}=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $R_{p,q}=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

If $p=q=2$ and if we assume that

$$mA^{2}\leqslant B^{2}\leqslant MA^{2}, \tag{2.28}$$

then by (2.24) we get

$$\delta_{\frac{1}{2}}(m,M)A^{2}\leqslant\frac{1}{2}\left(A^{2}+B^{2}\right)-A^{2}\sharp B^{2}\leqslant\Delta_{\frac{1}{2}}(m,M)A^{2}. \tag{2.29}$$

Assume that A and B satisfy the conditions

$$m_{1}I\leqslant A\leqslant M_{1}I,\ m_{2}I\leqslant B\leqslant M_{2}I \tag{2.30}$$

for some $0<m_{1}<M_{1}$ and $0<m_{2}<M_{2}$. We have from (2.30) that

$$m_{1}^{p}I\leqslant A^{p}\leqslant M_{1}^{p}I.$$

Then by (2.30) we also have

$$m_{1}^{p}M_{2}^{-q}I\leqslant m_{1}^{p}B^{-q}\leqslant B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\leqslant M_{1}^{p}B^{-q}\leqslant M_{1}^{p}m_{2}^{-q}I,$$

which implies that

$$m_{1}M_{2}^{-\frac{q}{p}}I\leqslant\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)^{\frac{1}{p}}\leqslant M_{1}m_{2}^{-\frac{q}{p}}I.$$

Now, on using the inequality (2.24) for $m=m_{1}M_{2}^{-\frac{q}{p}}$ and $M=M_{1}m_{2}^{-\frac{q}{p}}$, we get

$$\delta_{\nu}\left(m_{1}M_{2}^{-\frac{q}{p}},M_{1}m_{2}^{-\frac{q}{p}}\right)A^{p}\leqslant\frac{1}{p}A^{p}+\frac{1}{q}B^{q}-A^{p}\sharp_{\frac{1}{q}}B^{q} \tag{2.31}$$

$$\leqslant\Delta_{\nu}\left(m_{1}M_{2}^{-\frac{q}{p}},M_{1}m_{2}^{-\frac{q}{p}}\right)A^{p},$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

In particular, we have

$$\delta_{\frac{1}{2}}\left(m_1M_2^{-1},M_1m_2^{-1}\right)A^2\leqslant\frac{1}{2}\left(A^2+B^2\right)-A^2\sharp B^2 \leqslant\Delta_{\frac{1}{2}}\left(m_1M_2^{-1},M_1m_2^{-1}\right)A^2, \tag{2.32}$$

provided that A and B satisfy the conditions (2.30).

Further inequalities in terms of Specht's ratio and Kantorovich's constant may be obtained by using (2.26) and (2.27) respectively, however the details are not presented here.

3. Inequalities Related to McCarthy's

By the use of the spectral resolution of $P\geqslant 0$ and the Hölder inequality, C. A. McCarthy [12] proved that

$$\langle Px,x\rangle^p\leqslant\langle P^px,x\rangle\,,\ p\in(1,\infty) \tag{3.1}$$

and

$$\langle P^px,x\rangle\leqslant\langle Px,x\rangle^p\,,\ p\in(0,1) \tag{3.2}$$

for any $x\in H$ with $\|x\|=1$.

From the previous section, for positive numbers $a,\ b$ with $\frac{b}{a}\in[m,M]\subset(0,\infty)$ and $\nu\in[0,1]$ we can state the following scalar inequalities

$$\delta_\nu\left(m,M\right)a\leqslant(1-\nu)\,a+\nu b-a^{1-\nu}b^\nu\leqslant\Delta_\nu\left(m,M\right)a, \tag{3.3}$$

where $\Delta_\nu\left(m,M\right)$ and $\delta_\nu\left(m,M\right)$ are defined by (2.2) and (2.3) respectively.

We also have the scalar inequalities

$$r\times\begin{cases}\left(1-\sqrt{M}\right)^2a \text{ if } M<1,\\ 0 \text{ if } m\leqslant 1\leqslant M,\\ (\sqrt{m}-1)^2a \text{ if } 1<m,\end{cases} \tag{3.4}$$

$$\leqslant(1-\nu)\,a+\nu b-a^{1-\nu}b^\nu$$

$$\leqslant R\times\begin{cases}(1-\sqrt{m})^2a \text{ if } M<1,\\ \max\left\{(1-\sqrt{m})^2,\left(\sqrt{M}-1\right)^2\right\}a \text{ if } m\leqslant 1\leqslant M,\ ,\\ \left(\sqrt{M}-1\right)^2a \text{ if } 1<m,\end{cases}$$

