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GENERALIZATIONS AND REFINEMENTS OF STEČKIN–TYPE INEQUALITY FOR TANGENT AND SECANT FUNCTIONS

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Abstract. In this paper, we generalize and refine Stečkin-type inequality for the tangent function. We develop an inequality of Chen and Sándor for the secant function to produce a general form. We also present some refinements of the inequality for the secant function.

1. Introduction

It is known in the literature that, for $0 < x < \pi/2$,

$$\frac{4/\pi}{\pi - 2x} < \frac{\tan x}{x} < \frac{\pi}{\pi - 2x}. \tag{1.1}$$

The left-hand side inequality (1.1) was presented by Stečkin [18], while the right-hand side inequality (1.1) was proved by Ge [14]. This inequality is now known as Stečkin's inequality, see, e.g., [6, p. 246].

Becker and Stark [8] showed that for $0 < x < \pi/2$,

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2}, \tag{1.2}$$

or alternatively

$$\left(\frac{2}{\pi}\right)^2 \frac{2}{1-t^2} < \frac{\tan(\pi t/2)}{\pi t/2} < \frac{1}{1-t^2} \tag{1.3}$$

for $0 < t < 1$.

The inequalities (1.2) are shaper than the inequalities (1.1). The Becker-Stark inequality (1.2) has attracted much interest of many mathematicians and has motivated a large number of research papers (cf. [7, 9, 11, 13, 16, 19, 20, 21, 22] and the references cited therein).

Chen and Elezović [10] gave a unified treatment of the inequalities (1.1) and (1.2) and proved the following result:

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Let $p>0$ be a real number. Consider the following inequalities for $0<x<\pi/2$:

$$\frac{\pi^p}{\pi^p-(2x)^p}<\frac{\tan x}{x}<\frac{4p\pi^{p-2}}{\pi^p-(2x)^p}, \tag{1.4}$$

or alternatively

$$\frac{1}{1-t^p}<\frac{\tan(\pi t/2)}{\pi t/2}<\left(\frac{2}{\pi}\right)^2\frac{p}{1-t^p} \tag{1.5}$$

for $0<t<1$. The left-hand side of (1.5) holds if and only if $p\geqslant \pi^2/4$, while the reversed inequality holds if and only if $0<p\leqslant 2$. The right-hand side of (1.5) holds if and only if $p\geqslant 3$, while the reversed inequality holds if and only if $0<p\leqslant \pi^2/4$.

The choice $p=1$ in (1.4) yields Stečkin's inequality (1.1). The choice $p=2$ in (1.4) yields Becker-Stark inequality (1.2). The choice $p=3$ in (1.4) yields, for $0<x<\pi/2$,

$$\frac{\pi^3}{\pi^3-(2x)^3}<\frac{\tan x}{x}<\frac{12\pi}{\pi^3-(2x)^3}, \tag{1.6}$$

or alternatively

$$\frac{1}{1-t^3}<\frac{\tan(\pi t/2)}{\pi t/2}<\frac{12/\pi^2}{1-t^3} \tag{1.7}$$

for $0<t<1$.

Recently, Debnath [13] et al. refined Stečkin's inequality (1.1) and obtained Theorems 1.1 and 1.2 below.

Theorem 1.1 improves Stečkin's inequality, on a neighborhood of $\pi/2$.

THEOREM 1.1. *For every $x\in(0,\pi/2)$ it holds*

$$\frac{2}{\pi}-\frac{1}{2}\left(\frac{\pi}{2}-x\right)\leqslant \tan x-\frac{4}{\pi}\cdot\frac{x}{\pi-2x}\leqslant \frac{2}{\pi}-\frac{1}{3}\left(\frac{\pi}{2}-x\right). \tag{1.8}$$

Theorem 1.2 presents the following refinement of Stečkin's inequality which gives good results near the origin.

THEOREM 1.2. *For every $x\in(0,1)$, it holds*

$$\left(1-\frac{4}{\pi^2}\right)x-\frac{8}{\pi^3}x^2\leqslant \tan x-\frac{4}{\pi}\cdot\frac{x}{\pi-2x}\leqslant\left(1-\frac{4}{\pi^2}\right)x. \tag{1.9}$$

Chen and Sándor [12, Theorem 3.1(i)] proved in 2015 that for $0<|x|<\pi/2$,

$$\frac{\pi^2}{\pi^2-4x^2}<\sec x<\frac{4\pi}{\pi^2-4x^2}, \tag{1.10}$$

which is an analogous result to (1.2). In 2017, Chen and Paris [11] improved (1.10) and obtained the following inequalities:

$$\frac{\pi^2+\frac{28-8\pi}{\pi}x^2+\frac{16\pi-48}{\pi^3}x^4}{\pi^2-4x^2}<\sec x<\frac{\pi^2-\frac{8-\pi^2}{2}x^2-\frac{4\pi^3-128}{2\pi^3}x^4}{\pi^2-4x^2} \tag{1.11}$$

for $0 < x < \pi/2$.

Nishizawa [17] gave some inequalities with power exponential functions derived from the right hand side of Chen and Sándor's inequality (1.10).

In this paper, we generalize and refine the inequalities (1.1), (1.8) and (1.9). We develop (1.10) to produce a general form. We also present some refinements of (1.10).

The numerical values given have been calculated using the computer program MAPLE 13.

2. Lemmas

The Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined, respectively, by the generating functions:

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (|t| < 2\pi) \quad \text{and} \quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad (|t| < \pi).$$

The numbers $B_n = B_n(0)$ and $E_n = 2^n E_n(\frac{1}{2})$, which are known to be rational numbers and integers, respectively, are called Bernoulli and Euler numbers.

