Journal of Mathematical Inequalities
Volume 13, Number 2 (2019), 301–314 doi:10.7153/jmi-2019-13-22

THE ESTIMATE OF THE DIFFERENCE OF INITIAL SUCCESSIVE COEFFICIENTS OF UNIVALENT FUNCTIONS

ZHIGANG PENG AND MILUTIN OBRADOVIĆ

(*Communicated by S. Hencl*)

Abstract. Let $\mathscr{A}$ denote the family of all functions that are analytic in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ and satisfy $f(0) = 0 = f'(0) - 1$. Let S be the set of all functions $f \in \mathscr{A}$ that are univalent in $\mathbb{D}$. In this paper the sharp upper bounds of $|a_3 - a_2|$ and $|a_4 - a_3|$ for the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ being in several subclasses of S are presented.

1. Introduction

Let $\mathscr{A}$ denote the family of all functions that are analytic in the unit disk $\mathbb{D} :=\{z : |z| < 1\}$ and satisfy $f(0) = 0 = f'(0) - 1$. Let S be the set of all functions $f \in \mathscr{A}$ that are univalent in $\mathbb{D}$. Let S^* and K denote the subclasses of S consisting of starlike functions and convex functions, respectively. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ then $|a_n| \leqslant n$ and strict inequality holds for all n unless f is the Koebe function or one of its rotation. This is the famous conjecture of Bieberbach, first proposed by Bieberbach[2] in 1916 and finally proved by de Branges[1] in 1984. After Bieberbach conjecture was put forward, another coefficient problem which has attracted considerable attention is to estimate $||a_{n+1}| - |a_n||$, the difference of the moduli of successive coefficients of a function $f \in S$. Indeed, Hayman[4] proved $||a_{n+1}| - |a_n|| \leqslant A$ for $f \in S$, where $A \geqslant 1$ is an absolute constant. Pommerenke[17] conjecture that $||a_{n+1}| - |a_n|| \leqslant 1$ for $f \in S^*$ which was proved by Leung[6]. Z. Ye also estimated the difference of the moduli of successive coefficients of certain univalent functions[21, 22]. In addition to studying the bounds of $||a_{n+1}| - |a_n||$, some scholars are also interested in studying the bounds of $|a_{n+1} - a_n|$. Robertson[18] proved that $|a_{n+1} - a_n| \leqslant \frac{2n+1}{3}|a_2 - 1|$ for all $f \in K$. Recently, M. Li and T. Sugawa[7] estimated the bounds of $|a_3 - a_2|$ and $|a_4 - a_3|$ for $f \in K(p)$, where $K(p) = \{f : f \in K, f''(0) = p, 0 \leqslant p \leqslant 2\}$.

In the present paper the upper bounds of $|a_3 - a_2|$ and $|a_4 - a_3|$ for f belonging to various subclasses of S are studied.

Mathematics subject classification (2010): 30C45.
Keywords and phrases: Univalent function, successive coefficient, upper bounds, estimate.
The research of the corresponding author was supported by the Key Laboratory of Applied Mathematics in Hubei Province, China. The work of the second author was supported by MNZZS Grant, No. ON174017, Serbia.

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2. Preliminaries

Let $\mathscr{P}$ denote the class of all functions $p(z)$ analytic and having positive real part on $\mathbb{D}$, with the form

$$P(z)=1+\sum_{n=1}^{\infty} p_n z^n.$$

It is known that $|p_n|\leqslant 2$ for $p\in\mathscr{P}$ and $n=1,2,\cdots$[2].

In the course of the subsequent discussion, we need to make use of the following lemmas.

LEMMA 1. *Let* $-2\leqslant p_1\leqslant 2$ *and* $p_2,p_3\in\mathbb{C}$. *There exists a function* $P\in\mathscr{P}$ *with*

$$P(z)=1+p_1z+p_2z^2+p_3z^3+\ldots \tag{1}$$

if and only if

$$2p_2=p_1^2+x(4-p_1^2) \tag{2}$$

and

$$4p_3=p_1^3+2(4-p_1^2)p_1x-(4-p_1^2)p_1x^2+2(4-p_1^2)(1-|x|^2)y \tag{3}$$

for some $x,y\in\mathbb{C}$ *with* $|x|\leqslant 1$ *and* $|y|\leqslant 1$.

Lemma 1 is due to Libera and Złotkiewicz[8], one can also find it in [7].

LEMMA 2. *For given real numbers* a,b,c, *let*

$$Y(a,b,c)=max_{z\in\overline{\mathbb{D}}}\left(|a+bz+cz^2|+1-|z|^2\right). \tag{4}$$

If $a\geqslant 0$ *and* $c\geqslant 0$, *then*

$$Y(a,b,c)=\begin{cases} a+|b|+c, & |b|\geqslant 2(1-c)\\ 1+a+\frac{b^2}{4(1-c)}, & |b|<2(1-c)\end{cases}$$

The maximum in the definition of $Y(a,b,c)$ *is attained at* $z=\pm 1$ *in the first case according as* $b=\pm|b|$.

