ADDITIVE REFINEMENTS AND REVERSES OF YOUNG’S OPERATOR INEQUALITY WITH APPLICATIONS

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Abstract. In this paper we obtain some new additive refinements and reverses of Young’s operator inequality. Applications related to the Hölder-McCarthy inequality for positive operators and for trace class operators on Hilbert spaces are given as well.

1. Introduction

Throughout this paper $A$, $B$ are positive operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators

$$A \triangledown \nu B := (1 - \nu)A + \nu B,$$

the weighted operator arithmetic mean and

$$A^{\#} \nu B := A^{1/2} \left( A^{-1/2}BA^{-1/2} \right)^{\nu} A^{1/2},$$

the weighted operator geometric mean. When $\nu = \frac{1}{2}$ we write $A \triangledown B$ and $A^{\#}B$ for brevity, respectively.

The famous Young inequality for scalars says that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu}b^{\nu} \leq (1 - \nu)a + \nu b \quad (1.1)$$

with equality if and only if $a = b$. The inequality (1.1) is also called $\nu$-weighted arithmetic-geometric mean inequality.

We recall that Specht’s ratio is defined by [13]

$$S(h) := \begin{cases} \frac{1}{h^{\frac{1}{n-1}}} & \text{if } h \in (0, 1) \cup (1, \infty), \\ e^{\ln \left( \frac{1}{n-1} \right)} & \text{if } h = 1. \end{cases} \quad (1.2)$$


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It is well known that \( \lim_{h \to 1} S(h) = 1 \), \( S(h) = S\left(\frac{1}{h}\right) > 1 \) for \( h > 0 \), \( h \neq 1 \). The function is decreasing on \( (0, 1) \) and increasing on \( (1, \infty) \).

The following inequality provides a refinement and a multiplicative reverse for Young’s inequality

\[
S\left(\left(\frac{a}{b}\right)^r\right) a^{1-v} b^v \leq (1-v) a + vb \leq S\left(\frac{a}{b}\right) a^{1-v} b^v, \tag{1.3}
\]

where \( a, b > 0, \quad v \in [0, 1], \quad r = \min \{1-v, v\} \).

The second inequality in (1.3) is due to Tominaga [14] while the first one is due to Furuichi [6].

The operator version is as follows [6], [14]:

**THEOREM 1.** For two positive operators \( A, B \) and positive real numbers \( m, m', M, M' \) satisfying either of the following conditions:

(i) \( 0 < ml \leq A \leq m'l \leq M' l \leq B \leq MI \),

(ii) \( 0 < ml \leq B \leq m'l \leq M' l \leq A \leq MI \),

we have

\[
S \left( (h')^r \right) A^*_{\nu} B \leq A \nabla_{\nu} B \leq S (h) A^*_{\nu} B, \tag{1.4}
\]

where \( h := \frac{M}{m}, \quad h' := \frac{M'}{m'} \) and \( \nu \in [0, 1] \).

We consider the Kantorovich’s constant defined by

\[
K(h) := \frac{(h + 1)^2}{4h}, \quad h > 0. \tag{1.5}
\]

The function \( K \) is decreasing on \( (0, 1) \) and increasing on \( [1, \infty) \), \( K(h) \geq 1 \) for any \( h > 0 \) and \( K\left(\frac{1}{h}\right) = K\left(\frac{1}{h}\right) \) for any \( h > 0 \).

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich’s constant holds.

\[
K^r \left( \frac{a}{b} \right) a^{1-v} b^v \leq (1-v) a + vb \leq K^r \left( \frac{a}{b} \right) a^{1-v} b^v, \tag{1.6}
\]

where \( a, b > 0, \quad v \in [0, 1], \quad r = \min \{1-v, v\} \) and \( R = \max \{1-v, v\} \).

The first inequality in (1.6) was obtained by Zou et al. in [15] while the second by Liao et al. [11].

The operator version is as follows [15], [11]:

**THEOREM 2.** For two positive operators \( A, B \) and positive real numbers \( m, m', M, M' \) satisfying either of the following conditions:

(i) \( 0 < ml \leq A \leq m'l \leq M' l \leq B \leq MI \),

(ii) \( 0 < ml \leq B \leq m'l \leq M' l \leq A \leq MI \),

we have

\[
K^r \left( h' \right) A^*_{\nu} B \leq A \nabla_{\nu} B \leq K^r (h) A^*_{\nu} B, \tag{1.7}
\]

where \( h := \frac{M}{m}, \quad h' := \frac{M'}{m'}, \quad \nu \in [0, 1], \quad r = \min \{1-v, v\} \) and \( R = \max \{1-v, v\} \).
Kittaneh and Manasrah [8], [9] provided a refinement and an additive reverse for Young inequality as follows:

\[ r \left( \sqrt{a} - \sqrt{b} \right)^2 \leq (1 - \nu)a + \nu b - a^{1 - \nu}b^\nu \leq R \left( \sqrt{a} - \sqrt{b} \right)^2 \]  \hspace{1cm} (1.8)

where \( a, b > 0, \nu \in [0, 1], \ r = \min \{1 - \nu, \nu\} \) and \( R = \max \{1 - \nu, \nu\} \). The case \( \nu = \frac{1}{2} \) reduces (1.8) to an identity.

For some operator versions of (1.8) see [8] and [9]. Other recent results for operators may be found in [1]-[5].

Motivated by the above results we establish in this paper some new additive refinements and reverses of Young’s operator inequality. Applications related to the Hölder-McCarthy inequality for positive operators and for trace class operators on Hilbert spaces are given as well.

2. Additive Reverses

We consider the function \( f_\nu : [0, \infty) \rightarrow [0, \infty) \) defined for \( \nu \in (0, 1) \) by

\[ f_\nu (x) = 1 - \nu + \nu x - x^\nu. \]  \hspace{1cm} (2.1)

The following lemma holds.

**Lemma 1.** For any \( x \in [m, M] \subset [0, \infty) \) we have

\[ \max_{x \in [m, M]} f_\nu (x) = \Delta_\nu (m, M) := \left\{ \begin{array}{ll} f_\nu (m) & \text{if } M < 1, \\ \max \{f_\nu (m), f_\nu (M)\} & \text{if } m \leq 1 \leq M, \\ f_\nu (M) & \text{if } 1 < m \end{array} \right. \]  \hspace{1cm} (2.2)

and

\[ \min_{x \in [m, M]} f_\nu (x) = \delta_\nu (m, M) := \left\{ \begin{array}{ll} f_\nu (M) & \text{if } M < 1, \\ 0 & \text{if } m \leq 1 \leq M, \\ f_\nu (m) & \text{if } 1 < m. \end{array} \right. \]  \hspace{1cm} (2.3)

**Proof.** The function \( f_\nu \) is differentiable and

\[ f'_\nu (x) = \nu \left( 1 - x^{\nu - 1} \right) = \nu \frac{x^{1 - \nu} - 1}{x^{1 - \nu}}, \]

which shows that the function \( f_\nu \) is decreasing on \([0, 1]\) and increasing on \([1, \infty)\), \( f_\nu (0) = 1 - \nu, \ f_\nu (1) = 0 \) and the equation \( f_\nu (x) = 1 - \nu \) for \( x > 0 \) has the unique solution \( x_\nu = \nu^{\frac{1}{\nu - 1}} > 1 \).