The following lemmas will be useful in our present investigation.

LEMMA 2.1. *The following elementary power series expansions hold (see* [15, pp. 42-43]*):*

$$\tan x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!} x^{2k-1}, \qquad |x| < \frac{\pi}{2}, \tag{2.1}$$

$$\cot x = \frac{1}{x} - \sum_{j=1}^{\infty} \frac{2^{2j}|B_{2j}|}{(2j)!} x^{2j-1}, \qquad |x| < \pi, \tag{2.2}$$

$$\sec x = \sum_{j=0}^{\infty} \frac{|E_{2j}|}{(2j)!} x^{2j}, \qquad |x| < \frac{\pi}{2}, \tag{2.3}$$

$$\csc x = \frac{1}{x} + \sum_{j=1}^{\infty} \frac{(2^{2j}-2)|B_{2j}|}{(2j)!} x^{2j-1}, \qquad |x| < \pi. \tag{2.4}$$

LEMMA 2.2. *For all* $n \in \mathbb{N} := \{1, 2, \ldots\}$,

$$\frac{2}{(2\pi)^{2n}(1-2^{\alpha-2n})} < \frac{|B_{2n}|}{(2n)!} \leqslant \frac{2}{(2\pi)^{2n}\left(1-2^{\beta-2n}\right)}, \tag{2.5}$$

with the best possible constants

$$\alpha = 0 \quad \textit{and} \quad \beta = 2 + \frac{\ln(1-6/\pi^2)}{\ln 2} = 0.6491\ldots. \tag{2.6}$$

Lemma 2.2 was proved by Alzer [2]. Lemma 2.2 improves the following inequalities (see [1, p. 805]):

$$\frac{2}{(2\pi)^{2n}} < \frac{|B_{2n}|}{(2n)!} < \frac{2}{(2\pi)^{2n}\left(1-2^{1-2n}\right)}, \qquad n \in \mathbb{N}. \tag{2.7}$$

LEMMA 2.3. *For all* $n \in \mathbb{N}$,

$$\frac{\pi^2(2^{2n+2}-1)}{(2n+2)!}|B_{2n+2}| < \frac{(2^{2n}-1)}{(2n)!}|B_{2n}|. \tag{2.8}$$

The inequality (2.8) can be found in [22].

LEMMA 2.4. *For all* $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,

$$\frac{4^{n+1}}{\pi^{2n+1}}\left(\frac{1}{1+3^{-1-2n}}\right) < \frac{|E_{2n}|}{(2n)!} < \frac{4^{n+1}}{\pi^{2n+1}}. \tag{2.9}$$

The inequality (2.9) can be found in [1, p. 805].

LEMMA 2.5. *For all* $n \in \mathbb{N}$,

$$\frac{|E_{2n}|}{(2n)!} > \left(\frac{2}{\pi}\right)^2 \frac{|E_{2n-2}|}{(2n-2)!}. \tag{2.10}$$

Proof. It is well known that

$$(-1)^n E_{2n} > 0.$$

Using the series expansion (see [15, p. 592])

$$E_{2n}(x) = (-1)^n \frac{4(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin\left((2k+1)\pi x\right)}{(2k+1)^{2n+1}},$$

we have

$$\frac{|E_{2n}|}{(2n)!} = \frac{4^{n+1}}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}}.$$

We then obtain that

$$\begin{aligned}\frac{\pi^2|E_{2n}|}{(2n)!} - \frac{4|E_{2n-2}|}{(2n-2)!} &= \frac{4^{n+1}}{\pi^{2n-1}} \sum_{k=1}^{\infty} (-1)^{k-1} \left\{\frac{1}{(2k+1)^{2n-1}} - \frac{1}{(2k+1)^{2n+1}}\right\} \\ &= \frac{4^{n+1}}{\pi^{2n-1}} \sum_{k=1}^{\infty} (-1)^{k-1} u_k(n),\end{aligned}$$

where

$$u_k(n) = \frac{4k(k+1)}{(2k+1)^{2n+1}}, \qquad k,n \in \mathbb{N}.$$

We find that

$$\begin{aligned} u_k(n) - u_{k+1}(n) &= \frac{4k(k+1)}{(2k+1)^{2n+1}} - \frac{4(k+1)(k+2)}{(2k+3)^{2n+1}} \\ &= \frac{4k(k+1)}{(2k+3)^{2n+1}} \left\{ \left(1+\frac{2}{2k+1}\right)^{2n+1} - \left(1+\frac{2}{k}\right) \right\} \\ &> \frac{4k(k+1)}{(2k+3)^{2n+1}} \left\{ \left(1+\frac{2}{2k+1}\right)^{3} - \left(1+\frac{2}{k}\right) \right\} \\ &= \frac{8(k+1)^2(4k^2+8k-1)}{(2k+1)^3(2k+3)^{2n+1}} > 0. \end{aligned}$$

We then obtain that

$$\frac{\pi^2|E_{2n}|}{(2n)!} - \frac{4|E_{2n-2}|}{(2n-2)!} = \frac{4^{n+1}}{\pi^{2n-1}} \left\{ \Big(u_1(n) - u_2(n)\Big) + \Big(u_3(n) - u_4(n)\Big) + \ldots \right\} > 0.$$