Lemma 2 is due to R. Ohno and T. Sugawa[14], one can also find it in [7].

3. Main Results

Let $\mathscr{G}$ denote the class functions f from $\mathscr{A}$ satisfying the conditions

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)<\frac{3}{2},\quad z\in\mathbb{D},$$

It is known that $\mathscr{G}\subset S$ and $|\frac{1}{2}f''(0)|=|a_2|\leqslant\frac{1}{2}$ for $f=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{G}$ [19, 15, 10, 5]. Now, let

$$\mathscr{G}(p)=\{f\in\mathscr{G},\ f''(0)=p\},$$

where p is a given number satisfying $-1\leqslant p\leqslant 1$.

THEOREM 1. *Let* $0 \leqslant p \leqslant 1$ *and let* $f(z) = z + a_2z^2 + a_3z^3 + \dots$ *be in the class* $\mathscr{G}(p)$. *Then the next sharp inequalities hold:*

$$|a_3 - a_2| \leqslant \frac{1}{6}(-p^2 + 3p + 1) \tag{5}$$

$$|a_4 - a_3| \leqslant \frac{1}{24}(1 - p^2)(3p + 4) \tag{6}$$

Proof. Since

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2}, \quad z \in \mathbb{D},$$

it is follows that

$$\operatorname{Re}\left(1 - 2\frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

We can put

$$1 - 2\frac{zf''(z)}{f'(z)} = P(z),$$

where P is given by (1) and satisfy $\operatorname{Re}P(z) > 0, z \in \mathbb{D}$. From the last relation we have

$$f'(z) - 2zf''(z) = P(z)f'(z). \tag{7}$$

By using the Taylor representations for the functions f and P and comparing the coefficients of z^n $(n = 1,2,3)$ in both sides of (7), we obtain

$$a_2 = -\frac{p_1}{4},\ a_3 = -\frac{1}{12}p_2 - \frac{1}{6}a_2p_1,\ a_4 = -\frac{1}{24}p_3 - \frac{1}{8}a_3p_1 - \frac{1}{12}a_2p_2. \tag{8}$$

Since, $2a_2 = f''(0) = p$, we have $p_1 = -4a_2 = -2p$ by (8). In view of these facts and Lemma 1, we have

$$\begin{aligned} p_2 &= 2(p^2 + (1-p^2)x), \\ p_3 &= -2p^3 - 4(1-p^2)px + 2(1-p^2)px^2 + 2(1-p^2)(1-|x|^2)y, \end{aligned} \tag{9}$$

where $x, y \in \mathbb{C}$ with $|x| \leqslant 1$ and $|y| \leqslant 1$.
From the relations (8) and (9) and by some simple calculations, we have

$$\begin{aligned} |a_3 - a_2| &= \left|-\frac{1}{6}(1-p^2)x - \frac{1}{2}p\right| \\ &\leqslant \frac{1}{6}(1-p^2) + \frac{1}{2}p = \frac{1}{6}(-p^2 + 3p + 1), \end{aligned}$$

where equality occurs if $x = 1$. Also, we have

$$\begin{aligned} |a_4 - a_3| &= \left|-\frac{1}{12}(1-p^2)(1-|x|^2)y + [\frac{1}{24}(1-p^2)p + \frac{1}{6}(1-p^2)]x - \frac{1}{12}(1-p^2)px^2\right| \\ &\leqslant \frac{1}{12}(1-p^2)\left(1 - |x|^2 + |-\frac{1}{2}(p+4)x + px^2|\right) \\ &\leqslant \frac{1}{12}(1-p^2)Y(a,b,c), \end{aligned}$$

where $Y(a,b,c)$ is given in (4) and with

$$a=0,\, b=-\frac{1}{2}(p+4),\, c=p.$$

Since $0\leqslant p\leqslant 1$, we have that $|b|\geqslant 2(1-c)$. Then by using Lemma 2 we get

$$Y(a,b,c)=\frac{3}{2}p+2.$$

Therefore

$$|a_4-a_3|\leqslant \frac{1}{12}(1-p^2)Y(a,b,c)=\frac{1}{24}(1-p^2)(3p+4)$$

The equality holds for $x=-1$.