Therefore, by considering the 3 possible situations for the location of the interval \([m, M]\) and the number 1 we get the desired bounds (2.2) and (2.3). \( \square \)
REMARK 1. We have the inequalities

\[ 0 \leq f_\nu(x) \leq 1 - \nu \quad \text{for any } x \in \left[0, \nu^{\frac{1}{\nu - 1}}\right] \]

and

\[ 1 - \nu \leq f_\nu(x) \quad \text{for any } x \in \left[\nu^{\frac{1}{\nu - 1}}, \infty\right). \]

THEOREM 3. Assume that \( A, B \) are positive invertible operators and the constants \( M > m > 0 \) are such that

\[ mA \leq B \leq MA. \quad (2.4) \]

Let \( \nu \in [0,1] \), then we have the inequalities

\[ \delta_\nu(m,M)A \leq A\nabla_\nu B - A_\nu B \leq \Delta_\nu(m,M)A, \quad (2.5) \]

where \( \Delta_\nu(m,M) \) and \( \delta_\nu(m,M) \) are defined by (2.2) and (2.3), respectively.

Proof. From Lemma 1 we have the double inequality

\[ \delta_\nu(m,M) \leq 1 - \nu + \nu x - x^\nu \leq \Delta_\nu(m,M) \quad (2.6) \]

for any \( x \in [m,M] \).

If \( X \) is an operator such that \( mI \leq X \leq MI \), then by (2.6) and the continuous functional calculus, we have

\[ \delta_\nu(m,M)I \leq (1 - \nu)I + \nu X - X^\nu \leq \Delta_\nu(m,M)I. \quad (2.7) \]

If the condition (2.4) holds, then by multiplying in both sides with \( A^{-1/2} \) we get \( mI \leq A^{-1/2}BA^{-1/2} \leq MI \) and by taking \( X = A^{-1/2}BA^{-1/2} \) in (2.7) we get

\[ \delta_\nu(m,M)I \leq (1 - \nu)I + \nu A^{-1/2}BA^{-1/2} - \left( A^{-1/2}BA^{-1/2} \right)^\nu \quad (2.8) \]

\[ \leq \Delta_\nu(m,M)I. \]

Now, if we multiply (2.8) in both sides with \( A^{1/2} \) we get the desired result (2.5). \( \square \)

COROLLARY 1. For two positive operators \( A, B \) and positive real numbers \( m, m', M, M' \) put \( h = \frac{M}{m} \) and \( h' = \frac{M'}{m'} \).

If

(i) \( 0 < mI \leq A \leq m'I < M'I \leq B \leq MI \),

then

\[ f_\nu\left(h'\right)A \leq A\nabla_\nu B - A_\nu B \leq f_\nu\left(h\right)A. \quad (2.9) \]

If

(ii) \( 0 < mI \leq B \leq m'I < M'I \leq A \leq MI \),

then

\[ f_\nu\left((h')^{-1}\right)A \leq A\nabla_\nu B - A_\nu B \leq f_\nu\left(h^{-1}\right)A. \quad (2.10) \]
Proof. If (i) is valid, then we have

\[ A < \frac{M'}{m'} A = h' A \leq B \leq h A = \frac{M}{m} A, \]

and by (2.5) we have for \( 1 < h' \leq h \)

\[ f_v (h') A \leq A \nabla_v B - A^2_v B \leq f_v (h) A, \]

and the inequality (2.9) is proved.

If (ii) is valid, then we have

\[ \frac{1}{h} A \leq B \leq \frac{1}{h'} A < A \]

and by (2.5) for \( \frac{1}{h} \leq \frac{1}{m} < 1 \) we also have

\[ f_v \left( \frac{1}{h'} \right) A \leq A \nabla_v B - A^2_v B \leq f_v \left( \frac{1}{h} \right) A, \]

and the inequality (2.10) is proved. \( \square \)

We have the following simpler bounds:

**Corollary 2.** With the assumptions of Theorem 3 we have

\[
\begin{align*}
\begin{cases}
(1 - \sqrt{M})^2 A & \text{if } M < 1, \\
0 & \text{if } m \leq 1 \leq M, \\
(\sqrt{m} - 1)^2 A & \text{if } 1 < m,
\end{cases}
\end{align*}
\]

\[ \leq A \nabla_v B - A^2_v B \]

\[
\begin{align*}
\begin{cases}
(1 - \sqrt{m})^2 A & \text{if } M < 1, \\
\max \left\{ (1 - \sqrt{m})^2, (\sqrt{M} - 1)^2 \right\} & \text{if } m \leq 1 \leq M, \\
(\sqrt{M} - 1)^2 A & \text{if } 1 < m,
\end{cases}
\end{align*}
\]

where \( \nu \in [0, 1] \), \( r = \min \{ 1 - \nu, \nu \} \) and \( R = \max \{ 1 - \nu, \nu \} \).

Proof. From the inequality (1.8) we have for \( b = t \) and \( a = 1 \) that

\[ r (\sqrt{t} - 1)^2 \leq f_v (t) = 1 - \nu + \nu t - t^\nu \leq R (\sqrt{t} - 1)^2 \]

for any \( t \in [0, 1] \).
Then we have

\[ \Delta_\nu(m,M) \leq R \times \begin{cases} (1 - \sqrt{m})^2 & \text{if } M < 1, \\ \max \left\{ (1 - \sqrt{m})^2, (\sqrt{M} - 1)^2 \right\} & \text{if } m \leq 1 \leq M, \\ (\sqrt{M} - 1)^2 & \text{if } 1 < m \end{cases} \]

and

\[ \delta_\nu(m,M) \geq r \times \begin{cases} (1 - \sqrt{M})^2 & \text{if } M < 1, \\ 0 & \text{if } m \leq 1 \leq M, \\ (\sqrt{m} - 1)^2 & \text{if } 1 < m, \end{cases} \]

which by Theorem 3 proves the corollary. \( \square \)

**Remark 2.** With the assumptions of Corollary 1, we have, in the case (i), that

\[ r \left( \sqrt{h'} - 1 \right)^2 A \leq A\nabla_\nu B - A^\parallel_\nu B \leq R \left( \sqrt{h} - 1 \right)^2 A, \quad (2.12) \]

and in the case (ii), that

\[ \frac{r \left( 1 - \sqrt{h'} \right)^2}{h'} A \leq A\nabla_\nu B - A^\parallel_\nu B \leq \frac{R \left( 1 - \sqrt{h} \right)^2}{h} A. \quad (2.13) \]