The proof is complete. □

LEMMA 2.6. (see [3, 4, 5]) *Let* $-\infty < a < b < \infty$*, and let* $f,\ g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$*, differentiable on* (a,b)*. Let* $g'(x) \neq 0$ *on* (a,b)*. If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b)*, then so are*

$$[f(x) - f(a)]/[g(x) - g(a)] \text{ and } [f(x) - f(b)]/[g(x) - g(b)].$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

3. Generalizations and refinements of (1.8) and (1.9)

Using the expansion (2.2), we find that, for $0 < t < \pi/2$,

$$\cot t - \frac{4}{\pi} \cdot \frac{\frac{\pi}{2} - t}{\pi - 2(\frac{\pi}{2} - t)} = \cot t - \frac{1}{t} + \frac{2}{\pi} = \frac{2}{\pi} - \sum_{j=1}^{\infty} \frac{2^{2j}|B_{2j}|}{(2j)!} t^{2j-1}. \tag{3.1}$$

Replacing t by $\frac{\pi}{2} - x$ in (3.1), we obtain that, for $0 < x < \pi/2$,

$$\tan x - \frac{4}{\pi} \cdot \frac{x}{\pi - 2x} = \frac{2}{\pi} - \sum_{j=1}^{\infty} \frac{2^{2j}|B_{2j}|}{(2j)!} \left(\frac{\pi}{2} - x\right)^{2j-1}, \tag{3.2}$$

that is,

$$\tan x - \frac{4}{\pi} \cdot \frac{x}{\pi - 2x} = \frac{2}{\pi} - \frac{1}{3}\left(\frac{\pi}{2} - x\right) - \frac{1}{45}\left(\frac{\pi}{2} - x\right)^3 - \frac{2}{945}\left(\frac{\pi}{2} - x\right)^5 - \ldots. \tag{3.3}$$

REMARK 3.1. It follows by truncation of (3.2) that

$$\tan x - \frac{4}{\pi}\cdot\frac{x}{\pi-2x} < \frac{2}{\pi} - \sum_{j=1}^{N}\frac{2^{2j}|B_{2j}|}{(2j)!}\left(\frac{\pi}{2}-x\right)^{2j-1} \tag{3.4}$$

for $0 < t < \pi/2$ and $N \in \mathbb{N}$. This improves the upper bound of (1.8).

Theorem 3.1 improves the lower bound of (1.8).

THEOREM 3.1. *The following inequalities hold:*

$$\frac{2}{\pi} - \frac{4}{\pi^2}\left(\frac{\pi}{2}-x\right) < \tan x - \frac{4}{\pi}\cdot\frac{x}{\pi-2x} < \frac{2}{\pi} - \frac{1}{3}\left(\frac{\pi}{2}-x\right), \qquad 0 < x < \frac{\pi}{2}, \tag{3.5}$$

and the constants $\frac{4}{\pi^2}$ *and* $\frac{1}{3}$ *are the best possible.*

Proof. Clearly, the right-hand side of (3.5) holds. We now prove the left-hand side of (3.5). Replacing x by $\frac{\pi}{2}-x$ in the left-hand side of (3.5) leads to equivalent inequality:

$$\cot x - \frac{1}{x} + \frac{4}{\pi^2}x > 0, \qquad 0 < x < \frac{\pi}{2}. \tag{3.6}$$

After some elementary computations, (3.6) can be rewritten as the right-hand side of (1.2).

If we write (3.5) as

$$\frac{4}{\pi^2} > \frac{\tan x - \frac{4}{\pi}\cdot\frac{x}{\pi-2x} - \frac{2}{\pi}}{x-\frac{\pi}{2}} > \frac{1}{3},$$

we find that

$$\lim_{x\to 0^+}\frac{\tan x - \frac{4}{\pi}\cdot\frac{x}{\pi-2x} - \frac{2}{\pi}}{x-\frac{\pi}{2}} = \frac{4}{\pi^2} = 0.40528\ldots \quad \text{and} \quad \lim_{x\to\pi/2^-}\frac{\tan x - \frac{4}{\pi}\cdot\frac{x}{\pi-2x} - \frac{2}{\pi}}{x-\frac{\pi}{2}} = \frac{1}{3}.$$

Hence, the inequalities (3.5) hold, and the constants $\frac{4}{\pi^2}$ and $\frac{1}{3}$ are the best possible. The proof is complete. □

We obtain by (2.1) that, for $0 < x < \pi/2$,

$$\begin{aligned}\tan x - \frac{4}{\pi}\cdot\frac{x}{\pi-2x} &= \tan x - \frac{4x}{\pi^2}\cdot\frac{1}{1-\frac{2x}{\pi}}\\ &= \sum_{j=1}^{\infty}\frac{2^{2j}(2^{2j}-1)|B_{2j}|}{(2j)!}x^{2j-1} - \sum_{j=1}^{\infty}\left(\frac{2}{\pi}\right)^{j+1}x^j = \sum_{j=1}^{\infty}a_j x^{2j-1} - \sum_{j=1}^{\infty}b_j x^j\\ &= (a_1-b_1)x - b_2x^2 + (a_2-b_3)x^3 - b_4x^4 + \cdots + (a_j - b_{2j-1})x^{2j-1} - b_{2j}x^{2j} + \cdots,\end{aligned} \tag{3.7}$$

where

$$a_j = \frac{2^{2j}(2^{2j}-1)|B_{2j}|}{(2j)!} \quad \text{and} \quad b_j = \left(\frac{2}{\pi}\right)^{j+1}. \tag{3.8}$$

That is,

$$\tan x - \frac{4}{\pi}\cdot\frac{x}{\pi-2x} = \left(1-\frac{4}{\pi^2}\right)x - \frac{8}{\pi^3}x^2 + \left(\frac{1}{3}-\frac{16}{\pi^4}\right)x^3 - \frac{32}{\pi^5}x^4 + \ldots \tag{3.9}$$

for $0 < x < \pi/2$.