$$\begin{cases} [S(M^r)-1]M^\nu a \text{ if } M<1, \\ 0 \text{ if } m\leqslant 1\leqslant M, \\ [S(m^r)-1]m^\nu a \text{ if } 1<m \end{cases} \tag{3.5}$$

$$\leqslant (1-\nu)a+\nu b-a^{1-\nu}b^\nu$$

$$\leqslant \begin{cases} [S(m)-1]m^\nu a \text{ if } M<1, \\ \max\{[S(m)-1]m^\nu, [S(M)-1]M^\nu\} a \text{ if } m\leqslant 1\leqslant M, \\ [S(M)-1]M^\nu a \text{ if } 1<m \end{cases}$$

and

$$\begin{cases} [K^r(M)-1]M^\nu a \text{ if } M<1, \\ 0 \text{ if } m\leqslant 1\leqslant M, \\ [K^r(m)-1]m^\nu a \text{ if } 1<m \end{cases} \tag{3.6}$$

$$\leqslant (1-\nu)a+\nu b-a^{1-\nu}b^\nu$$

$$\leqslant \begin{cases} \left[K^R(m)-1\right]m^\nu a \text{ if } M<1, \\ \max\left\{\left[K^R(m)-1\right]m^\nu, \left[K^R(M)-1\right]M^\nu\right\} a \text{ if } m\leqslant 1\leqslant M, \\ \left[K^R(M)-1\right]M^\nu a \text{ if } 1<m, \end{cases}$$

where $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

THEOREM 4. *Let P and operator such that*

$$zI\leqslant P\leqslant ZI \tag{3.7}$$

for some constants $Z>z>0$.

Then for any $x\in H$ with $\|x\|=1$ we have

$$0\leqslant 1-\frac{\langle P^\lambda x,x\rangle}{\langle Px,x\rangle^\lambda}\leqslant \max\left\{f_\lambda\left(\frac{z}{Z}\right), f_\lambda\left(\frac{Z}{z}\right)\right\}, \tag{3.8}$$

where $\lambda\in[0,1]$ and the function $f_\lambda:[0,\infty)\to[0,\infty)$ is defined by

$$f_\lambda(t)=1-\lambda+\lambda t-t^\lambda. \tag{3.9}$$

Proof. If $u,v\in[z,Z]$ then $\frac{u}{v}\in\left[\frac{z}{Z},\frac{Z}{z}\right]$ and by (3.3) we have

$$0\leqslant(1-\lambda)v+\lambda u-v^{1-\lambda}u^{\lambda}\leqslant\max\left\{f_{\lambda}\left(\frac{z}{Z}\right),f_{\lambda}\left(\frac{Z}{z}\right)\right\}v$$

for any $\lambda\in[0,1]$.

Fix $v\in[z,Z]$, then by using the functional calculus for the operator P with $zI\leqslant P\leqslant ZI$ we have

$$0\leqslant(1-\lambda)vI+\lambda P-v^{1-\lambda}P^{\lambda}\leqslant\max\left\{f_{\lambda}\left(\frac{z}{Z}\right),f_{\lambda}\left(\frac{Z}{z}\right)\right\}v \tag{3.10}$$

for any $\lambda\in[0,1]$.

The inequality (3.10) implies that

$$0\leqslant(1-\lambda)v+\lambda\langle Px,x\rangle-v^{1-\lambda}\left\langle P^{\lambda}x,x\right\rangle\leqslant\max\left\{f_{\lambda}\left(\frac{z}{Z}\right),f_{\lambda}\left(\frac{Z}{z}\right)\right\}v, \tag{3.11}$$

for any $x\in H$ with $\|x\|=1$, for any $\lambda\in[0,1]$ and for any $v\in[z,Z]$.

If we take in (3.11) $v=\langle Px,x\rangle\in[z,Z]$, for $x\in H$ with $\|x\|=1$, then we have

$$0\leqslant\langle Px,x\rangle-\langle Px,x\rangle^{1-\lambda}\left\langle P^{\lambda}x,x\right\rangle\leqslant\max\left\{f_{\lambda}\left(\frac{z}{Z}\right),f_{\lambda}\left(\frac{Z}{z}\right)\right\}\langle Px,x\rangle,$$

which, by division with $\langle Px,x\rangle^{1-\lambda}>0$ produces

$$0\leqslant\langle Px,x\rangle^{\lambda}-\left\langle P^{\lambda}x,x\right\rangle\leqslant\max\left\{f_{\lambda}\left(\frac{z}{Z}\right),f_{\lambda}\left(\frac{Z}{z}\right)\right\}\langle Px,x\rangle^{\lambda}$$