If we denote by

$$\mathscr{G}^+=\bigcup_{0\leqslant p\leqslant 1}\mathscr{G}_p=\{f:f\in\mathscr{G},f''(0)\geqslant 0\},$$

then by using (5) and (6) and a simple calculation, we easily get

$$sup_{f\in\mathscr{G}^+}|a_3(f)-a_2(f)|=\frac{1}{2}$$

and

$$sup_{f\in\mathscr{G}^+}|a_4(f)-a_3(f)|=\frac{260+43\sqrt{43}}{2916}=0.1858...$$

where $a_n(f)\,(n=2,3,4)$ are the Taylor coefficients of $f(z)$. □

As usual, let $\mathscr{U}$ denote the set of all $f\in\mathscr{A}$ satisfying the condition

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<1$$

for $z\in\mathbb{D}$. It is known that $\mathscr{U}\subset S$[16]. In recent years, the properties of $\mathscr{U}$ were studied in detail[11, 12, 13, 3]. Let

$$\mathscr{U}_p=\{f\in\mathscr{U},f''(0)=p\},$$

where p is a given number with $-4\leqslant p\leqslant 4$ (Noticing that for $f\in\mathscr{U}$, we have $|\frac{1}{2}f''(0)|=|a_2(f)|\leqslant 2$).

THEOREM 2. *Let $0\leqslant p\leqslant 4$ and let $f(z)=z+a_2z^2+a_3z^3+...$ be in the class $\mathscr{U}_p$. Then we have the following sharp inequalities :*

$$|a_3-a_2|\leqslant\begin{cases}1+\frac{p}{4}(2-p),\ 0\leqslant p\leqslant 2\\ 1+\frac{p}{4}(p-2),\ 2\leqslant p\leqslant 4.\end{cases}\tag{10}$$

$$|a_4-a_3|\leqslant\begin{cases}\frac{1}{4}(-p^3+6p^2-8p+8),\ 0\leqslant p\leqslant 2\\ \frac{1}{8}(p^3-2p^2+8p-8),\ 2\leqslant p\leqslant 4.\end{cases}\tag{11}$$

Proof. If $f \in \mathcal{U}$, then

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1, |z| < 1.$$

It is equivalent to

$$\mathrm{Re}\left(2\left(\frac{f(z)}{z}\right)^2 \frac{1}{f'(z)} - 1\right) > 0, \quad z \in \mathbb{D}.$$

So, we can put

$$2\left(\frac{f(z)}{z}\right)^2 \frac{1}{f'(z)} - 1 = P(z),$$

where P is given by (1) and satisfy $\mathrm{Re}P(z) > 0, z \in \mathbb{D}$. From the last relation we have

$$2\left(\frac{f(z)}{z}\right)^2 - f'(z) = P(z)f'(z). \tag{12}$$

By using the relation (12) and the Taylor expansions of functions f and P, we obtain

$$p_1 = 0, a_3 = a_2^2 - \frac{1}{2}p_2, a_4 = -\frac{1}{4}p_3 - \frac{1}{2}a_2p_2 + a_2a_3. \tag{13}$$

Since $2a_2 = p$, we have $a_2 = \frac{p}{2}$. Also, since $p_1 = 0$, it follows from Lemma 1 that

$$p_2 = 2x, p_3 = 2(1-|x|^2)y \tag{14}$$

for some $x, y \in \mathbb{C}$ with $|x| \leqslant 1$ and $|y| \leqslant 1$.
By using the all previous facts, we obtain that

$$a_3 = \frac{1}{4}p^2 - x, a_4 = \frac{1}{8}p^3 - px - \frac{1}{2}(1-|x|^2)y.$$

Now, we have

$$|a_3 - a_2| = \left|-x + \frac{1}{4}p^2 - \frac{p}{2}\right| \leqslant 1 + \frac{p}{4}|p-2|,$$

or equivalently,

$$|a_3 - a_2| \leqslant \begin{cases} 1 + \frac{p}{4}(2-p), \, 0 \leqslant p \leqslant 2 \\ 1 + \frac{p}{4}(p-2), \, 2 \leqslant p \leqslant 4. \end{cases}$$

Also we have

$$\begin{aligned} |a_4 - a_3| &= \left|\frac{1}{8}p^3 - px - \frac{1}{2}(1-|x|^2)y - \frac{1}{4}p^2 + x\right| \\ &\leqslant \frac{1}{2}\left(1 - |x|^2 + \left|\frac{1}{4}p^2(p-2) + 2(1-p)x\right|\right) \\ &\leqslant \frac{1}{2}Y(a,b,c), \end{aligned}$$

where $Y(a,b,c)$ is given in (4). Since

$$\left|\frac{1}{4}p^2(p-2)+2(1-p)x\right| = \left|\frac{1}{4}p^2(2-p)+2(p-1)x\right|,$$

