GENERALIZED 'USEFUL' NON–SYMMETRIC DIVERGENCE MEASURES AND INEQUALITIES

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Abstract. In the present paper we consider one parameter generalizations of some 'useful' non - symmetric divergence measures. All the generalizations considered can be written as particular cases of Csiszar f-divergence. Under some conditions of probability distributions, inequalities among generalized 'useful' relative divergence measures are obtained.

1. Introduction

Let $\Delta_n^+ = \{P = (p_1, p_2, ..., p_n), p_i \geqslant 0, \sum_{i=1}^n p_i = 1\}$, be a set of all possible discrete probability distributions of a random variable X having utility distribution $U = \{(u_1, u_2, ..., u_n); u_i > 0 \text{ for all } i\}$ attached to each $P \in \Delta_n^+$ such that $u_i > 0$ is utility of an event having probability of occurrence $p_i > 0$.

The following measure of 'useful' directed divergence or 'useful' relative information is given by

$$I(P;Q;U) = \frac{\sum_{i=1}^n u_i p_i \log\left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^n u_i p_i} \tag{1}$$

It may be noted (1) that is a generalization of the measure

$$H(P;U) = -\frac{\sum_{i=1}^n u_i p_i \log p_i}{\sum_{i=1}^n u_i p_i} \tag{2}$$

It can be observed that (2) is not symmetric in P and Q and its symmetric version is given by

$$J(P;Q;U) = I(P;Q;U) + I(Q;P;U)$$

$$= \frac{\sum_{i=1}^n u_i p_i \log\left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^n u_i p_i} + \frac{\sum_{i=1}^n u_i q_i \log\left(\frac{q_i}{p_i}\right)}{\sum_{i=1}^n u_i q_i}$$

$$= \frac{\sum_{i=1}^n u_i(p_i - q_i) \log\left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^n u_i p_i}, \tag{3}$$

where $\sum_{i=1}^{n} u_i p_i = \sum_{i=1}^{n} u_i q_i$.

When utilities are ignored or $u_i = 1$ for each i, (3) reduces to

$$J(P;Q) = \sum_{i=1}^{n} (p_i - q_i) \log\left(\frac{p_i}{q_i}\right). \tag{4}$$

Which is[11] divergence measure and it is written as J-divergence. In the next section we give some generalized 'useful' non-Symmetric Divergence Measures.

2. Generalized 'useful' non–symmetric divergence measures

Some generalized 'useful' non-symmetric measures of information are given below:

'Useful' χ^2–divergence

$$\chi^2(P;Q;U) = \frac{\sum_{i=1}^{n} u_i (p_i - q_i)^2 / q_i}{\sum_{i=1}^{n} u_i p_i} = \frac{\sum_{i=1}^{n} \left(u_i p_i^2 / q_i\right)}{\sum_{i=1}^{n} u_i p_i} - 1. \tag{5}$$

'Useful' relative information

$$L(P;Q;U) = \frac{\sum_{i=1}^{n} u_i p_i \log\left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^{n} u_i p_i}. \tag{6}$$

'Useful' relative Jensen-Shannon divergence

$$S(P;Q;U) = \frac{\sum_{i=1}^{n} u_i p_i \log\left(\frac{2p_i}{p_i+q_i}\right)}{\sum_{i=1}^{n} u_i p_i}. \tag{7}$$

'Useful' relative arithmetic-geometric divergence

$$T(P;Q;U) = \frac{\sum_{i=1}^{n} u_i \left(\frac{p_i+q_i}{2}\right) \log\left(\frac{p_i+q_i}{2p_i}\right)}{\sum_{i=1}^{n} u_i p_i}. \tag{8}$$

'Useful' relative j- divergence (Dragomir et al. [8])

$$G(P;Q;U) = \frac{\sum_{i=1}^{n} u_i (p_i - q_i) \log\left(\frac{p_i+q_i}{2q_i}\right)}{\sum_{i=1}^{n} u_i p_i}. \tag{9}$$

Symmetric versions of above 'useful' measures are given by

$$\psi(P;Q;U) = \chi^2(P;Q;U) + \chi^2(Q;P;U) \tag{10}$$

$$F(P;Q;U) = L(P;Q;U) + L(Q;P;U) = G(P;Q;U) + G(Q;P;U) \tag{11}$$

$$K(P;Q;U) = \frac{1}{2}[S(P;Q;U) + S(Q;P;U)] \qquad (12)$$

and

$$J(P;Q;U) = \frac{1}{2}[T(P;Q;U) + T(Q;P;U)]. \qquad (13)$$

After simplification, we can write

$$F(P;Q;U) = 4[K(P;Q;U) + J(Q;P;U)] \qquad (14)$$

and

$$G(Q;P;U) = \frac{1}{2}[S(P;Q;U) + T(P;Q;U)].$$

When utilities are ignored all the above measures reduces to [2],[14],[12],[15, 16],[22],[8].