that is equivalent to the desired result (3.8). □

REMARK 5. *If* $1<\frac{Z}{z}\leqslant\lambda^{\frac{1}{\lambda-1}}$ *with* $\lambda\in(0,1)$ *then by Remark 1 we have that* $\max\left\{f_{\lambda}\left(\frac{z}{Z}\right),f_{\lambda}\left(\frac{Z}{z}\right)\right\}\leqslant1-\lambda$ *and by (3.8) we get*

$$\lambda\langle Px,x\rangle^{\lambda}\leqslant\left\langle P^{\lambda}x,x\right\rangle\left(\leqslant\langle Px,x\rangle^{\lambda}\right) \tag{3.12}$$

for any $x\in H$ *with* $\|x\|=1$.

COROLLARY 5. *With the assumptions of Theorem 4 and if* $T=\max\{\lambda,1-\lambda\}$ *for* $\lambda\in(0,1)$, *then we have*

$$0\leqslant1-\frac{\langle P^{\lambda}x,x\rangle}{\langle Px,x\rangle^{\lambda}}\leqslant\begin{cases}T\left(\sqrt{\frac{Z}{z}}-1\right)^{2},\\[2ex]\left[S\left(\frac{Z}{z}\right)-1\right]\left(\frac{Z}{z}\right)^{\lambda},\\[2ex]\left[K^{T}\left(\frac{Z}{z}\right)-1\right]\left(\frac{Z}{z}\right)^{\lambda}\end{cases}$$

for any $x\in H$ *with* $\|x\|=1$.

We have:

THEOREM 5. *Let A and B be two positive invertible operators, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m,\ M>0$ such that*

$$m^pB^q \leqslant A^p \leqslant M^pB^q. \tag{3.13}$$

Then we have

$$0 \leqslant 1-\frac{\left\langle B^q\sharp_{1/p}A^px,x\right\rangle}{\left\langle A^px,x\right\rangle^{1/p}\left\langle B^qx,x\right\rangle^{1/q}} \leqslant \max\left\{f_{\frac{1}{p}}\left(\left(\frac{m}{M}\right)^p\right), f_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^p\right)\right\}, \tag{3.14}$$

where the function $f_{\frac{1}{p}}:[0,\infty)\to[0,\infty)$ is defined by (3.9) for $\lambda=\frac{1}{p}$.

Proof. From the inequality (3.8) for $x=\frac{y}{\|y\|},\ y\neq 0$ we have

$$0 \leqslant 1-\frac{\left\langle P^\lambda y,y\right\rangle}{\left\langle y,y\right\rangle^{1-\lambda}\left\langle Py,y\right\rangle^\lambda} \leqslant \max\left\{f_\lambda\left(\frac{z}{Z}\right), f_\lambda\left(\frac{Z}{z}\right)\right\}, \tag{3.15}$$

provided that P satisfy the condition (3.7).

Now, from (3.13) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^pI \leqslant B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}} \leqslant M^pI$.

By writing the inequality (3.15) for $P=B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}},\ z=m^p,\ Z=M^p,\ \lambda=\frac{1}{p}$ and $y=B^{\frac{q}{2}}x$, with $x\in H,\ x\neq 0$, we have

$$\begin{aligned} 0 &\leqslant 1-\frac{\left\langle\left(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}\right)^{\frac{1}{p}}B^{\frac{q}{2}}x,B^{\frac{q}{2}}x\right\rangle}{\left\langle B^{\frac{q}{2}}x,B^{\frac{q}{2}}x\right\rangle^{\frac{1}{q}}\left\langle\left(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}\right)B^{\frac{q}{2}}x,B^{\frac{q}{2}}x\right\rangle^{\frac{1}{p}}} \\ &\leqslant \max\left\{f_{\frac{1}{p}}\left(\left(\frac{m}{M}\right)^p\right), f_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^p\right)\right\} \end{aligned}$$

that is equivalent to

$$\begin{aligned} 0 &\leqslant 1-\frac{\left\langle B^{\frac{q}{2}}\left(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}\right)^{\frac{1}{p}}B^{\frac{q}{2}}x,x\right\rangle}{\left\langle B^qx,x\right\rangle^{\frac{1}{q}}\left\langle A^px,x\right\rangle^{\frac{1}{p}}} \\ &\leqslant \max\left\{f_{\frac{1}{p}}\left(\left(\frac{m}{M}\right)^p\right), f_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^p\right)\right\} \end{aligned}$$

with $x\in H,\ x\neq 0$.