We find by the left-hand side of (2.5) that, for $j \in \mathbb{N}$,

$$a_j - b_{2j-1} > \frac{2^{2j+1}(2^{2j}-1)}{(2\pi)^{2j}(1-2^{-2j})} - \left(\frac{2}{\pi}\right)^{2j} = \left(\frac{2}{\pi}\right)^{2j} > 0 \tag{3.10}$$

so that (3.7) is an alternating series for $0 < x < \pi/2$. This fact motivated us to establish Theorem 3.2. Theorem 3.2 develops Theorem 1.2 to produce a general result.

THEOREM 3.2. *For all $0 < x < \pi/2$ and $N \in \mathbb{N}$, the following inequalities hold true:*

$$\sum_{j=1}^{N}\left\{-b_{2j-2}x^{2j-2} + (a_j - b_{2j-1})x^{2j-1}\right\} - b_{2N}x^{2N}$$
$$< \tan x - \frac{4}{\pi}\cdot\frac{x}{\pi-2x} < \sum_{j=1}^{N}\left\{-b_{2j-2}x^{2j-2} + (a_j - b_{2j-1})x^{2j-1}\right\}, \tag{3.11}$$

where $b_0 = 0$,

$$a_j = \frac{2^{2j}(2^{2j}-1)|B_{2j}|}{(2j)!} \quad \textit{and} \quad b_j = \left(\frac{2}{\pi}\right)^{j+1} \quad \textit{for} \quad j \in \mathbb{N}.$$

Proof. If we write (3.7) as

$$\tan x - \frac{4}{\pi}\cdot\frac{x}{\pi-2x} = \Big((a_1-b_1)x - b_2x^2\Big) + \Big((a_2-b_3)x^3 - b_4x^4\Big)$$
$$+\cdots+\Big((a_j-b_{2j-1})x^{2j-1} - b_{2j}x^{2j}\Big) + \cdots, \tag{3.12}$$

we find by (3.10) that, for $0 < x < \pi/2$ and $j \in \mathbb{N}$,

$$(a_j - b_{2j-1})x^{2j-1} - b_{2j}x^{2j} = \Big(a_j - b_{2j-1} - b_{2j}x\Big)x^{2j-1} > 0.$$

We then obtain the left-hand side of (3.11) by truncation of (3.12).

We now prove the right-hand side of (3.11). Let us denote

$$S(x) := \tan x - \frac{4}{\pi}\cdot\frac{x}{\pi-2x}, \tag{3.13}$$

$b_0 = 0$, and

$$c_n(x) = -b_{2n-2}x^{2n-2} + (a_n - b_{2n-1})x^{2n-1}, \qquad n \geqslant 1.$$

We should prove

$$S(x) < S_N(x) := \sum_{k=1}^{N} c_k(x), \qquad \forall N, \quad \forall 0 < x < \frac{\pi}{2}. \tag{3.14}$$

1) It holds $S(x) < c_1(x)$, that is,

$$\tan x - \frac{4}{\pi} \cdot \frac{x}{\pi - 2x} < \left(1 - \frac{4}{\pi^2}\right)x, \qquad 0 < x < \frac{\pi}{2}. \tag{3.15}$$

Using the right-hand side of (1.2), we find that for $0 < x < \pi/2$,

$$\begin{aligned}\frac{\tan x}{x} - \frac{4}{\pi} \cdot \frac{1}{\pi - 2x} - \left(1 - \frac{4}{\pi^2}\right) &< \frac{\pi^2}{\pi^2 - 4x^2} - \frac{4}{\pi} \cdot \frac{1}{\pi - 2x} - \left(1 - \frac{4}{\pi^2}\right) \\ &= -\frac{4x\Big(2\pi - (\pi^2 - 4)x\Big)}{\pi^2(\pi^2 - 4x^2)} < 0.\end{aligned}$$

This proves $S(x) < c_1(x)$.

2) Relation $c_n(x) < 0$ is equivalent to

$$\frac{a_n}{b_{2n-1}} < \frac{\pi}{2x} + 1. \tag{3.16}$$

Using (2.5) we have

$$\frac{a_n}{b_{2n-1}} < 2 + \frac{2^{\beta+1} - 2}{4^n - 2^\beta} \downarrow 2 \quad \text{as} \quad n \to \infty.$$

For any $x \in (0, \pi/2)$, we see that (3.16) is true for sufficiently large n. Therefore, for each $x \in (0, \pi/2)$ the sequence $(c_n(x))$ became eventually negative for sufficiently large n.

3) The sequence (c_n) changes its sign at most once. We have, using Lemma 2.3.

$$\begin{aligned} c_{n+1}(x) &= -b_{2n}x^{2n} + (a_{n+1} - b_{2n+1})x^{2n+1} \\ &= -\frac{2}{\pi}\left(\frac{2x}{\pi}\right)^{2n} + \left(\frac{4^{n+1}(4^{n+1}-1)|B_{2n+2}|}{(2n+2)!} - \left(\frac{2}{\pi}\right)^{2n+2}\right)x^{2n+1} \\ &< -\frac{2}{\pi}\left(\frac{2x}{\pi}\right)^{2n} + \left(\frac{4}{\pi^2} \cdot \frac{4^n(4^n-1)|B_{2n}|}{(2n)!} - \left(\frac{2}{\pi}\right)^{2n+2}\right)x^{2n+1} \\ &= \frac{4x^2}{\pi^2} c_n(x). \end{aligned}$$

Therefore, (c_n) is decreasing while being positive, and remains negative once it takes negative value.