[9] studied the measures (9), referred to subsequently as [21] 'useful' symmetric chi-square divergence. Measure (11) is known as 'useful' Jeffreys-Kullback-Leibler [12] J-divergence, the measure (12) is 'useful' Jensen-Shannon divergence studied by Sibson[16] and Burbea and Rao[3]. Measure(13) is 'useful' arithmetic and geometric mean divergence studied by Taneja [18]. More details on some of these measures can be found in Taneja[17, 18] and in the on line book by Taneja [20].

In the next section we will study the unified measures of 'useful' non symmetric relative information.

3. Generalized 'useful' non – symmetric divergence measures of type s

Let us consider the following generalization of (6) & call it 'useful' relative information of type s:

$$L_s(P;Q;U) = [s(s-1)]^{-1}\left[\frac{\sum_{i=1}^{n} u_i p_i^s q_i^{1-s}}{\sum_{i=1}^{n} u_i p_i} - 1\right], \quad s \neq 0,1; \qquad (15)$$

where $\sum_{i=1}^{n} u_i p_i = \sum_{i=1}^{n} u_i q_i$.
We have the following limiting cases:

$$\lim_{s \to 1} L_s(P;Q;U) = L(P;Q;U)$$
and
$$\lim_{s \to 0} L_s(P;Q;U) = L(Q;P;U),$$

provided $\sum_{i=1}^{n} u_i p_i = \sum_{i=1}^{n} u_i q_i$.
The measure (15) is a non-additive generalized 'useful' relative information measure & has some interesting particular cases which are discussed as given below:
(i) When $s = 1/2$ we have

$$L_{1/2}(P;Q;U) = 4\left[1 - B(P;Q;U)\right]$$

where $B(P;Q;U) = \frac{\sum_{i=1}^{n} u_i p_i^{1/2} q_i^{1/2}}{\sum_{i=1}^{n} u_i p_i}$ is 'useful' distance corresponding to [1] distance $B(P;Q) = \sum_{i=1}^{n} \sqrt{p_i q_i}$. Further

$$L_{1/2}(P;Q;U) = 4h(P;Q;U)$$

where $h(P;Q;U) = \frac{1}{2}\frac{\sum_{i=1}^{n} u_i (p_i^{1/2} - q_i^{1/2})^2}{\sum_{i=1}^{n} u_i p_i}$ is 'useful' discrimination measure corre-

sponding to [10] discrimination $h(P;Q) = \frac{1}{2}\sum_{i=1}^{n}\left(p_i^{1/2} - q_i^{1/2}\right)^2$.

(ii) When $s = 2$, we have

$$L_2(P;Q;U) = \frac{1}{2}\chi^2(P;Q;U).$$

(iii) When $s = -1$, we have

$$L_{-1}(P;Q;U) = \frac{1}{2}\chi^2(Q;P;U).$$

We can write measures (12) in the following unified way

$$\varphi_s(P;Q;U) = \begin{cases} L_s(P;Q;U) & s \neq 0,1 \\ L(Q;P;U) & s = 0 \\ L(P;Q;U) & s = 1, \end{cases} \tag{16}$$

provided $\sum_{i=1}^{n} u_i p_i = \sum_{i=1}^{n} u_i q_i$.

4. Generalized 'useful' unified relative JS and AG–divergence of type s

The following unified one parameter generalization of measures (7) and (8) simultaneously are considered:

$$\Omega_s(P;Q;U) = \begin{cases} ST_s(P;Q;U) = [s(s-1)]^{-1}\left[\frac{\sum_{i=1}^{n} u_i p_i \left(\frac{p_i+q_i}{2p_i}\right)^s}{\sum_{i=1}^{n} u_i p_i} - 1\right] & s \neq 0,1 \\ S(P;Q;U) = \frac{\sum_{i=1}^{n} u_i p_i \log\left(\frac{2p_i}{p_i+q_i}\right)}{\sum_{i=1}^{n} u_i p_i} & s = 0 \\ T(P;Q;U) = \frac{\sum_{i=1}^{n} u_i \left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2p_i}\right)}{\sum_{i=1}^{n} u_i p_i} & s = 1. \end{cases} \tag{17}$$

The adjoint of $\Omega_s(P;Q;U)$ written as $\Omega_s(Q;P;U)$ is obtained by interchanging P and Q, and p_i and q_i in the above expression. The measures $\Omega_s(Q;P;U)$ can also be obtained from (15) by replacing p_i by $\frac{p_i+q_i}{2}$.

We have the following particular cases of $\Omega_s(P;Q;U)$ and $\Omega_s(Q;P;U)$:
(i) $\Omega_{-1}(P;Q;U) = \Omega_{-1}(Q;P;U) = \frac{1}{4}\Delta(P;Q;U)$.

(ii) (a) $\Omega_0(P;Q;U) = S(P;Q;U)$.

    (b) $\Omega_0(Q;P;U) = S(Q;P;U)$.

(iii) (a) $\Omega_1(P;Q;U) = T(P;Q;U)$.

    (b) $\Omega_1(Q;P;U) = T(Q;P;U)$.

(iv) (a) $\Omega_2(P;Q;U) = \frac{1}{8}\chi^2(Q;P;U)$.