This is equivalent to the desired result (3.14). □

COROLLARY 6. *With the assumptions of Theorem 5 we have for* $x \in H,\ x \neq 0$, *that*

$$0 \leqslant 1 - \frac{\left\langle B^q \sharp_{1/p} A^p x, x \right\rangle}{\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q}} \leqslant \begin{cases} T_{p,q}\left(\left(\frac{M}{m}\right)^{\frac{p}{2}} - 1\right)^2, \\ \left[S\left(\left(\frac{M}{m}\right)^p\right) - 1\right]\frac{M}{m}, \\ \left[K^{T_{p,q}}\left(\left(\frac{M}{m}\right)^p\right) - 1\right]\frac{M}{m}, \end{cases}$$

where $T_{p,q} = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

4. Trace Inequalities

In the general case of Hilbert spaces $(H;\langle\cdot,\cdot\rangle)$, if $\{e_i\}_{i\in I}$ is an orthonormal basis of H, we say that a bounded linear operator $A \in \mathscr{B}(H)$ is *trace class* provided

$$\|A\|_1 := \sum_{i\in I} \langle |A|\, e_i, e_i\rangle < \infty. \tag{4.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathscr{B}_1(H)$ the set of trace class operators in $\mathscr{B}(H)$.

The following properties are also well known:

(i) We have

$$\|A\|_1 = \|A^*\|_1$$

for any $A \in \mathscr{B}_1(H)$;

(ii) $\mathscr{B}_1(H)$ is an *operator ideal* in $\mathscr{B}(H)$, i.e.

$$\mathscr{B}(H)\mathscr{B}_1(H)\mathscr{B}(H) \subseteq \mathscr{B}_1(H);$$

(iii) $(\mathscr{B}_1(H), \|\cdot\|_1)$ is a *Banach space*.

We define the *trace* of a trace class operator $A \in \mathscr{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{4.2}$$

where $\{e_i\}_{i\in I}$ is an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (4.2) converges absolutely and it is independent from the choice of basis.

The following results collect some properties of the trace:

(i) If $A \in \mathscr{B}_1(H)$ then $A^* \in \mathscr{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{4.3}$$

(ii) If $A \in \mathscr{B}_1(H)$ and $T \in \mathscr{B}(H)$, then $AT,\ TA \in \mathscr{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leqslant \|A\|_1 \|T\|; \tag{4.4}$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathscr{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;

(iv) $\mathscr{B}_{fin}(H)$, the space of operators of *finite rank*, is a dense subspace of $\mathscr{B}_1(H)$.

We have the following trace inequality:

THEOREM 6. *Let C be an operator with the property that*

$$zI \leqslant C \leqslant ZI \tag{4.5}$$

for some constants $z,$ Z with $Z > z > 0$ and $P \in \mathscr{B}_1(H),$ $P \geqslant 0$ with $\operatorname{tr}(P) > 0$. Then for any $\lambda \in [0,1]$ we have

$$0 \leqslant 1 - \frac{\operatorname{tr}\left(PC^{\lambda}\right)}{\operatorname{tr}^{1-\lambda}(P)\operatorname{tr}^{\lambda}(PC)} \leqslant \max\left\{f_{\lambda}\left(\frac{z}{Z}\right), f_{\lambda}\left(\frac{Z}{z}\right)\right\} \tag{4.6}$$

and the function f_{λ} is defined by (3.9).

Proof. As in the proof of Theorem 4, we have

$$0 \leqslant (1-\lambda)vI + \lambda C - v^{1-\lambda}C^{\lambda} \leqslant \max\left\{f_{\lambda}\left(\frac{z}{Z}\right), f_{\lambda}\left(\frac{Z}{z}\right)\right\}v$$

for any $\lambda \in [0,1]$.

This inequality implies that

$$\begin{aligned} 0 &\leqslant (1-\lambda)v\langle x,x\rangle + \lambda\langle Cx,x\rangle - v^{1-\lambda}\left\langle C^{\lambda}x,x\right\rangle \\ &\leqslant \max\left\{f_{\lambda}\left(\frac{z}{Z}\right), f_{\lambda}\left(\frac{Z}{z}\right)\right\}v\langle x,x\rangle, \end{aligned} \tag{4.7}$$

for any $x \in H,$ for any $\lambda \in [0,1]$ and for any $v \in [z,Z]$.