The following bounds in terms of Specht’s ratio can be stated as well:

**Corollary 3.** With the assumptions of Theorem 3 we have

\[ \begin{cases} [S(M') - 1]M^\nu A & \text{if } M < 1, \\ 0 & \text{if } m \leq 1 \leq M, \\ [S(m') - 1]m^\nu A & \text{if } 1 < m, \end{cases} \]

\[ \leq A\nabla_\nu B - A^\parallel_\nu B \]

\[ \begin{cases} [S(m) - 1]m^\nu A & \text{if } M < 1, \\ \max \{ [S(m) - 1]m^\nu, [S(M) - 1]M^\nu \} A & \text{if } m \leq 1 \leq M, \\ [S(M) - 1]M^\nu A & \text{if } 1 < m. \end{cases} \]
Proof. From the inequality (1.3) we have for \( a = 1 \) and \( b = t \) that

\[
S(t^r)t^v \leq 1 - v + vt \leq S(t)t^v,
\]

where \( t > 0, \ v \in [0, 1], \ r = \min \{1 - v, v\} \).

By subtracting \( t^v \) in the inequality (2.15) we get

\[
(0 \leq) [S(t^r) - 1]t^v \leq f_v(t) \leq [S(t) - 1]t^v,
\]

for any \( t > 0, \ v \in [0, 1] \).

Then we have

\[
\Delta_v(m, M) \leq \begin{cases} 
[S(m) - 1]m^v \text{ if } M < 1, \\
\max \{[S(m) - 1]m^v, [S(M) - 1]M^v\} \text{ if } m \leq 1 \leq M, \\
[S(M) - 1]M^v \text{ if } 1 < m 
\end{cases}
\]

and

\[
\delta_v(m, M) \geq \begin{cases} 
[S(M') - 1]M^v \text{ if } M < 1, \\
0 \text{ if } m \leq 1 \leq M, \\
[S(m') - 1]m^v \text{ if } 1 < m, 
\end{cases}
\]

which by Theorem 3 proves the corollary. ☐

**Remark 3.** With the assumptions of Corollary 1, we have in the case (i), that

\[
[S((h')^r) - 1] (h')^v A \leq A\nabla_v B - A\sharp_v B \leq [S(h) - 1] h^v A,
\]

and in the case (ii), that

\[
\frac{S((h')^r) - 1}{(h')^r} A \leq A\nabla_v B - A\sharp_v B \leq \frac{S(h) - 1}{h^v} A.
\]

From the inequality (1.6) we have

\[
(0 <) [K^r(t) - 1]t^v \leq 1 - v + vt - t^v \leq [K^R(t) - 1]t^v
\]

where \( t > 0, \ v \in [0, 1], \ r = \min \{1 - v, v\} \) and \( R = \max \{1 - v, v\} \).

We then have the following bounds in terms of Kantorovich’s constant:
COROLLARY 4. With the assumptions of Theorem 3 we have

\[
\begin{cases}
[K'(M) - 1] M^\nu A & \text{if } M < 1, \\
0 & \text{if } m \leq 1 \leq M, \\
[K'(m) - 1] m^\nu A & \text{if } 1 < m,
\end{cases}
\]

\[\leq A^\nabla^\nu B - A^\#_{3}^\nu B\]

\[
\begin{cases}
[K^R (m) - 1] m^\nu A & \text{if } M < 1, \\
\max \{ [K^R (m) - 1] m^\nu, [K^R (M) - 1] M^\nu \} A & \text{if } m \leq 1 \leq M, \\
[K^R (M) - 1] M^\nu A & \text{if } 1 < m.
\end{cases}
\]

REMARK 4. With the assumptions of Corollary 1, we have in the case (i), that

\[
[K' (h') - 1] (h')^\nu A \leq A^\nabla^\nu B - A_{3}^\nu B \leq [K^R (h) - 1] h^\nu A,
\]

and in the case (ii), that

\[
\frac{K^r (h') - 1}{(h')^r} A \leq A^\nabla^\nu B - A_{3}^\nu B \leq \frac{K^R (h) - 1}{h^\nu} A.
\]

Let \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Assume that the positive invertible operators \( A, B \) satisfy the condition

\[
mA^p \leq B^q \leq mA^p.
\]

Then by replacing \( A \) with \( A^p \), \( B \) with \( B^q \) and \( \nu = \frac{1}{q} \) in (2.5) we have

\[
\delta_{\frac{1}{q}} (m, M) A^p \leq \frac{1}{p} A^p + \frac{1}{q} B^q - A^{p}_{\frac{1}{q}} B^q \leq \Delta_{\frac{1}{q}} (m, M) A^p,
\]

where \( \Delta_{\frac{1}{q}} (m, M) \) and \( \delta_{\frac{1}{q}} (m, M) \) are defined by (2.2) and (2.3) respectively.

If the positive invertible operators \( A, B \) satisfy the condition (2.23), then from
(2.11) we get for

\[ r_{p,q} \times \begin{cases} (1 - \sqrt{M})^2 A^p & \text{if } M < 1, \\ 0 & \text{if } m \leq 1 \leq M, \\ (\sqrt{m} - 1)^2 A^p & \text{if } 1 < m \end{cases} \]  

(2.25)

\[ \leq \frac{1}{p} A^p + \frac{1}{q} B^q - A^p \frac{m}{M} B^q \]

\leq R_{p,q} \times \begin{cases} (1 - \sqrt{m})^2 A^p & \text{if } M < 1, \\ \max \left\{ (1 - \sqrt{m})^2, (\sqrt{M} - 1)^2 \right\} A^p & \text{if } m \leq 1 \leq M, \\ (\sqrt{M} - 1)^2 A^p & \text{if } 1 < m, \end{cases} \]

from (2.14) we get

\[ \begin{cases} [S (M^{p,q}) - 1] M^y A^p & \text{if } M < 1, \\ 0 & \text{if } m \leq 1 \leq M, \\ [S (m^{p,q}) - 1] m^y A^p & \text{if } 1 < m \end{cases} \]

(2.26)

\[ \leq \frac{1}{p} A^p + \frac{1}{q} B^q - A^p \frac{m}{M} B^q \]

\[ \leq \max \left\{ [S (m) - 1] m^y, [S (M) - 1] M^y \right\} A^p \text{ if } m \leq 1 \leq M, \]

\[ [S (M) - 1] M^y A^p \text{ if } 1 < m, \]
while from (2.20) we get

\[
\begin{cases}
[K^{r_{p,q}}(M) - 1] M^\nu A^p & \text{if } M < 1, \\
0 & \text{if } m \leq 1 \leq M, \\
[K^{r_{p,q}}(m) - 1] m^\nu A^p & \text{if } 1 < m
\end{cases}
\] (2.27)

\[
\leq \frac{1}{p} A^p + \frac{1}{q} B^q - A^{p+\frac{q}{2}} B^q
\]

\[
\begin{cases}
[K^{R_{p,q}}(m) - 1] m^\nu A^p & \text{if } m \leq 1 \leq M, \\
\max \left\{ \left[ K^{R_{p,q}}(m) - 1 \right] m^\nu, \left[ K^{R_{p,q}}(M) - 1 \right] M^\nu \right\} A^p & \text{if } 1 < m,
\end{cases}
\]

where \( r_{p,q} = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\} \) and \( R_{p,q} = \max \left\{ \frac{1}{p}, \frac{1}{q} \right\} \).