    (b) $\Omega_2(Q;P;U) = \frac{1}{8}\chi^2(P;Q;U)$.

The expression $\Delta(P;Q;U)$ appearing in part (i) is the well known 'useful' triangular discrimination, and is given by

$$\Delta(P;Q;U) = \frac{\sum_{i=1}^n u_i \frac{(p_i-q_i)^2}{p_i+q_i}}{\sum_{i=1}^n u_i p_i} \ .$$

5. Generalized 'useful' relative J–divergence of type s

We now propose the following one parameter generalization of the 'useful' relative J-divergence measures given by (9).

$$\zeta_s(P;Q;U) = \begin{cases} G_s(P;Q;U) = \dfrac{(s-1)^{-1}\sum_{i=1}^n u_i(p_i-q_i)\left(\frac{p_i+q_i}{2q_i}\right)^{s-1}}{\sum_{i=1}^n u_i p_i} & s \neq 1 \\[2em] G(P;Q;U) = \dfrac{\sum_{i=1}^n u_i(p_i-q_i)\ln\left(\frac{p_i+q_i}{2q_i}\right)}{\sum_{i=1}^n u_i p_i} & s = 1, \end{cases} \tag{18}$$

where $\sum_{i=1}^n u_i p_i = \sum_{i=1}^n u_i q_i$.

The adjoint of $\zeta_s(P;Q;U)$ written as $\zeta_s(Q;P;U)$ is obtained by interchanging P and Q, and p_i and q_i in the expression (18).

Some particular cases are:

(i) $\zeta_o(P;Q;U) = \zeta_0(Q;P;U) = \Delta(P;Q;U)$.

(ii) (a) $\zeta_1(P;Q;U) = D(P;Q;U)$.

    (b) $\zeta_1(Q;P;U) = D(Q;P;U)$.

(iii) (a) $\zeta_2(P;Q;U) = \frac{1}{2}\chi^2(P;Q;U)$.

    (b) $\zeta_2(Q;P;U) = \frac{1}{2}\chi^2(Q;P;U)$.

We observe that the generalized 'useful' relative information of type s, $\Phi_s(P;Q;U)$, contains the classical measures Bhattacharyya coefficient, 'useful' χ^2–divergence and 'useful' Hellinger discrimination. The generalized 'useful' unified relative JS and AG–divergence of type s, $\Omega_s(P;Q;U)$ and $\Omega_s(Q;P;U)$, contains 'useful' triangular discrimination and 'useful' χ^2–divergence, while the generalized 'useful' relative J– divergences of type s, $\zeta_s(P;Q;U)$ and $\zeta_s(Q;P;U)$, yield, in particular, 'useful' triangular discrimination and 'useful' χ^2-divergence.

6. Csiszar's 'useful' f- divergence and its particular cases

For a function $f : [0,\infty) \to \Re$, the 'useful' f-divergence measure is given by

$$C_f(P;Q;U) = \frac{\sum_{i=1}^n u_i q_i f\left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^n u_i p_i}, \tag{19}$$

for all $P, Q \in \Delta_n^+$.

The following result is well known in the literature.

THEOREM 1. [4, 5]. *If the function f is convex and normalized, i.e., $f(1) = 0$, then, $C_f(P;Q;U)$ and its adjoint $C_f(Q;P;U)$ are both nonnegative and convex in the pair of probability distributions $(P,Q) \in \Delta_n^+ \times \Delta_n^+$ and U is the utility distribution.*

EXAMPLE 1. (Relative information of type s). Let us consider

$$\varphi_s(x) = \begin{cases} [s(s-1)]^{-1} [x^s - 1 - s(x-1)] & s \neq 0, 1 \\ x - 1 - \log x & s = 0 \\ 1 - x + x \log x & s = 1. \end{cases} \tag{20}$$

For all $x > 0$ in (19). Then $C_f(P;Q;U) = \Phi_s(P;Q;U)$.

EXAMPLE 2. (Relative JS and AG – divergence of type s). Let us consider

$$\Psi_s(x) = \begin{cases} [s(s-1)]^{-1} \left[x \left(\frac{x+1}{2x} \right)^s - x - s \left(\frac{1-x}{2} \right) \right] & s \neq 0, 1 \\ \frac{1-x}{2} - x \log \left(\frac{x+1}{2x} \right) & s = 0 \\ \frac{x-1}{2} + \left(\frac{x+1}{2} \right) \log \left(\frac{x+1}{2x} \right) & s = 1. \end{cases} \tag{21}$$

For all $x > 0$ in (19). Then $C_f(P;Q;U) = \Omega_s(P;Q;U)$.

EXAMPLE 3. (Adjoint of Relative JS and AG- divergence of type s). Let us consider

$$\upsilon_s(x) = \begin{cases} [s(s-1)]^{-1} \left[\left(\frac{x+1}{2} \right)^s - 1 - s \left(\frac{x-1}{2} \right) \right] & s \neq 0, 1 \\ \frac{x-1}{2} + \log \left(\frac{2}{x+1} \right) & s = 0 \\ \frac{1-x}{2} + \frac{x+1}{2} \log \left(\frac{x+1}{2} \right) & s = 1. \end{cases} \tag{22}$$

For all $x > 0$ in (19). Then $C_f(P;Q;U) = \Omega_s(Q;P;U)$.