Denote by $n^*(x)$ the first index n for which $c_n(x) < 0$. Then, $c_k(x) \geqslant 0$ for $k < n^*(x)$ and $c_k(x) < 0$ for $k \geqslant n^*(x)$.

4) We have, $\forall N$,

$$S(x) = \sum_{k=1}^{\infty} c_k(x) = S_N(x) + \sum_{k=N+1}^{\infty} c_k(x).$$

Let us take any x, $0 < x < \pi/2$.

a) If $n^*(x) \leqslant N+1$, then $c_k(x) < 0$ for each $k \geqslant N+1$, therefore $S(x) < S_N(x)$.

b) If $n^*(x) > N+1$, then $c_k(x) > 0$ for each $k \leqslant N$, therefore $S(x) < S_N(x)$ since $S(x) < c_1(x)$.

Theorem is complete. □

REMARK 3.2. For every $j \in \mathbb{N}$, the function

$$x \longmapsto -b_{2j}x^{2j} + (a_{j+1} - b_{2j+1})x^{2j+1}$$

change its sign on $(0, \pi/2)$.

4. New results related to Stečkin's inequality

Noting that

$$\lim_{x\to 0} \frac{\tan x}{x}(\pi - 2x) = \pi$$

holds, we consider the expansion of function $\frac{\tan x}{x} - \frac{\pi}{\pi - 2x}$ near the origin. Using (2.1), we have

$$\begin{aligned}
\frac{\tan x}{x} - \frac{\pi}{\pi - 2x} &= \frac{\tan x}{x} - \frac{1}{1 - \frac{2}{\pi}x} \\
&= \sum_{j=1}^{\infty} \frac{2^{2j}(2^{2j}-1)|B_{2j}|}{(2j)!}x^{2j-2} - \sum_{j=0}^{\infty} \left(\frac{2}{\pi}\right)^j x^j \\
&= \sum_{j=1}^{\infty} \frac{2^{2j+2}(2^{2j+2}-1)|B_{2j+2}|}{(2j+2)!}x^{2j} - \sum_{j=1}^{\infty} \left(\frac{2}{\pi}\right)^j x^j \\
&= \sum_{j=1}^{\infty} \left\{ \frac{2^{2j+2}(2^{2j+2}-1)|B_{2j+2}|}{(2j+2)!} - \left(\frac{2}{\pi}\right)^{2j} \right\} x^{2j} - \sum_{j=1}^{\infty} \left(\frac{2}{\pi}\right)^{2j-1} x^{2j-1},
\end{aligned} \tag{4.1}$$

that is,

$$\frac{\tan x}{x} - \frac{\pi}{\pi - 2x} = -\frac{2}{\pi}x - \frac{12 - \pi^2}{3\pi^2}x^2 - \frac{8}{\pi^3}x^3 - \frac{2(120 - \pi^4)}{15\pi^4}x^4 - \dots. \tag{4.2}$$

We find from (2.7) that

$$\frac{2^{2j+2}(2^{2j+2}-1)|B_{2j+2}|}{(2j+2)!} < \frac{2\cdot 2^{2j+2}(2^{2j+2}-1)}{(2\pi)^{2j+2}\left(1-2^{1-2(j+1)}\right)}$$
$$= \left(\frac{2}{\pi}\right)^{2j}\left(\frac{2}{\pi}\right)^{2} 2\left(1+\frac{1}{2^{2j+2}-2}\right) < \left(\frac{2}{\pi}\right)^{2j}\left(\frac{2}{\pi}\right)^{2} 2\left(1+\frac{1}{2^{4}-2}\right)$$
$$= \left(\frac{2}{\pi}\right)^{2j}\frac{60}{7\pi^2} < \left(\frac{2}{\pi}\right)^{2j}, \qquad j\in\mathbb{N}. \tag{4.3}$$

We then obtain Theorem 4.1 by truncation of (4.1).

THEOREM 4.1. *For* $0<x<\pi/2$ *and* $N\in\mathbb{N}$,

$$\frac{\tan x}{x} < \frac{\pi}{\pi-2x} - \sum_{j=1}^{N}\left(\frac{2}{\pi}\right)^{2j-1} x^{2j-1}$$
$$- \sum_{j=1}^{N}\left\{\left(\frac{2}{\pi}\right)^{2j} - \frac{2^{2j+2}(2^{2j+2}-1)|B_{2j+2}|}{(2j+2)!}\right\}x^{2j}. \tag{4.4}$$

The choice $N=1$ in (4.4) yields

$$\frac{\tan x}{x} < \frac{\pi}{\pi-2x} - \frac{2}{\pi}x - \frac{12-\pi^2}{3\pi^2}x^2. \tag{4.5}$$

Noting that

$$\lim_{x\to\pi/2^-}\frac{\tan x}{x}(\pi-2x) = \frac{4}{\pi}$$

holds, we consider the expansion of function $\frac{\tan x}{x} - \frac{4/\pi}{\pi-2x}$, on a neighborhood of $\pi/2$.