If \( p = q = 2 \) and if we assume that

\[
m A^2 \leq B^2 \leq M A^2,
\] (2.28)

then by (2.24) we get

\[
\delta^\frac{1}{2}(m,M) A^2 \leq \frac{1}{2} (A^2 + B^2) - A^{2+\frac{q}{2}} B^2 \leq \Delta^\frac{1}{2}(m,M) A^2.
\] (2.29)

Assume that \( A \) and \( B \) satisfy the conditions

\[
m_1 I \leq A \leq M_1 I, \quad m_2 I \leq B \leq M_2 I
\] (2.30)

for some \( 0 < m_1 < M_1 \) and \( 0 < m_2 < M_2 \). We have from (2.30) that

\[
m_1^p I \leq A^p \leq M_1^p I.
\]

Then by (2.30) we also have

\[
m_1^p M_2^{-q} I \leq m_1^p B^{-q} \leq B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \leq M_1^p B^{-q} \leq M_1^p m_2^{-q} I,
\]

which implies that

\[
m_1 M_2^{-\frac{q}{p}} I \leq \left( B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \right)^{\frac{1}{p}} \leq M_1 m_2^{-\frac{q}{p}} I.
\]

Now, on using the inequality (2.24) for \( m = m_1 M_2^{-\frac{q}{p}} \) and \( M = M_1 m_2^{-\frac{q}{p}} \), we get

\[
\delta^v \left( m_1 M_2^{-\frac{q}{p}}, M_1 m_2^{-\frac{q}{p}} \right) A^p \leq \frac{1}{p} A^p + \frac{1}{q} B^q - A^{p+\frac{q}{2}} B^q
\]

\[
\leq \Delta^v \left( m_1 M_2^{-\frac{q}{p}}, M_1 m_2^{-\frac{q}{p}} \right) A^p,
\] (2.31)
where \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).

In particular, we have

\[
\delta_{\frac{1}{2}} (m_1 M_2^{-1}, M_1 m_2^{-1}) A^2 \leq \frac{1}{2} (A^2 + B^2) - A^2 B^2 \leq \Delta_{\frac{1}{2}} (m_1 M_2^{-1}, M_1 m_2^{-1}) A^2,
\]

provided that \( A \) and \( B \) satisfy the conditions (2.30).

Further inequalities in terms of Specht’s ratio and Kantorovich’s constant may be obtained by using (2.26) and (2.27) respectively, however the details are not presented here.

### 3. Inequalities Related to McCarthy’s

By the use of the spectral resolution of \( P \geq 0 \) and the Hölder inequality, C. A. McCarthy [12] proved that

\[
\langle Px, x \rangle^p \leq \langle P^p x, x \rangle, \ p \in (1, \infty)
\]

and

\[
\langle P^p x, x \rangle \leq \langle Px, x \rangle^p, \ p \in (0, 1)
\]

for any \( x \in H \) with \( \|x\| = 1 \).

From the previous section, for positive numbers \( a, b \) with \( \frac{b}{a} \in [m, M] \subset (0, \infty) \) and \( \nu \in [0, 1] \) we can state the following scalar inequalities

\[
\delta_{\nu} (m, M) a \leq (1 - \nu) a + \nu b - a^{1-\nu} b^\nu \leq \Delta_{\nu} (m, M) a,
\]

where \( \Delta_{\nu} (m, M) \) and \( \delta_{\nu} (m, M) \) are defined by (2.2) and (2.3) respectively.

We also have the scalar inequalities

\[
\max \left\{ (1 - \sqrt{m})^2 a \text{ if } M < 1, \right. \\
\left. (\sqrt{m} - 1)^2 a \text{ if } 1 < m, \right. \\
\left. \frac{1}{2} (A^2 + B^2) - A^2 B^2 \right. \\
\leq \frac{1}{2} (A^2 + B^2) - A^2 B^2 \leq \Delta_{\frac{1}{2}} (m_1 M_2^{-1}, M_1 m_2^{-1}) A^2,
\]

and

\[
\left. \max \left\{ (1 - \sqrt{m})^2, (\sqrt{M} - 1)^2 \right\} a \text{ if } m \leq 1 \leq M, \right. \\
\left. (\sqrt{M} - 1)^2 a \text{ if } 1 < m, \right. \\
\left. \frac{1}{2} (A^2 + B^2) - A^2 B^2 \right. \\
\leq \frac{1}{2} (A^2 + B^2) - A^2 B^2 \leq \Delta_{\frac{1}{2}} (m_1 M_2^{-1}, M_1 m_2^{-1}) A^2,
\]

provided that \( A \) and \( B \) satisfy the conditions (2.30).
\[
\begin{cases}
[S(M') - 1] M^\nu a & \text{if } M < 1, \\
0 & \text{if } m \leq 1 \leq M, \\
[S(m') - 1] m^\nu a & \text{if } 1 < m
\end{cases}
\]

(3.5)

\[
\leq (1 - \nu) a + \nu b - a^{1-\nu} b^\nu
\]

\[
\begin{cases}
[S(m) - 1] m^\nu a & \text{if } M < 1, \\
\max \{ [S(m) - 1] m^\nu, [S(M) - 1] M^\nu \} a & \text{if } m \leq 1 \leq M, \\
[S(M) - 1] M^\nu a & \text{if } 1 < m
\end{cases}
\]

and

\[
\begin{cases}
[K^r (M) - 1] M^\nu a & \text{if } M < 1, \\
0 & \text{if } m \leq 1 \leq M, \\
[K^r (m) - 1] m^\nu a & \text{if } 1 < m
\end{cases}
\]

(3.6)

\[
\leq (1 - \nu) a + \nu b - a^{1-\nu} b^\nu
\]

\[
\begin{cases}
[K^R (m) - 1] m^\nu a & \text{if } M < 1, \\
\max \{ [K^R (m) - 1] m^\nu, [K^R (M) - 1] M^\nu \} a & \text{if } m \leq 1 \leq M, \\
[K^R (M) - 1] M^\nu a & \text{if } 1 < m,
\end{cases}
\]

where \( r = \min \{ 1 - \nu, \nu \} \) and \( R = \max \{ 1 - \nu, \nu \} \).