EXAMPLE 4. (Relative J- divergence of type s). Let us consider

$$\xi_s(x) = \begin{cases} (s-1)^{-1}(x-1) \left[\left(\frac{x+1}{2} \right)^{s-1} - 1 \right] & s \neq 1 \\ (x-1) \log \left(\frac{x+1}{2} \right) & s = 1. \end{cases} \tag{23}$$

For all $x > 0$ in (19). Then $C_f(P;Q;U) = \zeta_s(P;Q;U)$.

EXAMPLE 5. (Adjoint of relative J-divergence of type s). Let us consider

$$\zeta_s(x) = \begin{cases} (s-1)^{-1}(1-x) \left[\left(\frac{x+1}{2x} \right)^{s-1} - 1 \right] & s \neq 1 \\ (1-x) \log \left(\frac{x+1}{2x} \right) & s = 1. \end{cases} \tag{24}$$

For all $x > 0$ in (19). Then $C_f(P;Q;U) = \zeta_s(Q;P;U)$.

By considering the second order derivative of the functions given by (20)-(24) with respect to x, and applying the Theorem 1, it can easily be checked that the measures

$\Phi_s(P;Q;U)$, $\Omega_s(P;Q;U)$, $\Omega_s(Q;P;U)$, $\zeta_s(P;Q;U)$ and $\zeta_s(Q;P;U)$ are nonnegative and convex in the pair of probability distributions $(P,Q) \in \Delta_n^+ \times \Delta_n^+$ respectively and U is the utility distribution, for all $s \in \Re$ for the measures $\Phi_s(P;Q;U)$, $\Omega_s(P;Q;U)$ and $\Omega_s(Q;P;U)$. And $0 \leqslant s \leqslant 4$ for the measure $\zeta_s(P;Q;U)$ and $\zeta_s(Q;P;U)$.

THEOREM 2. [23]. *Let $f_1, f_2 : I \subset \Re_+ \to \Re$ be two normalized functions, i.e.,* $f_1(1) = f_2(1) = 0$ *and satisfy assumptions:*
(i) f_1 and f_2 are twice differentiable on (r, R);
(ii) There exists the real constants m, M such that $0 \leqslant m < M$ and

$$m \leqslant \frac{f_1''(x)}{f_2''(x)} \leqslant M, \quad f_2''(x) > 0, \quad for \, all \, x \in (r,R) \tag{25}$$

If $P,Q \in \Delta_n^+$ are discrete probability distributions and U is the utility distribution attached to each $P \in \Delta_n^+$ such that $\sum_{i=1}^n u_i p_i = \sum_{i=1}^n u_i q_i$, then we have the inequalities:

$$m \, C_{f_2}(P;Q;U) \leqslant C_{f_1}(P;Q;U) \leqslant M \, C_{f_2}(P;Q;U). \tag{26}$$

Proof. Let us consider the functions $\eta_{m.s}(\cdot)$ and $\eta_{M.s}(\cdot)$ given by

$$\eta_m(x) = f_1(x) - m f_2(x) \tag{27}$$

and

$$\eta_M(x) = M f_2(x) - f_1(x), \tag{28}$$

respectively, where m and M are given by (25).

Since $f_1(x)$ and $f_2(x)$ are normalized, i.e., $f_1(1) = f_2(1) = 0$, then $\eta_{m.s}(\cdot)$ and $\eta_{M.s}(\cdot)$ are also normalized, i.e., $\eta_m(1) = 0$ and $\eta_M(1) = 0$. Also, the functions $f_1(x)$ and $f_2(x)$ are twice differentiable. Then in view of (25), we have

$$\eta_m''(x) = f_1''(x) - m f_2''(x) = f_2''(x) \left(\frac{f_1''(x)}{f_2''(x)} - m \right) \geqslant 0 \tag{29}$$

and

$$\eta_M''(x) = M f_2''(x) - f_1''(x) = f_2''(x) \left(M - \frac{f_1''(x)}{f_2''(x)} \right) \geqslant 0 \tag{30}$$

for all $x \in (r,R)$.

In view of (29) and (30), we can say that the functions $\eta_m(\cdot)$ *and* $\eta_M(\cdot)$ are convex on (r, R).