As x approaches $\pi/2$, with $x<\pi/2$, we find by Maple,

$$\frac{\tan x}{x} - \frac{4/\pi}{\pi-2x} = \frac{4}{\pi^2} + \frac{2(12-\pi^2)}{3\pi^3}\left(\frac{\pi}{2}-x\right) + \frac{4(12-\pi^2)}{3\pi^4}\left(\frac{\pi}{2}-x\right)^2$$
$$+ \frac{2(720-\pi^4-60\pi^2)}{45\pi^5}\left(\frac{\pi}{2}-x\right)^3 + \frac{4(720-\pi^4-60\pi^2)}{45\pi^6}\left(\frac{\pi}{2}-x\right)^4 + \dots. \tag{4.6}$$

Even though we can obtain as many coefficients as we please in the right-hand side of (4.6) by using Maple, here we aim at giving a formula for determining these coefficients.

Replacing x by $\frac{\pi}{2}-t$, it is sufficient to consider the expansion of function $\frac{\cot t}{\frac{\pi}{2}-t} - \frac{2}{\pi t}$ near the origin.

For $0<t<\pi/2$, we have

$$\frac{1}{\frac{\pi}{2}-t} = \frac{2}{\pi}\frac{1}{1-\frac{2}{\pi}t} = \sum_{j=0}^{\infty}\left(\frac{2}{\pi}\right)^{j+1} t^j. \tag{4.7}$$

We obtain from (2.2) and (4.7) that, for $0 < t < \pi/2$,

$$
\begin{aligned}
\frac{\cot t}{\frac{\pi}{2}-t}-\frac{2}{\pi t} &= \sum_{j=0}^{\infty}\left(\frac{2}{\pi}\right)^{j+2} t^j - \left(\sum_{j=0}^{\infty}\frac{2^{2j+2}|B_{2j+2}|}{(2j+2)!}t^{2j+1}\right)\sum_{j=0}^{\infty}\left(\frac{2}{\pi}\right)^{j+1} t^j \\
&= \sum_{j=0}^{\infty}\left(\frac{2}{\pi}\right)^{j+2} t^j - \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{2^{2k+2}|B_{2k+2}|}{(2k+2)!}\left(\frac{2}{\pi}\right)^{j-k+1} t^{j+k+1} \\
&= \sum_{\ell=0}^{\infty}\left(\frac{2}{\pi}\right)^{\ell+2} t^\ell - \sum_{\ell=1}^{\infty}\sum_{k=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\frac{2^{2k+2}|B_{2k+2}|}{(2k+2)!}\left(\frac{2}{\pi}\right)^{\ell-2k} t^{\ell} \\
&= \frac{4}{\pi^2}+\sum_{\ell=1}^{\infty}\left\{\left(\frac{2}{\pi}\right)^{\ell+2} - \sum_{k=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\frac{2^{2k+2}|B_{2k+2}|}{(2k+2)!}\left(\frac{2}{\pi}\right)^{\ell-2k}\right\} t^{\ell},
\end{aligned}
$$

or alternatively

$$
\frac{\cot t}{\frac{\pi}{2}-t}-\frac{2}{\pi t} = \frac{4}{\pi^2}+\sum_{\ell=1}^{\infty} c_\ell t^\ell, \tag{4.8}
$$

where

$$
c_\ell = \left(\frac{2}{\pi}\right)^{\ell+2} - \sum_{k=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\frac{2^{2k+2}|B_{2k+2}|}{(2k+2)!}\left(\frac{2}{\pi}\right)^{\ell-2k}, \qquad \ell\in\mathbb{N}. \tag{4.9}
$$

Setting $\ell = 2j+1$ and $\ell = 2j$ in (4.9), respectively, yields

$$
\begin{aligned}
c_{2j+1} &= \left(\frac{2}{\pi}\right)^{2j+3} - \sum_{k=0}^{j}\frac{2^{2k+2}|B_{2k+2}|}{(2k+2)!}\left(\frac{2}{\pi}\right)^{2j-2k+1} \\
&= \left(\frac{2}{\pi}\right)^{2j+3}\left\{1-\sum_{k=0}^{j}\frac{\pi^{2k+2}|B_{2k+2}|}{(2k+2)!}\right\}, \qquad j\in\mathbb{N}_0
\end{aligned} \tag{4.10}
$$

and

$$
\begin{aligned}
c_{2j} &= \left(\frac{2}{\pi}\right)^{2j+2} - \sum_{k=0}^{j-1}\frac{2^{2k+2}|B_{2k+2}|}{(2k+2)!}\left(\frac{2}{\pi}\right)^{2j-2k} \\
&= \left(\frac{2}{\pi}\right)^{2j+2}\left\{1-\sum_{k=1}^{j}\frac{\pi^{2k}|B_{2k}|}{(2k)!}\right\}, \qquad j\in\mathbb{N}_0
\end{aligned} \tag{4.11}
$$

(an empty sum is understood to be zero).

Uses of (4.10) and (4.11) are easily seen to generate the values

$$
\begin{aligned}
&c_0 = \frac{4}{\pi^2}, \quad c_1 = \frac{2(12-\pi^2)}{3\pi^3}, \quad c_2 = \frac{4(12-\pi^2)}{3\pi^4}, \\
&c_3 = \frac{2(720-\pi^4-60\pi^2)}{45\pi^5}, \quad c_4 = \frac{4(720-\pi^4-60\pi^2)}{45\pi^6}, \dots,
\end{aligned}
$$

which are the same coefficients as in (4.6).

Replacing t by $\frac{\pi}{2}-x$ in (4.8), we obtain Theorem 4.2.

THEOREM 4.2. *For $0<x<\pi/2$, we have*

$$\frac{\tan x}{x}-\frac{4/\pi}{\pi-2x}=\sum_{\ell=0}^{\infty}c_\ell\left(\frac{\pi}{2}-x\right)^\ell, \tag{4.12}$$

where the coefficients c_ℓ can be calculated using (4.10) *and* (4.11).