we can put $a=\frac{1}{4}p^2(2-p), b=2(p-1), c=0$ in case $0\leqslant p\leqslant 2$ and $a=\frac{1}{4}p^2(p-2), b=2(1-p), c=0$ in case $2\leqslant p\leqslant 4$. We have that $|b|\leqslant 2(1-c)$ in the first case and $|b|\geqslant 2(1-c)$ in the second case. By Lemma 2 we have

$$Y(a,b,c)=\begin{cases} 1+\frac{1}{4}p^2(2-p)+\frac{4(p-1)^2}{4}, \ 0\leqslant p\leqslant 2 \\ \frac{1}{4}p^2(p-2)+2(p-1), \ 2\leqslant p\leqslant 4 \end{cases}$$

and therefore

$$|a_4-a_3|\leqslant\begin{cases} \frac{1}{4}(-p^3+6p^2-8p+8), \ 0\leqslant p\leqslant 2 \\ \frac{1}{8}(p^3-2p^2+8p-8), \ 2\leqslant p\leqslant 4. \end{cases}$$

Now, let

$$\mathscr{U}^+=\bigcup_{0\leqslant p\leqslant 4}\mathscr{U}_p=\{f: f\in\mathscr{U}, f''(0)\geqslant 0\}.$$

Then, in view of (10) and (11), we easily get

$$sup_{f\in\mathscr{U}^+}|a_3(f)-a_2(f)|=3$$

and

$$sup_{f\in\mathscr{U}^+}|a_4(f)-a_3(f)|=7.$$

□

For a long time, the research on Bazilevic functions has attracted the attention of many scholars[20, 9, 23]. R.Singh[20] considered a subclass $\mathscr{B}_1(\alpha)$ of Bazilevic functions. $f\in\mathscr{B}_1(\alpha)$ if $f\in\mathscr{A}$ and

$$\operatorname{Re}\left\{\left(\frac{f(z)}{z}\right)^{\alpha-1}f'(z)\right\}>0, z\in\mathbb{D}, \alpha\geqslant 0.$$

It is well-known that $\mathscr{B}_1(\alpha)(\alpha\geqslant 0)$ is the subclass of S.
For $\alpha=1$ we have the class $\mathscr{R}$ defined by the condition

$$\operatorname{Re}\{f'(z)\}>0, z\in\mathbb{D}.$$

Further, let denote by $\mathscr{B}^{(2)}$ and $\mathscr{B}^{(3)}$ the classes given from $\mathscr{B}_1(\alpha)$ for $\alpha=2$ and $\alpha=3$, i.e. the classes of $\mathscr{A}$ satisfying the next conditions

$$\operatorname{Re}\left\{\frac{f(z)f'(z)}{z}\right\}>0, z\in\mathbb{D}$$

and

$$\operatorname{Re}\left\{\left(\frac{f(z)}{z}\right)^2 f'(z)\right\}>0, z\in\mathbb{D},$$

respectively. Also, let

$$\mathscr{R}_p = \{f \in \mathscr{R},\ f''(0) = p\},$$
$$\mathscr{B}_p^{(2)} = \{f \in \mathscr{B}^2,\ f''(0) = p\},$$
$$\mathscr{B}_p^{(3)} = \{f \in \mathscr{B}^3,\ f''(0) = p\}.$$

THEOREM 3. *Let* $0 \leqslant p \leqslant 2$ *and let* $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ *be in the class* $\mathscr{R}_p$. *Then we have the next sharp inequalities:*

$$|a_3 - a_2| \leqslant \frac{1}{6}(4 + 3p - 2p^2). \tag{15}$$

$$|a_4 - a_3| \leqslant \begin{cases} \frac{1}{18}(13 - 4p),\ 0 \leqslant p \leqslant \frac{5}{3} \\ \frac{1}{12}(-3p^3 + 4p^2 + 9p - 8),\ \frac{5}{3} \leqslant p \leqslant 2. \end{cases} \tag{16}$$

Proof. Since $f \in \mathscr{R}_p$, we can put

$$f'(z) = P(z), \tag{17}$$

where P is given by (1) with $\mathrm{Re}P(z) > 0, z \in \mathbb{D}$. By using the Taylor representations for the functions f and P and comparing the coefficients of $z^n (n = 1, 2, 3)$ in both sides of (17), we obtain

$$a_2 = \frac{1}{2}p_1,\ a_3 = \frac{1}{3}p_2,\ a_4 = \frac{1}{4}p_3. \tag{18}$$

Since $2a_2 = f''(0) = p$, it follows from (18) that $p_1 = 2a_2 = p$ and $|p| \leqslant 2$. By using Lemma 1, we have