According to Theorem 1, we have

$$C_{\eta m}(P;Q;U) = C_{f_1 - m f_2}(P;Q;U) = C_{f_1}(P;Q;U) - m C_{f_2}(P;Q;U) \geqslant 0, \tag{31}$$

and

$$C_{\eta M}(P;Q;U) = C_{M f_2 - f_1}(P;Q;U) = M C_{f_2}(P;Q;U) - C_{f_1}(P;Q;U) \geqslant 0. \qquad (32)$$

Combining (31) and (32), we get (26). □

7. Inequalities among generalized 'useful' relative divergences

THEOREM 3. *Let the generating functions given by (20)-(24) are twice differentiable in interval (r,R) with $0 < r \leqslant R$ and U is the utility distribution attached to each $P \in \Delta_n^+$ such that $\sum_{i=1}^n u_i p_i = \sum_{i=1}^n u_i q_i$,then we have the following inequalities among the generalized measures:*

(i) $\Omega_s(P;Q;U)$ and $\Phi_t(P;Q;U)$:

$$\frac{1}{4r^{t+1}} \left(\frac{r+1}{2r}\right)^{s-2} \Phi_t(P;Q;U) \leqslant \Omega_s(P;Q;U) \leqslant \frac{1}{4R^{t+1}} \left(\frac{R+1}{2R}\right)^{s-2} \Phi_t(P;Q;U),$$
$$s+t \leqslant 1, t \leqslant -1,$$

$$(33)$$

and

$$\frac{1}{4R^{t+1}} \left(\frac{R+1}{2R}\right)^{s-2} \Phi_t(P;Q;U) \leqslant \Omega_s(P;Q;U) \leqslant \frac{1}{4r^{t+1}} \left(\frac{r+1}{2r}\right)^{s-2} \Phi_t(P;Q;U),$$
$$s+t \geqslant 1, t \geqslant -1.$$

$$(34)$$

(ii) $\Omega_s(Q;P;U)$ and $\Phi_t(P;Q;U)$:

$$\frac{1}{4r^{t-2}} \left(\frac{r+1}{2}\right)^{s-2} \Phi_t(P;Q;U) \leqslant \Omega_s(Q;P;U) \leqslant \frac{1}{4R^{t-2}} \left(\frac{R+1}{2}\right)^{s-2} \Phi_t(P;Q;U),$$
$$s \geqslant t, t \leqslant 2,$$

$$(35)$$

and

$$\frac{1}{4R^{t-2}} \left(\frac{R+1}{2}\right)^{s-2} \Phi_t(P;Q;U) \leqslant \Omega_s(Q;P;U) \leqslant \frac{1}{4r^{t-2}} \left(\frac{r+1}{2}\right)^{s-2} \Phi_t(P;Q;U),$$
$$s \leqslant t, t \geqslant 2.$$

$$(36)$$

(iii) $\zeta_s(P;Q;U)$ and $\Phi_t(P;Q;U)$:

$$\left(\frac{r+1}{2}\right)^{s-3}\left(\frac{sr+4-s}{4r^{t-2}}\right)\Phi_t(P;Q;U) \leqslant \zeta_s(P;Q;U)$$

$$\leqslant \left(\frac{R+1}{2}\right)^{s-3}\left(\frac{sR+4-s}{4R^{t-2}}\right)\Phi_t(P;Q;U), \quad 0\leqslant s\leqslant 4, t\leqslant 2, s\geqslant t+1, \tag{37}$$

and

$$\left(\frac{R+1}{2}\right)^{s-3}\left(\frac{sR+4-s}{4R^{t-2}}\right)\Phi_t(P;Q;U) \leqslant \zeta_s(P;Q;U)$$

$$\leqslant \left(\frac{r+1}{2}\right)^{s-3}\left(\frac{sr+4-s}{4r^{t-2}}\right)\Phi_t(P;Q;U), \quad 0\leqslant s\leqslant 4, t\geqslant 2, s\leqslant t+1. \tag{38}$$

(iv) $\zeta_s(Q;P;U)$ and $\Phi_t(P;Q;U)$:

$$\left(\frac{r+1}{2r}\right)^{s-3}\left(\frac{(4-s)r+s}{4r^{t+2}}\right)\Phi_t(P;Q;U) \leqslant \zeta_s(Q;P;U)$$

$$\leqslant \left(\frac{R+1}{2R}\right)^{s-3}\left(\frac{(4-s)R+s}{4R^{t+2}}\right)\Phi_t(P;Q;U), \quad 0\leqslant s\leqslant 4,\ t\leqslant -1, s+t\leqslant 1, \tag{39}$$

and

$$\left(\frac{R+1}{2R}\right)^{s-3}\left(\frac{(4-s)R+s}{4R^{t+2}}\right)\Phi_t(P;Q;U) \leqslant \zeta_s(Q;P;U)$$

$$\leqslant \left(\frac{r+1}{2r}\right)^{s-3}\left(\frac{(4-s)r+s}{4r^{t+2}}\right)\Phi_t(P;Q;U), \quad 0\leqslant s\leqslant 4, t\geqslant -1, s+t\geqslant 2. \tag{40}$$