THEOREM 4.3. *For all $\ell\in\mathbb{N}_0$, $c_\ell>0$.*

Proof. In order to prove Theorem 4.3, it is sufficient to prove that

$$\sum_{k=1}^{j}\frac{\pi^{2k}|B_{2k}|}{(2k)!}<1, \quad j\in\mathbb{N}. \tag{4.13}$$

Since

$$\frac{\pi^{2k}|B_{2k}|}{(2k)!}=\frac{2\zeta(2k)}{4^k},$$

where $\zeta(s)$ denotes the zeta function, it is sufficient to prove that

$$S=\sum_{k=1}^{\infty}\frac{\zeta(2k)}{4^k}=\frac{1}{2}.$$

Now, interchanging the order of summation we have

$$S=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j^{2k}4^k}=\sum_{j=1}^{\infty}\frac{1}{4j^2-1}=\frac{1}{2}\sum_{j=1}^{\infty}\left(\frac{1}{2j-1}-\frac{1}{2j+1}\right)=\frac{1}{2}.$$

The proof is complete. □

COROLLARY 4.1. *For $0<x<\pi/2$ and $N\in\mathbb{N}_0$,*

$$\sum_{\ell=0}^{N}c_\ell\left(\frac{\pi}{2}-x\right)^\ell<\frac{\tan x}{x}-\frac{4/\pi}{\pi-2x}, \tag{4.14}$$

where the coefficients c_ℓ can be calculated using (4.10) *and* (4.11).

The choice $N=1$ in (4.14) yields

$$\frac{4/\pi}{\pi-2x}+\frac{4}{\pi^2}+\frac{2(12-\pi^2)}{3\pi^3}\left(\frac{\pi}{2}-x\right)<\frac{\tan x}{x}. \tag{4.15}$$

5. A general form of (1.10)

Theorem 5.1 below develops (1.10) to produce a general form.

THEOREM 5.1. *Let $p>0$ be a real number. Consider the following inequalities:*

$$\frac{\pi^p}{\pi^p-(2x)^p}<\sec x<\frac{2p\pi^{p-1}}{\pi^p-(2x)^p} \tag{5.1}$$

for $0<x<\pi/2$, or alternatively

$$\frac{1}{1-t^p}<\sec\frac{\pi t}{2}<\frac{(\frac{2}{\pi})p}{1-t^p} \tag{5.2}$$

for $0<t<1$. The left-hand side of (5.2) holds if and only if $p\geqslant 2$, while the reversed inequality holds if and only if $0<p\leqslant\pi/2$. The right-hand side of (5.2) holds if and only if $p\geqslant\pi/2$, while the reversed inequality holds if and only if $0<p\leqslant 1$.

Proof. The left-hand side of (5.2) can be written for $p>0$ as

$$\frac{\ln\left(1-\cos\frac{\pi t}{2}\right)}{\ln t}<p,\qquad 0<t<1.$$

For $0<t<1$, let

$$f_1(t)=\ln\left(1-\cos\frac{\pi t}{2}\right)\quad\text{and}\quad f_2(t)=\ln t,$$

and let

$$f(t)=\frac{f_1(t)}{f_2(t)}=\frac{\ln\left(1-\cos\frac{\pi t}{2}\right)}{\ln t}.$$

Then,

$$\frac{f_1'(t)}{f_2'(t)}=\frac{\pi}{2}\frac{t\sin(\frac{\pi t}{2})}{1-\cos(\frac{\pi t}{2})}=:g(t).$$

Differentiation yields

$$g'(t)=-\frac{\pi}{2}\cdot\frac{\frac{\pi t}{2}-\sin(\frac{\pi t}{2})}{1-\cos(\frac{\pi t}{2})}<0.$$

Therefore, the functions $g(t)$ and $f_1'(t)/f_2'(t)$ are strictly decreasing on $(0,1)$. By Lemma 2.6, the function

$$f(t)=\frac{f_1(t)}{f_2(t)}=\frac{f_1(t)-f_1(1)}{f_2(t)-f_2(1)}$$

is strictly decreasing on $(0,1)$. And hence, we have, $0<t<1$,

$$\frac{\pi}{2}=\lim_{u\to 1^-} f(u)<f(t)=\frac{\ln\left(1-\cos\frac{\pi t}{2}\right)}{\ln t}<\lim_{u\to 0^+} f(u)=2. \tag{5.3}$$

Hence, the left-hand side of (5.2) holds for $0<t<1$ if and only if $p\geqslant 2$, while the reversed inequality holds if and only if $0<p\leqslant \pi/2$.

By (5.3), we have, for $0<t<1$,

$$\frac{1}{1-t^{p_1}}<\sec\frac{\pi t}{2}<\frac{1}{1-t^{p_2}}, \tag{5.4}$$

where the constants $p_1=2$ and $p_2=\pi/2$ are the best possible, in the sense that $p_1=2$ can not be replaced by a smaller number, and $p_2=\pi/2$ can not be replaced by a larger number.

By the right-hand side of (5.4) and the monotonically increasing property of function $p\mapsto \frac{p}{1-t^p}$ (for $p\in\mathbb{R}$), we obtain that, for $p\geqslant \pi/2$,

$$\sec\frac{\pi t}{2}<\frac{1}{1-t^{\pi/2}}\leqslant\frac{(\frac{2}{\pi})p}{1-t^p}. \tag{5.5}$$

This shows that, for $p\geqslant\pi/2$, the right-hand side of (5.2) holds for $0<t<1$.