$$\begin{aligned} p_2 &= \tfrac{1}{2}[p^2 + (4 - p^2)x], \\ p_3 &= \tfrac{1}{4}[p^3 + 2(4 - p^2)px - (4 - p^2)px^2 + 2(4 - p^2)(1 - |x|^2)y] \end{aligned} \tag{19}$$

for some $x, y \in \mathbb{C}$ with $|x| \leqslant 1$ and $|y| \leqslant 1$.
Combining (18) with (19), we obtain

$$\begin{aligned} |a_3 - a_2| &= \left|\frac{1}{6}(4 - p^2)x - \frac{p}{6}(3 - p)\right| \\ &\leqslant \frac{1}{6}(4 - p^2) + \frac{p}{6}(3 - p) \\ &= \frac{1}{6}(4 + 3p - 2p^2) \end{aligned}$$

where equality occurs if $x = -1$. Similarly, we have

$$\begin{aligned} &|a_4 - a_3| \\ &= \left|\frac{1}{16}[p^3 + 2(4 - p^2)px - (4 - p^2)px^2 + 2(4 - p^2)(1 - |x|^2)y] - \frac{1}{6}[p^2 + (4 - p^2)x]\right| \\ &\leqslant \frac{1}{8}(4 - p^2)\left[1 - |x|^2 + \left|\frac{p^2(8/3 - p)}{2(4 - p^2)} + (4/3 - p)x + \frac{p}{2}x^2\right|\right] \\ &\leqslant \frac{1}{8}(4 - p^2)Y(a, b, c), \end{aligned}$$

where $Y(a,b,c)$ is given in (4) with

$$a=\frac{p^2(8/3-p)}{2(4-p^2)}, b=4/3-p, c=\frac{1}{2}p,$$

(for $p=2$, we have directly that $|a_4-a_3|=\frac{1}{6}$).
Noticing that for $p\in[0,2]$, $|b|\leqslant 2(1-c)$ is equivalent $0\leqslant p\leqslant\frac{5}{3}$, by Lemma 2 we have

$$Y(a,b,c)=\begin{cases} 1+\frac{p^2(8/3-p)}{2(4-p^2)}+\frac{(4/3-p)^2}{4(1-p/2)}, \ 0\leqslant p\leqslant\frac{5}{3} \\ \frac{p^2(8/3-p)}{2(4-p^2)}+p-\frac{4}{3}+\frac{1}{2}p, \ \frac{5}{3}\leqslant p<2. \end{cases}$$

Hence

$$|a_4-a_3|\leqslant\begin{cases} \frac{1}{18}(13-4p), \ 0\leqslant p\leqslant\frac{5}{3} \\ \frac{1}{12}(-3p^3+4p^2+9p-8), \ \frac{5}{3}\leqslant p\leqslant 2. \end{cases}$$

If we denote by

$$\mathscr{R}^+=\bigcup_{0\leqslant p\leqslant 2}\mathscr{R}_p=\{f: f\in\mathscr{R}, f''(0)\geqslant 0\},$$

then in view of (15) and (16), we easily get

$$sup_{f\in\mathscr{R}^+}|a_3(f)-a_2(f)|=\frac{41}{48}$$

and

$$sup_{f\in\mathscr{R}^+}|a_4(f)-a_3(f)|=\frac{13}{18}.$$

□

THEOREM 4. *Let $0\leqslant p\leqslant\frac{4}{3}$ and let $f(z)=z+a_2z^2+a_3z^3+\dots$ be in the class $\mathscr{B}^2_p$. Then we have the next sharp inequalities:*

$$|a_3-a_2|\leqslant\frac{1}{16}(-7p^2+8p+8), 0\leqslant p\leqslant\frac{4}{3}. \tag{20}$$

$$|a_4-a_3|\leqslant\begin{cases} \frac{1}{2560}(-85p^3-400p^2-260p+1424), \ 0\leqslant p\leqslant\frac{6}{5} \\ \frac{1}{80}(-5p^3-10p^2-4p+40), \ \frac{6}{5}\leqslant p\leqslant\frac{4}{3}. \end{cases} \tag{21}$$

Proof. From the definition of the class $\mathscr{B}^2_p$, we can put

$$\frac{f(z)f'(z)}{z}=P(z), \tag{22}$$

where $\mathrm{Re}P(z) > 0, z \in \mathbb{D}$, and P is given by (1). By using the Taylor representations for the functions f and P and comparing the coefficients of $z^n (n = 1,2,3)$ in both sides of (22), we obtain

$$a_2 = \frac{1}{3}p_1,\ a_3 = \frac{1}{4}p_2 - \frac{1}{2}a_2^2,\ a_4 = \frac{1}{5}p_3 - a_2a_3. \tag{23}$$