Now, if we take in (4.7) $x = P^{1/2}e,$ where $e \in H,$ then

$$\begin{aligned} 0 &\leqslant (1-\lambda)v\langle Pe,e\rangle + \lambda\left\langle P^{1/2}CP^{1/2}e,e\right\rangle - v^{1-\lambda}\left\langle P^{1/2}C^{\lambda}P^{1/2}e,e\right\rangle \\ &\leqslant \max\left\{f_{\lambda}\left(\frac{z}{Z}\right), f_{\lambda}\left(\frac{Z}{z}\right)\right\}v\langle Pe,e\rangle, \end{aligned} \tag{4.8}$$

for any $e \in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. If we take in (4.8) $e = e_i,$ $i \in I$ and by summing over $i \in I,$ then we get

$$\begin{aligned} 0 &\leqslant (1-\lambda)v\sum_{i\in I}\langle Pe_i,e_i\rangle + \lambda\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i,e_i\right\rangle \\ &\quad - v^{1-\lambda}\sum_{i\in I}\left\langle P^{1/2}C^{\lambda}P^{1/2}e_i,e_i\right\rangle \\ &\leqslant \max\left\{f_{\lambda}\left(\frac{z}{Z}\right), f_{\lambda}\left(\frac{Z}{z}\right)\right\}v\sum_{i\in I}\langle Pe_i,e_i\rangle, \end{aligned} \tag{4.9}$$

and by the properties of trace we have

$$\begin{aligned} 0 &\leqslant (1-\lambda)v\operatorname{tr}(P) + \lambda\operatorname{tr}(PC) - v^{1-\lambda}\operatorname{tr}\left(PC^{\lambda}\right) \\ &\leqslant \max\left\{f_{\lambda}\left(\frac{z}{Z}\right), f_{\lambda}\left(\frac{Z}{z}\right)\right\}v\operatorname{tr}(P), \end{aligned}$$

for any $\lambda \in [0,1]$ and for any $v \in [z,Z]$.

This inequality can be written as

$$0 \leqslant (1-\lambda)v + \lambda \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - v^{1-\lambda} \frac{\operatorname{tr}\left(PC^{\lambda}\right)}{\operatorname{tr}(P)} \leqslant \max\left\{ f_{\lambda}\left(\frac{z}{Z}\right), f_{\lambda}\left(\frac{Z}{z}\right)\right\} v, \tag{4.10}$$

for any $\lambda \in [0,1]$ and for any $v \in [z,Z]$.

Now, if we take in (4.10) $v = \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \in [z,Z]$, then we get

$$0 \leqslant (1-\lambda) \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} + \lambda \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\lambda} \frac{\operatorname{tr}\left(PC^{\lambda}\right)}{\operatorname{tr}(P)} \leqslant \max\left\{ f_{\lambda}\left(\frac{z}{Z}\right), f_{\lambda}\left(\frac{Z}{z}\right)\right\} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)},$$

namely

$$0 \leqslant \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\lambda} \frac{\operatorname{tr}\left(PC^{\lambda}\right)}{\operatorname{tr}(P)} \leqslant \max\left\{ f_{\lambda}\left(\frac{z}{Z}\right), f_{\lambda}\left(\frac{Z}{z}\right)\right\} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)},$$

and by multiplying with $\frac{\operatorname{tr}(P)}{\operatorname{tr}(PC)} > 0$ we get the desired result (4.6). □

In particular, we have:

COROLLARY 7. *With the assumptions of Theorem 6 and if* $T = \max\{\lambda, 1-\lambda\}$ *for* $\lambda \in (0,1)$, *then we have*

$$0 \leqslant 1 - \frac{\operatorname{tr}\left(PC^{\lambda}\right)}{\operatorname{tr}^{1-\lambda}(P)\operatorname{tr}^{\lambda}(PC)} \leqslant \begin{cases} T\left(\sqrt{\frac{Z}{z}} - 1\right)^2, \\ \left[S\left(\frac{Z}{z}\right) - 1\right]\left(\frac{Z}{z}\right)^{\lambda}, \\ \left[K^{T}\left(\frac{Z}{z}\right) - 1\right]\left(\frac{Z}{z}\right)^{\lambda}. \end{cases} \tag{4.11}$$

The following reverse of Hölder's trace inequality may be stated:

THEOREM 7. *Let* A *and* B *be two positive invertible operators,* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $m,\ M > 0$ *such that*

$$m^p B^q \leqslant A^p \leqslant M^p B^q. \tag{4.12}$$

If $B^q \in \mathscr{B}_1(H)$, *then*

$$0 \leqslant 1 - \frac{\operatorname{tr}\left(B^q \sharp_{1/p} A^p\right)}{\operatorname{tr}^{1/p}(A^p)\operatorname{tr}^{1/q}(B^q)} \leqslant \max\left\{ f_{\frac{1}{p}}\left(\left(\frac{m}{M}\right)^p\right), f_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^p\right)\right\}. \tag{4.13}$$

