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ON THE COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF EXTENDED NEGATIVELY DEPENDENT RANDOM VARIABLES

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Abstract. The authors investigate the complete convergence for weighted sums of extended negatively dependent (END) random variables. The main results obtained in the paper extend and improve the corresponding result of Zarei and Jabbari [Zarei, H., Jabbari, H., 2011. Complete convergence of weighted sums under negative dependence. Stat. Papers, 52, 413-418].

1. Introduction

The following concept of negatively orthant dependent (NOD) random variables was introduced by Ebrahimi and Ghosh (1981).

DEFINITION 1.1. The random variables $X_1,\cdots,X_k$ are said to be negatively upper orthant dependent (NUOD) if for all real $x_1,\cdots,x_k$,

$$P(X_i > x_i, i=1,2,\cdots,k) \leqslant \prod_{i=1}^{k} P(X_i > x_i),$$

and negatively lower orthant dependent (NLOD) if

$$P(X_i \leqslant x_i, i=1,2,\cdots,k) \leqslant \prod_{i=1}^{k} P(X_i \leqslant x_i).$$

Random variables $X_1,\cdots,X_k$ are said to be negatively orthant dependent (NOD) if they are both NUOD and NLOD.

Liu (2009) extended the above NOD dependent structure. She introduced a new dependent concept of extended negatively dependent (END) random variables.

DEFINITION 1.2. The random variables $\{X_i, i \geqslant 1\}$ are said to be END if for each $n=1,2,\cdots$ and all $x_1,\cdots,x_n$, there exists a constant $M>0$ such that both

$$P(X_i \leqslant x_i, i=1,2,\cdots,n) \leqslant M\prod_{i=1}^{n} P(X_i \leqslant x_i)$$

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and

$$P(X_i > x_i, i = 1, 2, \cdots, n) \leqslant M \prod_{i=1}^{n} P(X_i > x_i),$$

hold.

Obviously, the notion of NOD random variables is the special case $M = 1$ for the notion of END random variables. As stated in Liu (2009), the END structure is more wide than the NOD structure, because it includes not only some negative dependence structures but also some positive ones. On the other hand, Joag-Dev and Proschan (1983) mentioned that the negatively associated (NA) dependent structure must be NOD, but NOD is not necessarily NA. Therefore, we know that NA random variables are END random variables. Hence it is very interesting to investigate the limit theory of this wider END random variables.

As we know, since the concept of NOD random variables was presented by Ebrahimi and Ghosh (1981), the limit theorems of NOD random variables have been discussed by many researchers. Taylor et al. (2002) discussed the strong law of large numbers for arrays of rowwise NOD random variables, Volodin (2002) studied the Kolmogorov exponential inequality for NOD random variables, Amini and Bozorgnia (2003), Volodin et al. (2006), Gan and Chen (2008), Wu (2010), Wu and Zhu (2010), Qiu et al. (2011) studied the complete convergence for NOD random variables, Mi-Hwa Ko et al. (2005, 2006), Wang et al. (2011), Wu et al. (2013) investigated some strong limit theorems for sequences of NOD random variables.

On the other hand, as far as we know, some scholars also studied the limiting behaviour for sequences or arrays of END random variables. The authors can refer the readers to Chen et al. (2010), Qiu et al. (2013), Wu and Guan (2012), Wang and Wang (2013), Zhang (2014), Wu et al. (2015) and Shen (2017).

The following concept of the complete convergence was introduced by Hsu and Robbins (1947). A sequence of random variables $\{U_n, n \geqslant 1\}$ is said to converge completely to a constant θ if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \varepsilon) < \infty.$$

By the Borel-Cantelli lemma, it is clearly that the above complete convergence result implies that $U_n \to \theta$ almost surely. Hence, the complete convergence is an important tool in studying some strong limit convergence of sums of random variables.

Zarei and Jabbari (2011) studied the complete convergence for weighted sums of NOD random variables and presented the following result.