(v) $\Omega_s(Q;P;U)$ and $\Omega_t(P;Q;U)$:

$$r^{t+1}\left(\frac{r+1}{2}\right)^{s-t}\Omega_t(P;Q;U) \leqslant \Omega_s(Q;P;U) \leqslant R^{t+1}\left(\frac{R+1}{2}\right)^{s-t}\Omega_t(P;Q;U), \tag{41}$$

$$s\geqslant -1, t\geqslant -1,$$

and

$$R^{t+1}\left(\frac{R+1}{2}\right)^{s-t}\Omega_t(P;Q;U) \leqslant \Omega_s(Q;P;U) \leqslant r^{t+1}\left(\frac{r+1}{2}\right)^{s-t}\Omega_t(P;Q;U), \tag{42}$$

$$s\leqslant -1, t\leqslant -1.$$

(vi) $\zeta_s(P;Q;U)$ and $\Omega_t(P;Q;U)$:

$$r^{t+1}\left(\frac{r+1}{2}\right)^{s-t-1}(sr+4-s)\,\Omega_t(P;Q;U) \leqslant \zeta_s(P;Q;U)$$

$$\leqslant R^{t+1}\left(\frac{R+1}{2}\right)^{s-t-1}(sR+4-s)\Omega_t(P;Q;U), \quad 0\leqslant s\leqslant 4,\ t\geqslant -1. \tag{43}$$

(vii) $\zeta_s(Q;P;U)$ and $\Omega_t(P;Q;U)$:

$$\left(\frac{r+1}{2r}\right)^{s-t-1}\left(\frac{(4-s)r+s}{r}\right)\Omega_t(P;Q;U) \leqslant \zeta_s(Q;P;U)$$

$$\leqslant \left(\frac{R+1}{2R}\right)^{s-t-1}\left(\frac{(4-s)R+s}{R}\right)\Omega_t(P;Q;U),$$

$$0 \leqslant s \leqslant 4, t \geqslant s, t(4-s) \geqslant 6s - s^2 - 4, \tag{44}$$

and

$$\left(\frac{R+1}{2R}\right)^{s-t-1}\left(\frac{(4-s)R+s}{R}\right)\Omega_t(P;Q;U) \leqslant \zeta_s(Q;P;U)$$

$$\leqslant \left(\frac{r+1}{2r}\right)^{s-t-1}\left(\frac{(4-s)r+s}{r}\right)\Omega_t(P;Q;U),$$

$$0 \leqslant s \leqslant 4, t \leqslant s, t(4-s) \leqslant 6s - s^2 - 4. \tag{45}$$

(viii) $\zeta_s(P;Q;U)$ and $\Omega_t(Q;P;U)$:

$$\left(\frac{r+1}{2}\right)^{s-t-1}(sr+4-s)\,\Omega_t(Q;P;U) \leqslant \zeta_s(P;Q;U)$$

$$\leqslant \left(\frac{R+1}{2}\right)^{s-t-1}(sR+4-s)\Omega_t(Q;P;U), \quad 0 \leqslant s \leqslant 4, s \geqslant t, s(t-s+6) \geqslant 4(1+t), \tag{46}$$

and

$$\left(\frac{R+1}{2}\right)^{s-t-1}(sR+4-s)\,\Omega_t(Q;P;U) \leqslant \zeta_s(P;Q;U)$$

$$\leqslant \left(\frac{r+1}{2}\right)^{s-t-1}(sr+4-s)\Omega_t(Q;P;U), \quad 0 \leqslant s \leqslant 4, \ s \leqslant t, \ s(t-s+6) \leqslant 4(1+t). \tag{47}$$

(ix) $\zeta_s(Q;P;U)$ and $\Omega_t(Q;P;U)$:

$$\frac{1}{R^{s+1}}\left(\frac{R+1}{2}\right)^{s-t-1}[(4-s)R+s]\,\Omega_t(Q;P;U) \leqslant \zeta_s(Q;P;U)$$

$$\leqslant \frac{1}{r^{s+1}}\left(\frac{r+1}{2}\right)^{s-t-1}[(4-s)R+s]\,\Omega_t(Q;P;U), \quad 0 \leqslant s \leqslant 4, t \geqslant -1. \tag{48}$$

(x) $\zeta_s(Q;P;U)$ and $\zeta_t(P;Q;U)$:

$$\frac{1}{R^{s+1}} \left(\frac{R+1}{2}\right)^{s-t} \left(\frac{(4-s)R+s}{tR+4-t}\right) \zeta_t(P;Q;U) \leqslant \zeta_s(Q;P;U)$$

$$\leqslant \frac{1}{r^{s+1}} \left(\frac{r+1}{2}\right)^{s-t} \left(\frac{(4-s)r+s}{tr+4-t}\right) \zeta_t(P;Q;U), \quad 2 \leqslant s \leqslant 4, 2 \leqslant t \leqslant 4. \tag{49}$$

Proof. (i) Let us consider

$$g_{(\psi_s,\phi_t)}(x) = \frac{\psi_s''(x)}{\phi_t''(x)} = \frac{\frac{1}{4x^3}\left(\frac{x+1}{2x}\right)^{s-2}}{x^{t-2}} = \frac{1}{4x^{t+1}}\left(\frac{x+1}{2x}\right)^{s-2}, \tag{50}$$