**Theorem 4.** Let \( P \) and operator such that

\[
z I \preceq P \preceq Z I
\]

(3.7)

for some constants \( Z > z > 0 \).

Then for any \( x \in H \) with \( \|x\| = 1 \) we have

\[
0 \leq 1 - \frac{\langle P^\lambda x, x \rangle}{\langle P x, x \rangle^\lambda} \leq \max \left\{ f_\lambda \left( \frac{z}{Z} \right), f_\lambda \left( \frac{Z}{z} \right) \right\},
\]

(3.8)

where \( \lambda \in [0, 1] \) and the function \( f_\lambda : [0, \infty) \to [0, \infty) \) is defined by

\[
f_\lambda(t) = 1 - \lambda + \lambda t - t^\lambda.
\]

(3.9)
Proof. If \( u, v \in [z, Z] \) then \( \frac{u}{v} \in \left[ \frac{z}{Z}, \frac{Z}{z} \right] \) and by (3.3) we have

\[
0 \leq (1 - \lambda)v + \lambda u - v^{1-\lambda}u^\lambda \leq \max \left\{ f_{\lambda} \left( \frac{z}{Z} \right), f_{\lambda} \left( \frac{Z}{z} \right) \right\} v
\]

for any \( \lambda \in [0, 1] \).

Fix \( v \in [z, Z] \), then by using the functional calculus for the operator \( P \) with \( zI \leq P \leq ZI \) we have

\[
0 \leq (1 - \lambda)v + \lambda P - v^{1-\lambda}P^\lambda \leq \max \left\{ f_{\lambda} \left( \frac{z}{Z} \right), f_{\lambda} \left( \frac{Z}{z} \right) \right\} v
\]

(3.10)

for any \( \lambda \in [0, 1] \).

The inequality (3.10) implies that

\[
0 \leq (1 - \lambda)v + \lambda \langle Px, x \rangle - v^{1-\lambda} \langle P^\lambda x, x \rangle \leq \max \left\{ f_{\lambda} \left( \frac{z}{Z} \right), f_{\lambda} \left( \frac{Z}{z} \right) \right\} \langle Px, x \rangle,
\]

(3.11)

for any \( x \in H \) with \( \|x\| = 1 \), for any \( \lambda \in [0, 1] \) and for any \( v \in [z, Z] \).

If we take in (3.11) \( v = \langle Px, x \rangle \in [z, Z] \), for \( x \in H \) with \( \|x\| = 1 \), then we have

\[
0 \leq \langle Px, x \rangle - \langle Px, x \rangle^{1-\lambda} \langle P^\lambda x, x \rangle \leq \max \left\{ f_{\lambda} \left( \frac{z}{Z} \right), f_{\lambda} \left( \frac{Z}{z} \right) \right\} \langle Px, x \rangle \lambda
\]

which, by division with \( \langle Px, x \rangle^{1-\lambda} > 0 \) produces

\[
0 \leq \langle Px, x \rangle^\lambda - \langle P^\lambda x, x \rangle \leq \max \left\{ f_{\lambda} \left( \frac{z}{Z} \right), f_{\lambda} \left( \frac{Z}{z} \right) \right\} \langle Px, x \rangle^\lambda
\]

that is equivalent to the desired result (3.8). \( \square \)

REMARK 5. If \( 1 < \frac{z}{Z} \leq \frac{1}{\lambda - 1} \) with \( \lambda \in (0, 1) \) then by Remark 1 we have that \( \max \left\{ f_{\lambda} \left( \frac{z}{Z} \right), f_{\lambda} \left( \frac{Z}{z} \right) \right\} \leq 1 - \lambda \) and by (3.8) we get

\[
\lambda \langle Px, x \rangle^\lambda \leq \langle P^\lambda x, x \rangle \leq \langle Px, x \rangle^\lambda
\]

(3.12)

for any \( x \in H \) with \( \|x\| = 1 \).

COROLLARY 5. With the assumptions of Theorem 4 and if \( T = \max \{ \lambda, 1 - \lambda \} \) for \( \lambda \in (0, 1) \), then we have

\[
0 \leq 1 - \frac{\langle P^\lambda x, x \rangle}{\langle Px, x \rangle^\lambda} \leq \begin{cases} T \left( \sqrt{\frac{z}{Z}} - 1 \right)^2, \\
\left[ S \left( \frac{z}{Z} \right) - 1 \right] \left( \frac{z}{Z} \right)^\lambda, \\
\left[ K^T \left( \frac{z}{Z} \right) - 1 \right] \left( \frac{z}{Z} \right)^\lambda
\end{cases}
\]

for any \( x \in H \) with \( \|x\| = 1 \).
We have:

**THEOREM 5.** Let $A$ and $B$ be two positive invertible operators, $p$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $m$, $M > 0$ such that

$$m^p B^q I \leq A^p \leq M^p B^q I. \quad (3.13)$$

Then we have

$$0 \leq 1 - \frac{\langle B^q x, x \rangle^{1/p} \langle A^p x, x \rangle^{1/q}}{\langle A^p x, x \rangle^{1/p} \langle B^q x, x \rangle^{1/q}} \leq \max \left\{ f_{\frac{1}{p}} \left( \left( \frac{m}{M} \right)^p \right), f_{\frac{1}{p}} \left( \left( \frac{M}{m} \right)^p \right) \right\}, \quad (3.14)$$

where the function $f_{\frac{1}{p}} : [0, \infty) \rightarrow [0, \infty)$ is defined by (3.9) for $\lambda = \frac{1}{p}$.

**Proof.** From the inequality (3.8) for $x = \frac{y}{\|y\|}$, $y \neq 0$ we have

$$0 \leq 1 - \frac{\langle P^\lambda y, y \rangle}{\langle y, y \rangle^{1-\lambda} \langle P y, y \rangle^\lambda} \leq \max \left\{ f_\lambda \left( \frac{z}{Z} \right), f_\lambda \left( \frac{Z}{z} \right) \right\}, \quad (3.15)$$

provided that $P$ satisfy the condition (3.7).

Now, from (3.13) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^p I \leq B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \leq M^p I$.