Proof. Now, from (4.12) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^p I \leqslant B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \leqslant M^p I$. By writing the inequality (4.6) for $C = B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}$, $z = m^p$, $Z = M^p$, $\lambda = \frac{1}{p}$ and $P = B^q$ we get the desired result (4.13). □

Finally, we have

COROLLARY 8. *With the assumptions of Theorem 7 and if* $T_{p,q} = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, *then we have*

$$0 \leqslant 1 - \frac{\operatorname{tr}\left(B^q \sharp_{1/p} A^p\right)}{\operatorname{tr}^{1/p}\left(A^p\right) \operatorname{tr}^{1/q}\left(B^q\right)} \leqslant \begin{cases} T_{p,q}\left(\left(\frac{M}{m}\right)^{\frac{p}{2}} - 1\right)^2, \\ \left[S\left(\left(\frac{M}{m}\right)^p\right) - 1\right] \frac{M}{m}, \\ \left[K^{T_{p,q}}\left(\left(\frac{M}{m}\right)^p\right) - 1\right] \frac{M}{m}. \end{cases} \tag{4.14}$$

5. Other Upper and Lower Bounds

In [1] we proved the following reverses of Young's inequality

$$0 \leqslant (1-\nu)a + \nu b - a^{1-\nu} b^{\nu} \leqslant \nu(1-\nu)(a-b)(\ln a - \ln b) \tag{5.1}$$

and

$$1 \leqslant \frac{(1-\nu)a + \nu b}{a^{1-\nu} b^{\nu}} \leqslant \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right) - 1\right)\right], \tag{5.2}$$

for any $a,\ b > 0$ and $\nu \in [0,1]$, where K is Kantorovich's constant defined by (1.5).

The inequality (5.2) is equivalent to

$$\begin{aligned} 0 &\leqslant (1-\nu)a + \nu b - a^{1-\nu} b^{\nu} \\ &\leqslant \left(\exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right) - 1\right)\right] - 1\right) a^{1-\nu} b^{\nu} \end{aligned} \tag{5.3}$$

for any $a,\ b > 0$ and $\nu \in [0,1]$.

Therefore, by (2.2), (5.1) and (5.3) we have

$$\begin{aligned} \Delta_\nu(m,M) &= \begin{cases} f_\nu(m) \text{ if } M < 1, \\ \max\{f_\nu(m), f_\nu(M)\} \text{ if } m \leqslant 1 \leqslant M, \\ f_\nu(M) \text{ if } 1 < m \end{cases} \\ &\leqslant \nu(1-\nu) \times \begin{cases} (m-1)\ln m \text{ if } M < 1, \\ \max\{(m-1)\ln m, (M-1)\ln M\} \text{ if } m \leqslant 1 \leqslant M, \\ (M-1)\ln M \text{ if } 1 < m, \end{cases} \end{aligned} \tag{5.4}$$

and

$$\Delta_\nu(m,M) \leqslant \begin{cases} (\exp[4\nu(1-\nu)(K(m)-1)]-1)m^\nu \text{ if } M<1, \\ \max\{(\exp[4\nu(1-\nu)(K(m)-1)]-1)m^\nu, \\ (\exp[4\nu(1-\nu)(K(M)-1)]-1)M^\nu\} \text{ if } m \leqslant 1 \leqslant M, \\ (\exp[4\nu(1-\nu)(K(M)-1)]-1)M^\nu \text{ if } 1<m. \end{cases} \tag{5.5}$$

In [2] we also obtained the following refinements and reverses of Young's inequality

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\min\{a,b\} &\leqslant (1-\nu)a+\nu b-a^{1-\nu}b^\nu \\ &\leqslant \frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\max\{a,b\} \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} &\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \\ &\leqslant \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu} \\ &\leqslant \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right] \end{aligned} \tag{5.7}$$

for any $a,\ b>0$ and $\nu\in[0,1]$.

The inequality (5.7) is equivalent to

$$\begin{aligned} &\left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right]-1\right)a^{1-\nu}b^\nu \\ &\leqslant (1-\nu)a+\nu b-a^{1-\nu}b^\nu \\ &\leqslant \left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right]-1\right)a^{1-\nu}b^\nu \end{aligned} \tag{5.8}$$

for any $a,\ b>0$ and $\nu\in[0,1]$.