THEOREM A. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of NOD and identically distributed random variables with* $EX_1 = 0$*, and let* $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ *be an array of real numbers satisfying*

$$A_n = \sum_{k=1}^{n} a_{nk}^2 \leqslant Cn^{-\alpha}, \ |a_{nk}| \leqslant CA_n \tag{1.1}$$

for some $0 < C < \infty$ *and* $0 < \alpha < 1$. *If*

$$E|X_1|^{2/\alpha} < \infty, \tag{1.2}$$

then

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{k=1}^{n} a_{nk}X_k\right| \geqslant \varepsilon\right) < \infty \quad \textit{for all } \varepsilon > 0. \tag{1.3}$$

In this work, the authors will investigate the complete convergence for weighted sums of END random variables. We shall extend and improve Theorem A by considering END instead of NOD, the maximal partial sums instead of the common partial sums, and obtaining some stronger conclusions under the same or weaker conditions.

Throughout the current paper, C will be used to stand for various positive constants, which may differ from one place to another. The symbol $I(A)$ will be used to indicate the indicator function of A.

2. Main results

To prove our main results, we need the following important technical lemmas.

LEMMA 2.1. (Liu, 2009) *If random variables* $\{X_n, n \geqslant 1\}$ *are END, then* $\{f_n(X_n), n \geqslant 1\}$ *are still END, where* $\{f_n(\cdot), n \geqslant 1\}$ *are either all monotone increasing or all monotone decreasing.*

LEMMA 2.2. (Shen, 2011) *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of END random variables with* $EX_n = 0$ *and* $E|X_n|^p < \infty$ *for some* $p \geqslant 2$ *and any* $n \geqslant 1$. *Then there exist positive constants* C *depending only on* p *such that for any* $n \geqslant 1$,

$$E\left|\sum_{k=1}^{n} X_k\right|^p \leqslant C\left\{\sum_{k=1}^{n} E|X_k|^p + \left(\sum_{k=1}^{n} EX_k^2\right)^{p/2}\right\}.$$

By a similar way of Stout (1974, Theorem 2.3.1), Zhang (2014) obtained the following lemma, which is very important in the proof of our main results.

LEMMA 2.3. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of END random variables with* $EX_n = 0$ *and* $E|X_n|^p < \infty$ *for some* $p \geqslant 2$ *and any* $n \geqslant 1$. *Then there exist positive constants* C *depending only on* p *such that for any* $n \geqslant 1$,

$$E\left\{\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} X_k\right|\right\}^p \leqslant C\log^p n\left\{\sum_{k=1}^{n} E|X_k|^p + \left(\sum_{k=1}^{n} EX_k^2\right)^{p/2}\right\}.$$

Now we state our main results. The proofs will be presented in the next section.

THEOREM 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of END and identically distributed random variables with $EX_1 = 0$, and let $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of real numbers satisfying*

$$\sum_{k=1}^{n} a_{nk}^2 = O(n^{-\alpha}) \tag{2.1}$$

and

$$\max_{1 \leqslant k \leqslant n} |a_{nk}| = O(n^{-\alpha}) \tag{2.2}$$

for some $1/p \leqslant \alpha < 1$ and $p \geqslant 2$. If

$$E|X_1|^p < \infty, \tag{2.3}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\max_{1 \leqslant j \leqslant n} \left| \sum_{k=1}^{j} a_{nk} X_k \right| \geqslant \varepsilon \right) < \infty \quad \text{for all } \varepsilon > 0. \tag{2.4}$$

Letting $p = 2/\alpha$ in Theorem 2.1, we obtain the following conclusion.

COROLLARY 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of END and identically distributed random variables with $EX_1 = 0$, and let $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of real numbers satisfying (2.1) and (2.2) for some $0 < \alpha < 1$. If (1.2) holds, then*

$$\sum_{n=1}^{\infty} P\left(\max_{1 \leqslant j \leqslant n} \left| \sum_{k=1}^{j} a_{nk} X_k \right| \geqslant \varepsilon \right) < \infty \quad \text{for all } \varepsilon > 0. \tag{2.5}$$

REMARK 2.1. Since NOD implies END and (2.5) is more stronger than (1.3), Corollary 2.1 extends and improves Theorem A.