By writing the inequality (3.15) for $P = B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}$, $z = m^p$, $Z = M^p$, $\lambda = \frac{1}{p}$ and $y = B^{\frac{q}{2}} x$, with $x \in H$, $x \neq 0$, we have

$$0 \leq 1 - \frac{\left\langle \left( B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \right)^{\frac{1}{p}} B^{\frac{q}{2}} x, B^{\frac{q}{2}} x \right\rangle}{\left\langle B^{\frac{q}{2}} x, B^{\frac{q}{2}} x \right\rangle^{\frac{1}{q}} \left\langle \left( B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \right)^{\frac{1}{p}} B^{\frac{q}{2}} x, B^{\frac{q}{2}} x \right\rangle^{\frac{1}{p}}} \leq \max \left\{ f_{\frac{1}{p}} \left( \left( \frac{m}{M} \right)^p \right), f_{\frac{1}{p}} \left( \left( \frac{M}{m} \right)^p \right) \right\}$$

that is equivalent to

$$0 \leq 1 - \frac{\langle B^q x, x \rangle^{\frac{1}{p}}}{\langle B^q x, x \rangle^{\frac{1}{q}} \langle A^p x, x \rangle^{\frac{1}{p}}} \leq \max \left\{ f_{\frac{1}{p}} \left( \left( \frac{m}{M} \right)^p \right), f_{\frac{1}{p}} \left( \left( \frac{M}{m} \right)^p \right) \right\}$$

with $x \in H$, $x \neq 0$.

This is equivalent to the desired result (3.14).
COROLLARY 6. With the assumptions of Theorem 5 we have for $x \in H$, $x \neq 0$, that

$$0 \leq 1 - \frac{\langle B^{\sharp 1}_{p/q} A^p x, x \rangle}{\langle A^p x, x \rangle^{1/p} \langle B^{\sharp 1}_{q/p} x, x \rangle^{1/q}} \leq \begin{cases} T_{p,q} \left( \frac{M}{m} \right)^{\frac{p}{q}} - 1 \right)^2, \\ [S \left( \frac{M}{m} \right)^{p} - 1] \frac{M}{m}, \\ [K_{p,q} \left( \frac{M}{m} \right)^{p} - 1] \frac{M}{m}, \end{cases}$$

where $T_{p,q} = \max \left\{ \frac{1}{p}, \frac{1}{q} \right\}$.

4. Trace Inequalities

In the general case of Hilbert spaces $(H; \langle \cdot, \cdot \rangle)$, if $\{e_i\}_{i \in I}$ is an orthonormal basis of $H$, we say that a bounded linear operator $A \in \mathcal{B}(H)$ is trace class provided

$$\|A\|_1 := \sum_{i \in I} |\langle A e_i, e_i \rangle| < \infty. \quad (4.1)$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following properties are also well known:

(i) We have $$\|A\|_1 = \|A^*\|_1$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ is an operator ideal in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H) \mathcal{B}_1(H) \mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) $(\mathcal{B}_1(H), \|\cdot\|_1)$ is a Banach space.

We define the trace of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\text{tr}(A) := \sum_{i \in I} \langle A e_i, e_i \rangle, \quad (4.2)$$

where $\{e_i\}_{i \in I}$ is an orthonormal basis of $H$. Note that this coincides with the usual definition of the trace if $H$ is finite-dimensional. We observe that the series (4.2) converges absolutely and it is independent from the choice of basis.

The following results collect some properties of the trace:

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\text{tr}(A^*) = \overline{\text{tr}(A)}; \quad (4.3)$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT, TA \in \mathcal{B}_1(H)$ and

$$\text{tr}(AT) = \text{tr}(TA) \quad \text{and} \quad |\text{tr}(AT)| \leq \|A\|_1 \|T\|; \quad (4.4)$$

(iii) $\text{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\text{tr}\| = 1$;

(iv) $\mathcal{B}_{\text{fin}}(H)$, the space of operators of finite rank, is a dense subspace of $\mathcal{B}_1(H)$. We have the following trace inequality:
THEOREM 6. Let $C$ be an operator with the property that 
\[ zI \leq C \leq ZI \]  
(4.5)
for some constants $z, Z$ with $Z > z > 0$ and $P \in \mathcal{B}_1(H),\ P \geq 0$ with $\text{tr}(P) > 0$. Then for any $\lambda \in [0, 1]$ we have 
\[ 0 \leq 1 - \frac{\text{tr}(PC^\lambda)}{\text{tr}^1(\lambda)(PC)} \leq \max \left\{ f_\lambda \left( \frac{z}{Z} \right), f_\lambda \left( \frac{Z}{z} \right) \right\} \]  
(4.6)
and the function $f_\lambda$ is defined by (3.9).

Proof. As in the proof of Theorem 4, we have 
\[ 0 \leq (1 - \lambda)vI + \lambda C - v^1 - \lambda C^\lambda \leq \max \left\{ f_\lambda \left( \frac{z}{Z} \right), f_\lambda \left( \frac{Z}{z} \right) \right\} v \]  
for any $\lambda \in [0, 1]$.

This inequality implies that 
\[ 0 \leq (1 - \lambda)v \langle x, x \rangle + \lambda \langle Cx, x \rangle - v^1 - \lambda \langle C^\lambda x, x \rangle \]  
(4.7)
\[ \leq \max \left\{ f_\lambda \left( \frac{z}{Z} \right), f_\lambda \left( \frac{Z}{z} \right) \right\} v \langle x, x \rangle, \]
for any $x \in H$, for any $\lambda \in [0, 1]$ and for any $v \in [z, Z]$.

Now, if we take in (4.7) $x = P^{1/2}e$, where $e \in H$, then 
\[ 0 \leq (1 - \lambda)v \langle Pe, e \rangle + \lambda \langle P^{1/2}CP^{1/2}e, e \rangle - v^1 - \lambda \langle P^{1/2}C^\lambda P^{1/2}e, e \rangle \]  
(4.8)
\[ \leq \max \left\{ f_\lambda \left( \frac{z}{Z} \right), f_\lambda \left( \frac{Z}{z} \right) \right\} v \langle Pe, e \rangle, \]
for any $e \in H$.

Let $\{ei\}_{i \in I}$ be an orthonormal basis of $H$. If we take in (4.8) $e = ei$, $i \in I$ and by summing over $i \in I$, then we get 
\[ 0 \leq (1 - \lambda)v \sum_{i \in I} \langle Pe_i, e_i \rangle + \lambda \sum_{i \in I} \langle P^{1/2}CP^{1/2}e_i, e_i \rangle \]  
(4.9)
\[ - v^1 - \lambda \sum_{i \in I} \langle P^{1/2}C^\lambda P^{1/2}e_i, e_i \rangle \]
\[ \leq \max \left\{ f_\lambda \left( \frac{z}{Z} \right), f_\lambda \left( \frac{Z}{z} \right) \right\} v \sum_{i \in I} \langle Pe_i, e_i \rangle, \]
and by the properties of trace we have 
\[ 0 \leq (1 - \lambda)v \text{tr}(P) + \lambda \text{tr}(PC) - v^1 - \lambda \text{tr}(PC^\lambda) \]
\[ \leq \max \left\{ f_\lambda \left( \frac{z}{Z} \right), f_\lambda \left( \frac{Z}{z} \right) \right\} \text{tr}(P), \]
for any $\lambda \in [0, 1]$ and for any $v \in [z, Z]$.