Therefore, by (5.6) and (5.8) we have the upper bounds

$$\Delta_\nu(m,M) \leqslant \frac{1}{2}\nu(1-\nu)\begin{cases} (\ln m)^2 \text{ if } M<1, \\ \max\left\{(\ln m)^2, (\ln M)^2 M\right\} \text{ if } m\leqslant 1\leqslant M, \\ (\ln M)^2 M \text{ if } 1<m \end{cases} \tag{5.9}$$

and

$$\Delta_{\nu}(m,M) \leqslant \begin{cases} \left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{1}{m}-1\right)^2\right]-1\right)m^{\nu}\text{if } M<1, \\ \max\left\{\left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{1}{m}-1\right)^2\right]-1\right)m^{\nu}, \right. \\ \left.\left(\exp\left[\frac{1}{2}\nu(1-\nu)(M-1)^2\right]-1\right)M^{\nu}\right\} \text{ if } m\leqslant 1\leqslant M, \\ \left(\exp\left[\frac{1}{2}\nu(1-\nu)(M-1)^2\right]-1\right)M^{\nu} \text{ if } 1<m. \end{cases} \tag{5.10}$$

From (2.3), (5.6) and (5.8) we have the lower bounds

$$\delta_{\nu}(m,M) := \begin{cases} f_{\nu}(M) \text{ if } M<1, \\ 0 \text{ if } m\leqslant 1\leqslant M, \\ f_{\nu}(m) \text{ if } 1<m. \end{cases} \tag{5.11}$$

$$\geqslant \frac{1}{2}\nu(1-\nu)\times\begin{cases} (\ln M)^2 M \text{ if } M<1, \\ 0 \text{ if } m\leqslant 1\leqslant M, \\ (\ln m)^2 \text{ if } 1<m \end{cases}$$

and

$$\delta_{\nu}(m,M) \geqslant \begin{cases} \left(\exp\left[\frac{1}{2}\nu(1-\nu)(1-M)^2\right]-1\right)M^{\nu} \text{ if } M<1, \\ 0 \text{ if } m\leqslant 1\leqslant M, \\ \left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{1}{m}\right)^2\right]-1\right)m^{\nu} \text{ if } 1<m. \end{cases} \tag{5.12}$$

Assume that $A,\ B$ are positive invertible operators and the constants $M>m>0$ are such that $mA\leqslant B\leqslant MA$. If we use the second inequality in (2.5), then we have the following upper bounds for the difference $A\nabla_{\nu}B - A\sharp_{\nu}B$:

$$A\nabla_{\nu}B - A\sharp_{\nu}B \tag{5.13}$$

$$\leqslant \nu(1-\nu)\times\begin{cases} [(m-1)\ln m]A \text{ if } M<1, \\ \max\{(m-1)\ln m,(M-1)\ln M\}A \\ \text{if } m\leqslant 1\leqslant M, \\ [(M-1)\ln M]A \text{ if } 1<m, \end{cases}$$

$$A\nabla_\nu B - A\sharp_\nu B \tag{5.14}$$
$$\leqslant \begin{cases} \left(\exp\left[4\nu\left(1-\nu\right)\left(K\left(m\right)-1\right)\right]-1\right)Am^\nu \text{ if } M<1, \\ \max\left\{\left(\exp\left[4\nu\left(1-\nu\right)\left(K\left(m\right)-1\right)\right]-1\right)m^\nu, \right. \\ \left. \left(\exp\left[4\nu\left(1-\nu\right)\left(K\left(M\right)-1\right)\right]-1\right)M^\nu\right\}A \text{ if } m\leqslant 1\leqslant M, \\ \left(\exp\left[4\nu\left(1-\nu\right)\left(K\left(M\right)-1\right)\right]-1\right)AM^\nu \text{ if } 1<m, \end{cases}$$

$$A\nabla_\nu B - A\sharp_\nu B \tag{5.15}$$
$$\leqslant \frac{1}{2}\nu\left(1-\nu\right)\begin{cases} \left[\left(\ln m\right)^2\right]A \text{ if } M<1, \\ \max\left\{\left(\ln m\right)^2, \left(\ln M\right)^2 M\right\}A \text{ if } m\leqslant 1\leqslant M, \\ \left[\left(\ln M\right)^2 M\right]A \text{ if } 1<m \end{cases}$$