For the case $1/p \leqslant \alpha < 2/p$, we can remove the condition (2.2) of Theorem 2.1 and obtain the following theorem.

THEOREM 2.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of END and identically distributed random variables with $EX_1 = 0$, and let $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of real numbers satisfying (2.1) for some $1/p \leqslant \alpha < 2/p$ and $p \geqslant 2$. Then (2.3) implies (2.4).*

Letting $p = 1/\alpha$ in Theorem 2.2, we obtain the following strong convergence result.

COROLLARY 2.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of END and identically distributed random variables with $EX_1 = 0$, and let $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of real numbers satisfying (2.1) for some $0 < \alpha \leqslant 1/2$. If*

$$E|X_1|^{1/\alpha} < \infty, \tag{2.6}$$

then

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leqslant j \leqslant n} \left| \sum_{k=1}^{j} a_{nk} X_k \right| \geqslant \varepsilon \right) < \infty \quad \text{for all } \varepsilon > 0. \tag{2.7}$$

Moreover, we have

$$\sum_{k=1}^{n} a_{nk}X_k \to 0 \qquad a.s. \tag{2.8}$$

REMARK 2.2. By Theorem A and the Borel-Cantelli lemma, one get directly (2.8). However, the moment conditon (1.2) is stronger than (2.6). In addition, we remove the condition $|a_{nk}| \leqslant Cn^{-\alpha}$ of (1.1) in Theorem A. Therefore, to some extent, Corollary 2.2 improves Theorem A.

3. The proofs

Proof of Theorem 2.1. If $1 \leqslant \alpha p \leqslant 2$, we take $\beta > \max\{\alpha, \frac{1-\alpha}{p-1}\}$. If $\alpha p > 2$, we take $\max\{\alpha, \frac{1-\alpha}{p-1}\} < \beta < \alpha + \frac{\alpha(1-\alpha)}{\alpha p-2}$. Fixed $n \geqslant 1$, let

$$\begin{aligned} Y_{nk} &= -n^{\beta} I(X_k < -n^{\beta}) + X_k I(|X_k| \leqslant n^{\beta}) + n^{\beta} I(X_k > n^{\beta}), \\ Z_{nk} &= (X_k + n^{\beta}) I(X_k < -n^{\beta}) + (X_k - n^{\beta}) I(X_k > n^{\beta}). \end{aligned}$$

Then $Y_{nk} + Z_{nk} = X_k$, and $\{Y_{nk}, k \geqslant 1, n \geqslant 1\}$ and $\{Z_{nk}, k \geqslant 1, n \geqslant 1\}$ are both END by Lemma 2.1. Then

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| \geqslant \varepsilon\right) \\ \leqslant\ &\sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} P\big(|X_k| > n^{\beta}\big) \ + \ \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}Y_{nk}\right| \geqslant \varepsilon\right) \\ =:\ & I_1 + I_2. \end{aligned}$$

For I_1, from (2.3) and $\beta > \alpha$, we can gat

$$\begin{aligned} I_1 &\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-1} P\big(|X_1| > n^{\beta}\big) \\ &\leqslant C \sum_{n=1}^{\infty} n^{-1-(\beta-\alpha)p} E|X_1|^p I\big(|X_1| > n^{\beta}\big) < \infty. \end{aligned}$$

Next we prove $I_2 < \infty$. We first prove that

$$\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}EY_{nk}\right| \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

By (2.1) and Hölder inequality, we obtain

$$\sum_{k=1}^{n} |a_{nk}| \leqslant \left(\sum_{k=1}^{n} a_{nk}^2\right)^{1/2} \left(\sum_{k=1}^{n} 1\right)^{1/2} \leqslant Cn^{(1-\alpha)/2}. \tag{3.2}$$