This inequality can be written as

$$0 \leq (1 - \lambda)v + \lambda \frac{\text{tr}(PC)}{\text{tr}(P)} v - v^{-\lambda} \frac{\text{tr}(PC^\lambda)}{\text{tr}(P)} v$$

for any $\lambda \in [0, 1]$ and for any $v \in [z, Z]$.

Now, if we take in (4.10) $v = \frac{\text{tr}(PC)}{\text{tr}(P)} \in [z, Z]$, then we get

$$0 \leq (1 - \lambda)\frac{\text{tr}(PC)}{\text{tr}(P)} + \lambda \frac{\text{tr}(PC)}{\text{tr}(P)} - \left(\frac{\text{tr}(PC)}{\text{tr}(P)}\right)^{1-\lambda} \frac{\text{tr}(PC^\lambda)}{\text{tr}(P)}$$

namely

$$0 \leq \frac{\text{tr}(PC)}{\text{tr}(P)} - \left(\frac{\text{tr}(PC)}{\text{tr}(P)}\right)^{1-\lambda} \frac{\text{tr}(PC^\lambda)}{\text{tr}(P)}$$

and by multiplying with $\frac{\text{tr}(P)}{\text{tr}(PC)} > 0$ we get the desired result (4.6). □

In particular, we have:

**COROLLARY 7.** With the assumptions of Theorem 6 and if $T = \max \{ \lambda, 1 - \lambda \}$ for $\lambda \in (0, 1)$, then we have

$$0 \leq 1 - \frac{\text{tr}(PC^\lambda)}{\text{tr}(P) \text{tr}^{1-\lambda}(PC)} \leq \begin{cases} T \left(\sqrt{\frac{Z}{z}} - 1\right)^2, \\
S \left(\frac{Z}{z}\right) - 1 \left(\frac{Z}{z}\right)^\lambda, \\
K^T \left(\frac{Z}{z}\right) - 1 \left(\frac{Z}{z}\right)^\lambda. 
\end{cases}$$

The following reverse of Hölder’s trace inequality may be stated:

**THEOREM 7.** Let $A$ and $B$ be two positive invertible operators, $p$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $m$, $M > 0$ such that

$$m^p B^q \leq A^p \leq M^p B^q.$$  

If $B^q \in \mathcal{B}_1(H)$, then

$$0 \leq 1 - \frac{\text{tr}\left(B^q \#_1^p A^p\right)}{\text{tr}^{1/p}(A^p) \text{tr}^{1/q}(B^q)} \leq \max \left\{ f_{\frac{1}{p}} \left(\frac{m}{M}\right)^p, f_{\frac{1}{q}} \left(\frac{M}{m}\right)^p \right\}.$$  

(4.12)
Proof. Now, from (4.12) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^p I \leq B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \leq M^p I$. By writing the inequality (4.6) for $C = B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}$, $z = m^p$, $Z = M^p$, $\lambda = \frac{1}{p}$ and $P = B^q$ we get the desired result (4.13). □

Finally, we have

**COROLLARY 8.** With the assumptions of Theorem 7 and if $T_{p,q} = \max \left\{ \frac{1}{p}, \frac{1}{q} \right\}$, then we have

$$0 \leq 1 - \frac{\text{tr} (B^q\frac{1}{1/p} A^p)}{\text{tr}^{1/p} (A^p) \text{tr}^{1/q} (B^q)} \leq \begin{cases} T_{p,q} \left( \left( \frac{M}{m} \right)^{\frac{q}{2}} - 1 \right)^2, \\ [S \left( \left( \frac{M}{m} \right)^{p} - 1 \right) \frac{M}{m}, \\ [K^{T_{p,q}} \left( \left( \frac{M}{m} \right)^{p} - 1 \right) \frac{M}{m}, \\ (4.14) 

5. Other Upper and Lower Bounds

In [1] we proved the following reverses of Young’s inequality

$$0 \leq (1 - \nu) a + \nu b - a^{1 - \nu} b^\nu \leq \nu (1 - \nu) (a - b) (\ln a - \ln b) \quad (5.1)$$

and

$$1 \leq \frac{(1 - \nu) a + \nu b}{a^{1 - \nu} b^\nu} \leq \exp \left[ 4 \nu (1 - \nu) \left( K \left( \frac{a}{b} \right) - 1 \right) \right], \quad (5.2)$$

for any $a, b > 0$ and $\nu \in [0, 1]$, where $K$ is Kantorovich’s constant defined by (1.5). The inequality (5.2) is equivalent to

$$0 \leq (1 - \nu) a + \nu b - a^{1 - \nu} b^\nu \leq \left( \exp \left[ 4 \nu (1 - \nu) \left( K \left( \frac{a}{b} \right) - 1 \right) \right] - 1 \right) a^{1 - \nu} b^\nu \quad (5.3)$$

for any $a, b > 0$ and $\nu \in [0, 1]$.

Therefore, by (2.2), (5.1) and (5.3) we have

$$\Delta_{\nu} (m,M) = \begin{cases} f_{\nu} (m) \text{ if } M < 1, \\ \max \{ f_{\nu} (m), f_{\nu} (M) \} \text{ if } m \leq 1 \leq M, \\ f_{\nu} (M) \text{ if } 1 < M \\ \leq \nu (1 - \nu) \times \begin{cases} (m - 1) \ln m \text{ if } M < 1, \\ \max \{ (m - 1) \ln m, (M - 1) \ln M \} \text{ if } m \leq 1 \leq M, \\ (M - 1) \ln M \text{ if } 1 < M, \end{cases} \quad (5.4)$$
and

\[ \Delta_\nu(m, M) \leq \begin{cases} (\exp[4\nu(1-\nu)(K(m) - 1)] - 1)m^\nu & \text{if } M < 1, \\ \max\{(\exp[4\nu(1-\nu)(K(m) - 1)] - 1)m^\nu, \exp[4\nu(1-\nu)(K(M) - 1)] - 1)M^\nu\} & \text{if } m \leq 1 \leq M, \\ \exp[4\nu(1-\nu)(K(M) - 1)] - 1)M^\nu & \text{if } 1 < m. \end{cases} \] (5.5)

In [2] we also obtained the following refinements and reverses of Young’s inequality

\[ \frac{1}{2} \nu(1-\nu)(\ln a - \ln b)^2 \min\{a, b\} \leq (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq \frac{1}{2} \nu(1-\nu)(\ln a - \ln b)^2 \max\{a, b\} \] (5.6)

and

\[ \exp\left[\frac{1}{2} \nu(1-\nu)\left(1 - \frac{\min\{a, b\}}{\max\{a, b\}}\right)^2\right] \leq (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq \exp\left[\frac{1}{2} \nu(1-\nu)\left(\frac{\max\{a, b\}}{\min\{a, b\}} - 1\right)^2\right] \] (5.7)

for any \( a, b > 0 \) and \( \nu \in [0, 1] \).