Since $2a_2 = f''(0) = p$ and $|p_1| \leqslant 2$, it follows from (23) that $p_1 = 3a_2 = \frac{3}{2}p$ and $|p| \leqslant \frac{4}{3}$. In view of these facts and Lemma 1, we have

$$\begin{aligned} p_2 &= \tfrac{9}{8}(p^2 + (\tfrac{16}{9} - p^2)x), \\ p_3 &= \tfrac{9}{32}\left(3p^3 + 6(\tfrac{16}{9} - p^2)px - 3(\tfrac{16}{9} - p^2)px^2 + 4(\tfrac{16}{9} - p^2)(1 - |x|^2)y\right) \end{aligned} \tag{24}$$

for some $x, y \in \mathbb{C}$ with $|x| \leqslant 1$ and $|y| \leqslant 1$.
Combining (23) with (24), we obtain

$$\begin{aligned} |a_3 - a_2| &= \left|\frac{9}{32}(\frac{16}{9} - p^2)x + \frac{5}{32}p^2 - \frac{p}{2}\right| \\ &\leqslant \frac{9}{32}(\frac{16}{9} - p^2) + \frac{5}{32}p\left|p - \frac{16}{5}\right| \\ &= \frac{1}{16}(-7p^2 + 8p + 8), \end{aligned}$$

where equality occurs if $x = -1$. Similarly, we have

$$\begin{aligned} |a_4 - a_3| &= \left|\frac{29}{320}p^3 - \frac{5}{32}p^2 + (\frac{16}{9} - p^2)[\frac{63}{320}px - \frac{27}{160}px^2 + \frac{9}{40}(1 - |x|^2)y - \frac{9}{32}x]\right| \\ &\leqslant \frac{16 - 9p^2}{40}\left[1 - |x|^2 + \left|\frac{p^2(50 - 29p)}{8(16 - 9p^2)} + \frac{1}{8}(10 - 7p)x + \frac{3}{4}px^2\right|\right] \\ &\leqslant \frac{16 - 9p^2}{40}Y(a,b,c), \end{aligned}$$

where $Y(a,b,c)$ is given in (4) with

$$a = \frac{p^2(50 - 29p)}{8(16 - 9p^2)},\ b = \frac{1}{8}(10 - 7p),\ c = \frac{3}{4}p.$$

(for $p = \frac{4}{3}$, we have directly that $|a_4 - a_3| = \frac{17}{270}$).
Since $p \in [0, 4/3]$, $|b| \leqslant 2(1 - c)$ is equivalent to $0 \leqslant p \leqslant \frac{6}{5}$, by Lemma 2, we have

$$Y(a,b,c) = \begin{cases} 1 + \frac{p^2(50-29p)}{8(16-9p^2)} + \frac{(10-7p)^2}{64(4-3p)},\ 0 \leqslant p \leqslant \frac{6}{5} \\ \frac{p^2(50-29p)}{8(16-9p^2)} + \frac{1}{8}(10 - 7p) + \frac{3}{4}p,\ \frac{6}{5} \leqslant p < \frac{4}{3}. \end{cases}$$

Therefore

$$|a_4 - a_3| \leqslant \begin{cases} \frac{1}{2560}(-85p^3 - 400p^2 - 260p + 1424),\ 0 \leqslant p \leqslant \frac{6}{5} \\ \frac{1}{80}(-5p^3 - 10p^2 - 4p + 40),\ \frac{6}{5} \leqslant p \leqslant \frac{4}{3}. \end{cases}$$

Let

$$\mathscr{B}^{(2)+} = \bigcup_{0 \leqslant p \leqslant \frac{4}{3}} \mathscr{B}_p^{(2)} = \{f : f \in \mathscr{B}^{(2)}, f''(0) \geqslant 0\}.$$

Then by using (20) and (21) we easily get

$$sup_{f \in \mathscr{B}^{(2)+}} |a_3(f) - a_2(f)| = \frac{9}{14} = 0.64...$$

and

$$sup_{f \in \mathscr{B}^{(2)+}} |a_4(f) - a_3(f)| = \frac{1424}{2560} = 0.556...$$

□

THEOREM 5. *Let* $0 \leqslant p \leqslant 1$ *and let* $f(z) = z + a_2 z^2 + a_3 z^3 + ...$ *be in the class* $\mathscr{B}_p^3$. *Then we have the next sharp inequalities:*

$$|a_3 - a_2| \leqslant \frac{1}{20}(-11p^2 + 10p + 8). \tag{25}$$

$$|a_4 - a_3| \leqslant \begin{cases} \frac{1}{600}(-53p^3 - 174p^2 - 24p + 272), 0 \leqslant p \leqslant \frac{2}{3} \\ \frac{1}{120}(-25p^3 - 30p^2 + 8p + 48), \frac{2}{3} \leqslant p \leqslant 1. \end{cases} \tag{26}$$