Then from $EX_1 = 0$, $|Z_{nk}| \leqslant |X_k| I(|X_k| > n^{\beta})$ and $\beta > \frac{1-\alpha}{p-1}$, we have

$$\begin{aligned}\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}EY_{nk}\right| &= \max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}EZ_{nk}\right| \\ &\leqslant \sum_{k=1}^{n} |a_{nk}|E|X_k|I(|X_k| > n^{\beta}) \\ &\leqslant n^{-\beta(p-1)}\sum_{k=1}^{n} |a_{nk}|E|X_k|^p I(|X_k| > n^{\beta}) \\ &\leqslant Cn^{-\beta(p-1)+(1-\alpha)/2}E|X_1|^p I(|X_1| > n^{\beta}) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Therefore, while n is sufficiently large, we obtain $\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}EY_{nk}\right| \leqslant \varepsilon/2$. Hence, for $\max\{p, \frac{2(\alpha p-1)}{\alpha}\} < q < p + \frac{1-\alpha}{\beta-\alpha}$, by the Markov inequality and Lemma 2.3, we have

$$\begin{aligned}I_2 &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}(Y_{nk}-EY_{nk})\right| \geqslant \varepsilon/2\right) \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}(Y_{nk}-EY_{nk})\right|\right)^q \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}\log^q n\sum_{k=1}^{n} |a_{nk}|^q E|Y_{nk}-EY_{nk}|^q \\ &\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2}\log^q n\left(\sum_{k=1}^{n} a_{nk}^2 E(Y_{nk}-EY_{nk})^2\right)^{q/2} \\ &=: I_3 + I_4.\end{aligned}$$

We first show $I_3 < \infty$. Note that

$$\begin{aligned}I_3 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}\log^q n\sum_{k=1}^{n} |a_{nk}|^q E|Y_{nk}|^q \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-2}\log^q n\sum_{k=1}^{n} |a_{nk}|^q E|X_k|^q I(|X_k| \leqslant n^{\beta}) \\ &\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2+\beta q}\log^q n\sum_{k=1}^{n} |a_{nk}|^q P(|X_k| > n^{\beta}) \\ &=: I_3' + I_3''.\end{aligned}$$

We have by (2.1), (2.2), (2.3) and $\alpha p - 1 + \beta q - \beta p - \alpha q + \alpha < 0$ that

$$\begin{aligned}I_3'' &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\beta q-\beta p}\log^q n\sum_{k=1}^{n} |a_{nk}|^q E|X_k|^p \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\beta q-\beta p}\log^q n\left(\max_{1\leqslant k\leqslant n}|a_{nk}|\right)^{q-2}\sum_{k=1}^{n} a_{nk}^2\end{aligned}$$

$$\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\beta q-\beta p-\alpha q+\alpha}\log^q n<\infty.$$

By $q>p$ and similar argument as in the proof of $I_3''<\infty$, we obtain

$$\begin{aligned}
I_3' &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\beta q-\beta p}\log^q n\sum_{k=1}^{n}|a_{nk}|^q E|X_k|^p I(|X_k|\leqslant n^{\beta})\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\beta q-\beta p-\alpha q+\alpha}\log^q n<\infty.
\end{aligned}$$

Then we prove that $I_4<\infty$. We can find that

$$\begin{aligned}
I_4 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}\log^q n\left(\sum_{k=1}^{n} a_{nk}^2 EY_{nk}^2\right)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}\log^q n\left(\sum_{k=1}^{n} a_{nk}^2 EX_k^2 I(|X_k|\leqslant n^{\beta})\right)^{q/2}\\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2}\log^q n\left(\sum_{k=1}^{n} a_{nk}^2 n^{2\beta}P(|X_k|>n^{\beta})\right)^{q/2}\\
&=: I_4'+I_4''.
\end{aligned}$$