For all $x \in (0,\infty)$.
From (50) one has

$$g'_{(\psi_s,\phi_t)}(x) = -\left(\frac{x+1}{2x}\right)^{s-2} \frac{x(t+1)+t+s-1}{4x^{t+2}(x+1)} \begin{cases} \geqslant 0 & t \leqslant -1, s+t \leqslant 1 \\ \leqslant 0 & t \geqslant -1, s+t \geqslant 1. \end{cases} \tag{51}$$

In view of (51) we conclude the followings

$$m = \inf_{x \in [r,R]} g_{(\psi_s,\phi_t)}(x) = \begin{cases} \frac{1}{4r^{t+1}}\left(\frac{r+1}{2r}\right)^{s-2} & s+t \leqslant 1, t \leqslant -1 \\ \frac{1}{4R^{t+1}}\left(\frac{R+1}{2R}\right)^{s-2} & s+t \geqslant 1, t \geqslant -1 \end{cases} \tag{52}$$

and

$$M = \sup_{x \in [r,R]} g_{(\psi_s,\phi_t)}(x) = \begin{cases} \frac{1}{4R^{t+1}}\left(\frac{R+1}{2R}\right)^{s-2} & s+t \leqslant 1, t \leqslant -1 \\ \frac{1}{4R^{t+1}}\left(\frac{r+1}{2r}\right)^{s-2} & s+t \geqslant 1, t \geqslant -1. \end{cases} \tag{53}$$

Now (51) and (53) together with (26) give the inequalities (33) and (34).
The proof of other parts (ii)-(x) follows on similar lines. $\quad\square$

PARTICULAR CASES. Here we have considered some particular cases of the inequalities (33)-(49) given below:
Take $t = \frac{1}{2}$, $s = 2$ in (35) or in (37), one gets

$$r \leqslant \frac{\sqrt[3]{\chi^2(P;Q;U)^2}}{4\sqrt[3]{h(P;Q;U)^2}} \leqslant R.$$

Take $t = \frac{1}{2}$, $s = 2$ in (34) or $t = \frac{1}{2}$, $s = 2$ in (40), one gets

$$r \leqslant \frac{4\sqrt[3]{h(P;Q;U)^2}}{\sqrt[3]{\chi^2(Q;P;U)^2}} \leqslant R.$$

Take $t = 2$, $s = 2$ in (34) or in (40) or in (41) or in (43) or in (48) or in (49) or $t = -1$, $s = 2$ in (35) or in (37), one gets

$$r \leqslant \frac{\sqrt[3]{\chi^2(P;Q;U)}}{\sqrt[3]{\chi^2(Q;P;U)}} \leqslant R.$$

Take $t = 1$, $s = 2$ in (35) or in (37), one gets

$$r \leqslant \frac{\chi^2(P;Q;U)}{2K(P;Q;U)} \leqslant R.$$

Take $t = 0$, $s = 2$ in (34) or in (40), one gets

$$r \leqslant \frac{2K(Q;P;U)}{\chi^2(Q;P;U)} \leqslant R.$$

Take $t = 1$, $s = 2$ in (34) or in (40), one gets

$$r \leqslant \frac{\sqrt{2K(P;Q;U)}}{\sqrt{\chi^2(Q;P;U)}} \leqslant R.$$

Take $t = 0$, $s = 2$ in (35) or in (37), one gets

$$r \leqslant \frac{\sqrt{\chi^2(P;Q;U)}}{\sqrt{2K(Q;P;U)}} \leqslant R.$$

Take $t = 0$, $s = 0$ in (41), one gets

$$r \leqslant \frac{F(Q;P;U)}{F(P;Q;U)} \leqslant R.$$

Take $t = 1$, $s = 1$ in (41), one gets

$$r \leqslant \frac{\sqrt{G(Q;P;U)}}{\sqrt{G(P;Q;U)}} \leqslant R.$$

Take t=2, s= -1 in (34) or t=2, s=0 in (40) or (47) or in (48) or t= -1, s=2 in (41) or in (43), one gets

$$r \leqslant \frac{\sqrt[3]{4\chi^2(P;Q;U)} - \sqrt[3]{\Delta(P;Q;U)}}{\sqrt[3]{\Delta(P;Q;U)}} \leqslant R.$$

Take t= -1, s= -1 in (33) or in (35) t= -1, s=0 in (37) or t=2, s= -1 in (41) or t=2, s=0 in (43) or in (45) or t= -1, s=2 in (45) or in (48), one gets

$$r \leqslant \frac{\sqrt[3]{\Delta(P;Q;U)}}{\sqrt[3]{4\chi^2(Q;P;U)} - \sqrt[3]{\Delta(P;Q;U)}} \leqslant R.$$

Take t=0, s=1 in (34), one gets

$$r \leqslant \frac{K(Q;P;U) - 2G(P;Q;U)}{2G(P;Q;U)} \leqslant R.$$

Take t=1, s=1 in (34), one gets

$$r \leqslant \frac{2G(Q;P;U)}{K(P;Q;U) - 2G(P;Q;U)} \leqslant R.$$

Take t=1, s=0 in (34), one gets

$$r \leqslant \frac{\sqrt{K(P;Q;U)} - \sqrt{F(P;Q;U)}}{\sqrt{F(P;Q;U)}} \leqslant R.$$