Proof. The hypothesis $f \in \mathscr{B}_p^3$ implies that there exists a function P, defined by (1) and satisfying $\mathrm{Re}P(z) > 0, z \in \mathbb{D}$, such that

$$\left(\frac{f(z)}{z}\right)^2 f'(z) = P(z). \tag{27}$$

By using the Taylor representations for the functions f and P and comparing the coefficients of $z^n (n = 1, 2, 3)$ in both sides of (27), we obtain

$$a_2 = \frac{1}{4}p_1, a_3 = \frac{1}{5}p_2 - a_2^2, a_4 = \frac{1}{6}p_3 - 2a_2a_3 - \frac{1}{3}a_2^3. \tag{28}$$

Since $2a_2 = f''(0) = p$ and $|p_1| \leqslant 2$, by (28) we have $p_1 = 4a_2 = 2p$ and $|p| \leqslant 1$. By using these facts and Lemma 1, we get

$$\begin{aligned} p_2 &= 2[p^2 + (1-p^2)x], \\ p_3 &= 2p^3 + 4(1-p^2)px - 2(1-p^2)px^2 + 2(1-p^2)(1-|x|^2)y \end{aligned} \tag{29}$$

for some $x, y \in \mathbb{C}$ with $|x| \leqslant 1$ and $|y| \leqslant 1$.
Combining (28) with (29), we obtain

$$\begin{aligned} |a_3 - a_2| &= \left|\frac{2}{5}(1-p^2)x - \frac{3}{20}p(10/3 - p)\right| \\ &\leqslant \frac{1}{20}(-11p^2 + 10p + 8), \end{aligned}$$

where equality occurs if $x=-1$. Similarly, we also have

$$\begin{aligned}|a_4-a_3| &= \left|\frac{1}{6}p_3-2a_2a_3-\frac{1}{3}a_2^3-a_3\right|\\ &\leqslant \frac{1}{3}(1-p^2)\left[1-|x|^2+\left|\frac{18p^2-17p^3}{40(1-p^2)}+\frac{2}{5}(3-2p)x+px^2\right|\right]\\ &\leqslant \frac{1}{3}(1-p^2)Y(a,b,c),\end{aligned}$$

where $Y(a,b,c)$ is given in (4) with

$$a=\frac{18p^2-17p^3}{40(1-p^2)},\, b=\frac{2}{5}(3-2p),\, c=p.$$

(for $p=1$ we have directly that $|a_4-a_3|=\frac{1}{120}$).
Since for $p\in[0,1]$, $|b|\leqslant 2(1-c)$ is equivalent to $0\leqslant p\leqslant\frac{2}{3}$, by using Lemma 2 we have

$$Y(a,b,c)=\begin{cases}1+\frac{p^2(18-17p)}{40(1-p^2)}+\frac{(3-2p)^2}{25(1-p)},\ 0\leqslant p\leqslant\frac{2}{3}\\ \frac{p^2(18-17p)}{40(1-p^2)}+\frac{1}{5}p+\frac{6}{5},\ \frac{2}{3}\leqslant p<1.\end{cases}$$

And therefore

$$|a_4-a_3|\leqslant\begin{cases}\frac{1}{600}(-53p^3-174p^2-24p+272),\ 0\leqslant p\leqslant\frac{2}{3}\\ \frac{1}{120}(-25p^3-30p^2+8p+48),\ \frac{2}{3}\leqslant p\leqslant 1.\end{cases}$$

Let

$$\mathscr{B}^{(3)+}=\bigcup_{0\leqslant p\leqslant 1}\mathscr{B}_p^{(3)}=\{f: f\in\mathscr{B}^{(3)}, f''(0)\geqslant 0\}..$$

In view of (25) and (26), we easily get

$$sup_{f\in\mathscr{B}^{(3)+}}|a_3(f)-a_2(f)|=\frac{113}{220}=0.5136...$$

and

$$sup_{f\in\mathscr{B}^{(3)+}}|a_4(f)-a_3(f)|=\frac{34}{75}=0.4533.$$

□

In [24] the authors introduced the class Ω which consists of all functions $f\in\mathscr{A}$ satisfying

$$|zf'(z)-f(z)|<\frac{1}{2},(|z|<1).$$

It is proved that $\Omega\subset S^*$. Now, let

$$\Omega_p=\{f: f\in\Omega, f''(0)=p\},$$

where $|p|\leqslant 1$ (Noting that $|a_n|\leqslant\frac{1}{2(n-1)}$ when $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\Omega$[24]).