and

$$A\nabla_\nu B - A\sharp_\nu B \tag{5.16}$$
$$\leqslant \begin{cases} \left(\exp\left[\frac{1}{2}\nu\left(1-\nu\right)\left(\frac{1}{m}-1\right)^2\right]-1\right)Am^\nu \text{if } M<1, \\ \max\left\{\left(\exp\left[\frac{1}{2}\nu\left(1-\nu\right)\left(\frac{1}{m}-1\right)^2\right]-1\right)m^\nu, \right. \\ \left. \left(\exp\left[\frac{1}{2}\nu\left(1-\nu\right)\left(M-1\right)^2\right]-1\right)M^\nu\right\}A \text{ if } m\leqslant 1\leqslant M, \\ \left(\exp\left[\frac{1}{2}\nu\left(1-\nu\right)\left(M-1\right)^2\right]-1\right)AM^\nu \text{ if } 1<m. \end{cases}$$

If we use the first inequality in (2.5), then we have the following lower bounds for the difference $A\nabla_\nu B - A\sharp_\nu B$:

$$A\nabla_\nu B - A\sharp_\nu B \tag{5.17}$$
$$\geqslant \frac{1}{2}\nu\left(1-\nu\right)\times\begin{cases} \left[\left(\ln M\right)^2 M\right]A\text{if } M<1, \\ 0 \text{ if } m\leqslant 1\leqslant M, \\ \left[\left(\ln m\right)^2\right]A \text{ if } 1<m \end{cases}$$

and

$$
A\nabla_\nu B - A\sharp_\nu B \tag{5.18}
$$

$$
\geqslant \begin{cases} \left(\exp\left[\frac{1}{2}\nu(1-\nu)(1-M)^2\right]-1\right)AM^\nu \text{ if } M<1, \\ 0 \text{ if } m \leqslant 1 \leqslant M, \\ \left(\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{1}{m}\right)^2\right]-1\right)Am^\nu \text{ if } 1<m. \end{cases}
$$

The interested reader may state other inequalities by using Theorems 4-7, however the details are nor presented here.

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REFERENCES

[1] S. S. DRAGOMIR, *Some new reverses of Young's operator inequality*, Preprint RGMIA Res. Rep. Coll., **18** (2015), Art. 130. [http://rgmia.org/papers/v18/v18a130.pdf].

[2] S. S. DRAGOMIR, *On new refinements and reverses of Young's operator inequality*, Preprint RGMIA Res. Rep. Coll., **18** (2015), Art. 135. [http://rgmia.org/papers/v18/v18a135.pdf].

[3] S. S. DRAGOMIR, *Some inequalities for operator weighted geometric mean*, Preprint RGMIA Res. Rep. Coll., **18** (2015), Art. 139. [http://rgmia.org/papers/v18/v18a139.pdf].

[4] S. S. DRAGOMIR, *Refinements and reverses of Hölder-McCarthy operator inequality*, Preprint RGMIA Res. Rep. Coll., **18** (2015), Art. 143. [http://rgmia.org/papers/v18/v18a143.pdf].

[5] S. S. DRAGOMIR, *Some reverses and a refinement of Hölder operator inequality*, Preprint RGMIA Res. Rep. Coll., **18** (2015), Art. 147. [http://rgmia.org/papers/v18/v18a147.pdf].

[6] S. FURUICHI, *Refined Young inequalities with Specht's ratio*, J. Egyptian Math. Soc., **20**(2012), 46–49.

[7] S. FURUICHI, *On refined Young inequalities and reverse inequalities*, J. Math. Inequal., **5** (2011), 21–31.

[8] F. KITTANEH AND Y. MANASRAH, *Improved Young and Heinz inequalities for matrix*, J. Math. Anal. Appl., **361** (2010), 262–269.

[9] F. KITTANEH AND Y. MANASRAH, *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear Algebra, **59** (2011), 1031–1037.

[10] F. KUBO AND T. ANDO, *Means of positive operators*, Math. Ann., **264** (1980), 205–224.

[11] W. LIAO, J. WU AND J. ZHAO, *New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant*, Taiwanese J. Math., **19** (2015), No. 2, pp. 467–479.

[12] C. A. MCCARTHY, c_p, Israel J. Math., **5** (1967), 249–271.

[13] W. SPECHT, *Zer Theorie der elementaren Mittel*, Math. Z., **74** (1960), pp. 91–98.

[14] M. TOMINAGA, *Specht's ratio in the Young inequality*, Sci. Math. Japon., **55** (2002), 583–588.
[15] G. ZUO, G. SHI AND M. FUJII, *Refined Young inequality with Kantorovich constant*, J. Math. Inequal., **5** (2011), 551–556.

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