The inequality (5.7) is equivalent to

\[ \left(\exp\left[\frac{1}{2} \nu(1-\nu)\left(1 - \frac{\min\{a, b\}}{\max\{a, b\}}\right)^2\right] - 1\right)a^{1-\nu}b^\nu \leq (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq \left(\exp\left[\frac{1}{2} \nu(1-\nu)\left(\frac{\max\{a, b\}}{\min\{a, b\}} - 1\right)^2\right] - 1\right)a^{1-\nu}b^\nu \] (5.8)

for any \( a, b > 0 \) and \( \nu \in [0, 1] \).

Therefore, by (5.6) and (5.8) we have the upper bounds

\[ \Delta_\nu(m, M) \leq \frac{1}{2} \nu(1-\nu) \max\left\{ (\ln m)^2, (\ln M)^2 M \right\} \text{ if } m \leq 1 \leq M, \] (5.9)

\[ (\ln M)^2 M \text{ if } 1 < m \]
\[ \Delta_\nu(m, M) \leq \begin{cases} 
\left( \exp \left[ \frac{1}{2} \nu \left( 1 - \nu \right) \left( \frac{1}{m} - 1 \right)^2 \right] - 1 \right) m^\nu & \text{if } M < 1, \\
\max \left\{ \left( \exp \left[ \frac{1}{2} \nu \left( 1 - \nu \right) \left( \frac{1}{m} - 1 \right)^2 \right] - 1 \right) m^\nu, \\
\left( \exp \left[ \frac{1}{2} \nu \left( 1 - \nu \right) (M - 1)^2 \right] - 1 \right) M^\nu \right\} & \text{if } m \leq 1 \leq M, \\
\left( \exp \left[ \frac{1}{2} \nu \left( 1 - \nu \right) (M - 1)^2 \right] - 1 \right) M^\nu & \text{if } 1 < m. 
\end{cases} \tag{5.10} \]

From (2.3), (5.6) and (5.8) we have the lower bounds

\[ \delta_\nu(m, M) := \begin{cases} f_\nu(M) & \text{if } M < 1, \\
0 & \text{if } m \leq 1 \leq M, \\
f_\nu(m) & \text{if } 1 < m. 
\end{cases} \tag{5.11} \]

and

\[ \delta_\nu(m, M) \geq \frac{1}{2} \nu (1 - \nu) \times \begin{cases} (\ln M)^2 M & \text{if } M < 1, \\
\max \{ (m - 1) \ln m, (M - 1) \ln M \} & \text{if } m \leq 1 \leq M, \\
(\ln m)^2 & \text{if } 1 < m. 
\end{cases} \tag{5.12} \]

Assume that \( A, B \) are positive invertible operators and the constants \( M > m > 0 \) are such that \( mA \leq B \leq MA \). If we use the second inequality in (2.5), then we have the following upper bounds for the difference \( A \nabla_\nu B - A^\dagger_\nu B \):

\[ A \nabla_\nu B - A^\dagger_\nu B \leq \nu (1 - \nu) \times \begin{cases} [(m - 1) \ln m] A & \text{if } M < 1, \\
\max \{ (m - 1) \ln m, (M - 1) \ln M \} A & \text{if } m \leq 1 \leq M, \\
[(M - 1) \ln M] A & \text{if } 1 < m, 
\end{cases} \tag{5.13} \]
\[
A \nabla \nu B - A^\# \nu B = \left\{
\begin{align*}
\exp[4\nu (1 - \nu) (K(m) - 1)] - 1) Am^\nu & \quad \text{if } M < 1, \\
\max \left\{ \exp[4\nu (1 - \nu) (K(m) - 1)] - 1 \right\} m^\nu, \\
\exp[4\nu (1 - \nu) (K(M) - 1)] - 1) M^\nu \right\} A & \quad \text{if } m \leq 1 \leq M, \\
\exp[4\nu (1 - \nu) (K(M) - 1)] - 1) AM^\nu & \quad \text{if } 1 < m,
\end{align*}\right.
\]

\[
A \nabla \nu B - A^\# \nu B \leq \frac{1}{2} \nu (1 - \nu) \left\{
\begin{align*}
\max \left\{ \left(\ln m\right)^2, \left(\ln M\right)^2 M \right\} A & \quad \text{if } m \leq 1 \leq M, \\
\left(\ln m\right)^2 A & \quad \text{if } M < 1, \\
0 & \quad \text{if } 1 < m
\end{align*}\right.
\]

and

\[
A \nabla \nu B - A^\# \nu B \leq \frac{1}{2} \nu (1 - \nu) \left\{
\begin{align*}
\exp \left[ \frac{1}{2} \nu \left( \frac{1}{m} - 1 \right)^2 \right] - 1 \right) Am^\nu & \quad \text{if } M < 1, \\
\max \left\{ \exp \left[ \frac{1}{2} \nu \left( \frac{1}{m} - 1 \right)^2 \right] - 1 \right\} m^\nu, \\
\exp \left[ \frac{1}{2} \nu \left( \frac{1}{m} - 1 \right)^2 \right] - 1 \right) M^\nu \right\} A & \quad \text{if } m \leq 1 \leq M, \\
\exp \left[ \frac{1}{2} \nu \left( \frac{1}{m} - 1 \right)^2 \right] - 1 \right) AM^\nu & \quad \text{if } 1 < m.
\end{align*}\right.
\]

If we use the first inequality in (2.5), then we have the following lower bounds for the difference \(A \nabla \nu B - A^\# \nu B\):

\[
A \nabla \nu B - A^\# \nu B \geq \frac{1}{2} \nu (1 - \nu) \left\{
\begin{align*}
\left(\ln M\right)^2 M & \quad \text{if } M < 1, \\
0 & \quad \text{if } m \leq 1 \leq M, \\
\left(\ln m\right)^2 A & \quad \text{if } 1 < m
\end{align*}\right.
\]
and

\[ A^{\nabla}vB - A^{\#}vB \]

\[
\begin{cases}
\left( \exp \left[ \frac{1}{2} \nu \left( 1 - \nu \right) \left( 1 - M \right)^2 \right] - 1 \right) AM^\nu \text{ if } M < 1, \\
0 \text{ if } m \leq 1 \leq M, \\
\left( \exp \left[ \frac{1}{2} \nu \left( 1 - \nu \right) \left( 1 - \frac{1}{m} \right)^2 \right] - 1 \right) Am^\nu \text{ if } 1 < m.
\end{cases}
\]

The interested reader may state other inequalities by using Theorems 4-7, however the details are not presented here.

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