THEOREM 6. *Let* $0 \leqslant p \leqslant 1$ *and let* $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ *be in the class* Ω_p. *Then we have the following sharp inequalities:*

$$|a_3 - a_2| \leqslant \frac{1}{4} + \frac{1}{2}p - \frac{1}{4}p^2, 0 \leqslant p \leqslant 1. \tag{30}$$

$$|a_4 - a_3| \leqslant \begin{cases} \frac{1}{96}(-16p^2 + 9p + 25),\ 0 \leqslant p \leqslant \frac{1}{4} \\ \frac{1}{12}(-2p^3 - 3p^2 + 2p + 3),\ \frac{1}{4} \leqslant p \leqslant 1. \end{cases} \tag{31}$$

Proof. By the definition of Ω, $f \in \Omega$ if and only if there exists a function $P(z)$ defined by (1) with $\operatorname{Re} P(z) > 0, z \in \mathbb{D}$, such that

$$2[P(z) + 1][zf'(z) - f(z)] = z[P(z) - 1] \tag{32}$$

By using the Taylor representations for the functions f and P and comparing the coefficients of $z^n (n = 2, 3, 4)$ in both sides of (32), we obtain

$$a_2 = \frac{1}{4}p_1,\ a_3 = \frac{1}{8}p_2 - \frac{1}{4}a_2 p_1,\ a_4 = \frac{1}{12}p_3 - \frac{1}{3}a_3 p_1 - \frac{1}{6}a_2 p_2. \tag{33}$$

Since $2a_2 = f''(0) = p$ and $|p_1| \leqslant 2$, by (33) we have $p_1 = 4a_2 = 2p$ and $|p| \leqslant 1$. In view of these facts and Lemma 1, we get

$$\begin{aligned} p_2 &= 2[p^2 + (1 - p^2)x], \\ p_3 &= 2p^3 + 4(1 - p^2)px - 2(1 - p^2)px^2 + 2(1 - p^2)(1 - |x|^2)y \end{aligned} \tag{34}$$

for some $x, y \in \mathbb{C}$ with $|x| \leqslant 1$ and $|y| \leqslant 1$.
Combining (33) with (34), we obtain

$$\begin{aligned} |a_3 - a_2| &= \left|\frac{1}{4}(1 - p^2)x - \frac{1}{2}p\right| \\ &\leqslant \frac{1}{4} + \frac{1}{2}p - \frac{1}{4}p^2, \end{aligned}$$

where equality occurs if $x = -1$. Similarly, we also have

$$\begin{aligned} |a_4 - a_3| &= \left|\frac{1}{6}(1 - p^2)(1 - |x|^2)y - \frac{1}{6}(1 - p^2)px^2 - \frac{1}{4}(1 - p^2)x\right| \\ &\leqslant \frac{1}{6}(1 - p^2)\left[1 - |x|^2 + \left|\frac{3}{2}x + px^2\right|\right] \\ &\leqslant \frac{1}{6}(1 - p^2)Y(a, b, c), \end{aligned}$$

where $Y(a, b, c)$ is given in (4) with

$$a = 0, b = \frac{3}{2}, c = p.$$

Since for $p \in [0,1]$, $|b| \leqslant 2(1-c)$ is equivalent to $0 \leqslant p \leqslant \frac{1}{4}$, by using Lemma 2 we have

$$Y(a,b,c) = \begin{cases} 1 + \frac{9}{16(1-p)}, \ 0 \leqslant p \leqslant \frac{1}{4} \\ \frac{3}{2} + p, \ \frac{1}{4} \leqslant p \leqslant 1. \end{cases}$$

And therefore

$$|a_4 - a_3| \leqslant \begin{cases} \frac{1}{96}(-16p^2 + 9p + 25), \ 0 \leqslant p \leqslant \frac{1}{4} \\ \frac{1}{12}(-2p^3 - 3p^2 + 2p + 3), \ \frac{1}{4} \leqslant p \leqslant 1. \end{cases}$$

Let

$$\Omega^+ = \bigcup_{0 \leqslant p \leqslant 1} \Omega_p = \{f : f \in \Omega, f''(0) \geqslant 0\}.$$

In view of (30) and (31), we easily get

$$sup_{f \in \Omega^+} |a_3(f) - a_2(f)| = \frac{1}{2}$$

and

$$sup_{f \in \Omega^+} |a_4(f) - a_3(f)| = \frac{27 + 7\sqrt{21}}{216} = 0.2735...$$

□

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(Received March 13, 2018)

Zhigang Peng
Hubei Key Laboratory of Applied Mathematics
Faculty of Mathematics and Statistics, Hubei University
Wuhan 430062, P. R. China
e-mail: pengzhigang@hubu.edu.cn

Milutin Obradović
Department of Mathematics
Faculty of Civil Engineering, University of Belgrade
Bulevar Kralja Aleksandra 73, 11000
e-mail: obrad@grf.bg.ac.rs