By (2.1), (2.3) and $q>2(\alpha p-1)/\alpha$, we have

$$\begin{aligned}
I_4' &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}\log^q n\left(\sum_{k=1}^{n} a_{nk}^2\right)^{q/2}\left(EX_1^2 I(|X_1|\leqslant n^{\beta})\right)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q/2}\log^q n<\infty
\end{aligned}$$

and

$$\begin{aligned}
I_4'' &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}\log^q n\left(\sum_{k=1}^{n} a_{nk}^2 n^{2\beta}P(|X_1|>n^{\beta})\right)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-(p-2)\beta q/2}\log^q n\left(\sum_{k=1}^{n} a_{nk}^2\right)^{q/2}\left(E|X_1|^p I(|X_1|\leqslant n^{\beta})\right)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q/2-(p-2)\beta q/2}\log^q n<\infty.
\end{aligned}$$

The proof is completed. □

Proof of Theorem 2.2. We take $\beta > \max\{\alpha, \frac{1-\alpha}{p-1}\}$ and $\max\{p, \frac{2(\alpha p-1)}{\alpha}\} < q < \frac{(\beta-\alpha)p+1}{\beta-\alpha/2}$. Following the notations and the methods of the proof in Theorem 2.1, $I_1 < \infty$, (3.1) and $I_4 < \infty$ hold. So we only need to show $I_3' < \infty$ and $I_3'' < \infty$.

Elementary Jensen's inequality and (2.1) imply that for any $0 < 2 < q$,

$$\sum_{k=1}^{n} |a_{nk}|^q \leqslant \left(\sum_{k=1}^{n} a_{nk}^2\right)^{q/2} \leqslant Cn^{-\frac{\alpha q}{2}}. \tag{3.3}$$

Then by $q > p$, (2.3), (3.3) and $(\beta-\alpha/2)q - (\beta-\alpha)p < 1$, we have

$$\begin{aligned} I_3' &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\beta q-\beta p}\log^q n \sum_{k=1}^{n} |a_{nk}|^q E|X_k|^p I(|X_k| \leqslant n^{\beta}) \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\beta q-\beta p-\alpha q/2}\log^q n < \infty \end{aligned}$$

and

$$\begin{aligned} I_3'' &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\beta q-\beta p}\log^q n \sum_{k=1}^{n} |a_{nk}|^q E|X_k|^p \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\beta q-\beta p-\alpha q/2}\log^q n < \infty. \end{aligned}$$

The proof is completed. □

Proof of Corollary 2.2. Take $p = 1/\alpha$ in Theorem 2.2, one can get directly (2.7). Then we only need to show (2.8). It follows from (2.7) that

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| \geqslant \varepsilon\right) \\ &= \sum_{m=0}^{\infty}\sum_{n=2^m}^{2^{m+1}-1} n^{-1} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| \geqslant \varepsilon\right) \\ &\geqslant \frac{1}{2}\sum_{m=1}^{\infty} P\left(\max_{1\leqslant j\leqslant 2^m}\left|\sum_{k=1}^{j} a_{nk}X_k\right| \geqslant \varepsilon\right). \end{aligned}$$

By the Borel-Cantelli Lemma, we have

$$\lim_{m\to\infty}\max_{1\leqslant j\leqslant 2^m}\left|\sum_{k=1}^{j} a_{nk}X_k\right| = 0 \qquad \text{a.s.} \tag{3.4}$$

For all given positive integers n, there exists a positive integer m_0 such that $2^{m_0-1} \leqslant n < 2^{m_0}$. We obtain by (3.4) that

$$\left|\sum_{k=1}^{n} a_{nk}X_k\right| \leqslant \max_{2^{m_0-1}\leqslant n<2^{m_0}}\left|\sum_{k=1}^{n} a_{nk}X_k\right|$$

$$\leqslant \max_{1 \leqslant j < 2^{m_0}} \left| \sum_{k=1}^{j} a_{nk} X_k \right| \to 0 \text{ a.s.} \quad \text{ as } m_0 \to \infty.$$

The proof is completed. □

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