Take t=0, s=0 in (34), one gets

$$r \leqslant \frac{\sqrt{F(Q;P;U)}}{\sqrt{K(Q;P;U)} - \sqrt{F(Q;P;U)}} \leqslant R.$$

Take t=1, s= -1 in (41) or in t=1, s=0 in (43) or (44), one gets

$$r \leqslant \frac{\sqrt{\Delta(P;Q;U)}}{4\sqrt{G(P;Q;U)} - \sqrt{\Delta(P;Q;U)}} \leqslant R.$$

Take t= -1, s=1 in (41) or t=1, s=0 in (47) or in (48), one gets

$$r \leqslant \frac{4\sqrt{G(Q;P;U)} - \sqrt{\Delta(P;Q;U)}}{\sqrt{\Delta(P;Q;U)}} \leqslant R.$$

Take t= 0, s= -1 in (41 or t=0, s=0 in (43) or in (45), one gets

$$r \leqslant \frac{\Delta(P;Q;U)}{8F(P;Q;U) - \Delta(P;Q;U)} \leqslant R.$$

Take t= -1, s= 0 in (41) or t=0, s=0 in (47) or in (48), one gets

$$r \leqslant \frac{8F(P;Q;U) - \Delta(P;Q;U)}{\Delta(P;Q;U)} \leqslant R.$$

Take t=1, s=1 in (47), one gets

$$r \leqslant \frac{6G(Q;P;U) - D(P;Q;U)}{D(P;Q;U) - 2G(Q;P;U)} \leqslant R.$$

Take t=1, s=1 in (45), one gets

$$r \leqslant \frac{D(Q;P;U) - 2G(P;Q;U)}{6G(P;Q;U) - D(Q;P;U)} \leqslant R.$$

Take t= -1, s= 1 in (33) or t=1, s=2 in (45), one gets

$$r \leqslant \frac{4G(P;Q;U)}{\chi^2(Q;P;U) - 4G(P;Q;U)} \leqslant R.$$

Take t= 0, s= 1 in (33) or t=1, s=2 in (45), one gets

$$r \leqslant \frac{F(P;Q;U)}{D(Q;P;U) - 3F(P;Q;U)} \leqslant R.$$

Take t= -1, s= 0 in (33) or t=0, s=2 in (45), one gets

$$r \leqslant \frac{\sqrt{2F(P;Q;U)}}{\sqrt{\chi^2(Q;P;U)} - \sqrt{2F(P;Q;U)}} \leqslant R.$$

Take t=0, s=1 in (43), one gets

$$r \leqslant \frac{\sqrt{4D(P;Q;U) + 9F(P;Q;U)} - 3\sqrt{F(P;Q;U)}}{2\sqrt{F(P;Q;U)}} \leqslant R.$$

Take t=0, s=1 in (48), one gets

$$r \leqslant \frac{2\sqrt{F(Q;P;U)}}{\sqrt{4D(Q;P;U) + 9F(Q;P;U)} - 3\sqrt{F(Q;P;U)}} \leqslant R.$$

Take t=1, s=0 in (41), one gets

$$r \leqslant \frac{2\sqrt{F(Q;P;U)}}{\sqrt{8G(Q;P;U) + F(Q;P;U)} - \sqrt{F(Q;P;U)}} \leqslant R.$$

Take t=0, s=1 in (41), one gets

$$r \leqslant \frac{\sqrt{8G(Q;P;U) + F(P;Q;U)} - \sqrt{F(P;Q;U)}}{2\sqrt{F(P;Q;U)}} \leqslant R.$$

Take t=0, s=1 in (35), one gets

$$r \leqslant \frac{2\sqrt{G(Q;P;U)}}{\sqrt{2K(Q;P;U) + G(Q;P;U)} - \sqrt{G(Q;P;U)}} \leqslant R.$$

Take t=1, s=1 in (34), one gets

$$r \leqslant \frac{\sqrt{2K(P;Q;U) + G(P;Q;U)} - \sqrt{G(P;Q;U)}}{2\sqrt{G(P;Q;U)}} \leqslant R.$$

Take t= -1, s=1 in (43), one gets

$$r \leqslant \frac{\sqrt{8D(P;Q;U) + \Delta(P;Q;U)} - 2\sqrt{\Delta(P;Q;U)}}{\sqrt{\Delta(P;Q;U)}} \leqslant R.$$

Take t= -1, s=1 in (45) or in (48), one gets

$$r \leqslant \frac{\sqrt{\Delta(P;Q;U)}}{\sqrt{8D(Q;P;U) + \Delta(P;Q;U)} - 2\sqrt{\Delta(P;Q;U)}} \leqslant R.$$

Take t=2, s=1 in (47), one gets

$$r \leqslant \frac{5\sqrt{\chi^2(P;Q;U)} - \sqrt{16D(P;Q;U) + \chi^2(P;Q;U)}}{\sqrt{16D(P;Q;U) + \chi^2(P;Q;U)} - \sqrt{\chi^2(P;Q;U)}} \leqslant R.$$

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