As t approaches 0, with $t>0$, we find that

$$\sec\frac{\pi t}{2}-\frac{(\frac{2}{\pi})p}{1-t^p}=\frac{\pi-2p}{\pi}+\frac{1}{8}\pi^2t^2-\frac{2p}{\pi}t^{2p}+\dots.$$

It then follows that it is necessary to have $p\geqslant\pi/2$ for $\sec\frac{\pi t}{2}-\frac{(\frac{2}{\pi})p}{1-t^p}$ to be negative on $(0,1)$. Hence, the right-hand side of (5.2) holds if and only if $p\geqslant\pi/2$.

We now show that the right-hand side of (5.2) is reversed if and only if $0<p\leqslant 1$. We first prove that

$$\frac{\frac{2}{\pi}}{1-t}<\sec\frac{\pi t}{2}. \tag{5.6}$$

Replacing t by $1-u$ leads to equivalent inequality:

$$\sin\frac{\pi u}{2}<\frac{\pi u}{2},\qquad 0<u<1,$$

which is true. Hence, (5.6) holds.

By (5.6) and the monotonically increasing property of function $p\mapsto\frac{p}{1-t^p}$ (for $p\in\mathbb{R}$), we obtain that, for $0<p\leqslant 1$,

$$\frac{(\frac{2}{\pi})p}{1-t^p}\leqslant\frac{\frac{2}{\pi}}{1-t}<\sec\frac{\pi t}{2}.$$

This shows that, for $0<p\leqslant 1$, the right-hand side of (5.2) is reversed.

As t approaches 1, with $t<1$, we find that

$$\sec\frac{\pi t}{2}-\frac{(\frac{2}{\pi})p}{1-t^p}=\frac{1-p}{\pi}+\frac{2+\pi^2-2p^2}{12\pi}(1-t)+O\Big((1-t)^2\Big).$$

It then follows that it is necessary to have $p\leqslant 1$ for $\sec\frac{\pi t}{2}-\frac{(\frac{2}{\pi})p}{1-t^p}$ to be positive on $(0,1)$. Hence, the right-hand side of (5.2) is reversed for $0<t<1$ if and only if $0<p\leqslant 1$. The proof is complete. □

REMARK 5.1. In order to ensure that the lower bound of (5.2) is positive, we restrict $p>0$. In Theorem 5.1, we do not think about the case $p=0$, since

$$\lim_{p\to 0^+}\frac{1}{1-t^p}=\infty.$$

REMARK 5.2. Computing limit of the upper bound in (5.2) yields

$$\lim_{p\to 0}\frac{(\frac{2}{\pi})p}{1-t^p}=\frac{2}{\pi\ln(1/t)}. \tag{5.7}$$

For $p=0$, the right-hand side of (5.2) is reversed, which is understood as

$$\sec\frac{\pi t}{2}>\frac{2}{\pi\ln(1/t)},\qquad 0<t<1. \tag{5.8}$$

In fact, the right-hand side of (5.2) is reversed for all $p\leqslant 1$.

6. Refinements of (1.10)

Noting that

$$\lim_{x\to 0}\sec x\big(\pi^2-4x^2\big)=\pi^2$$

holds, we consider the expansion of function $\sec x-\frac{\pi^2}{\pi^2-4x^2}$ near the origin.

Using (2.3), we have

$$\begin{aligned}\sec x-\frac{\pi^2}{\pi^2-4x^2}&=\sum_{j=0}^{\infty}\frac{|E_{2j}|}{(2j)!}x^{2j}-\frac{1}{1-(\frac{2}{\pi}x)^2}=\sum_{j=0}^{\infty}\frac{|E_{2j}|}{(2j)!}x^{2j}-\sum_{j=0}^{\infty}\left(\frac{2}{\pi}\right)^{2j}x^{2j}\\&=\sum_{j=1}^{\infty}\left\{\frac{|E_{2j}|}{(2j)!}-\left(\frac{2}{\pi}\right)^{2j}\right\}x^{2j},\end{aligned} \tag{6.1}$$

that is,

$$\sec x-\frac{\pi^2}{\pi^2-4x^2}=\frac{\pi^2-8}{2\pi^2}x^2+\frac{5\pi^4-384}{24\pi^4}x^4+\frac{61\pi^6-46080}{720\pi^6}+\dots. \tag{6.2}$$

We find by the left-hand side of (2.9) that

$$\frac{|E_{2j}|}{(2j)!} - \left(\frac{2}{\pi}\right)^{2j} > \frac{4^{j+1}}{\pi^{2j+1}}\left(\frac{1}{1+3^{-1-2j}}\right) - \left(\frac{2}{\pi}\right)^{2j}$$
$$= \left(\frac{2}{\pi}\right)^{2j} \frac{(4-\pi)3^{2n+1}-\pi}{\pi(3^{2j+1}+1)} > 0, \qquad j \in \mathbb{N}. \tag{6.3}$$

We then obtain Theorem 6.1 by truncation of (6.1).

THEOREM 6.1. *For* $0 < x < \pi/2$ *and* $N \in \mathbb{N}$,

$$\frac{\pi^2}{\pi^2 - 4x^2} + \sum_{j=1}^{N} \left\{ \frac{|E_{2j}|}{(2j)!} - \left(\frac{2}{\pi}\right)^{2j} \right\} x^{2j} < \sec x. \tag{6.4}$$

The inequality (6.4) improves the lower bounds of (1.10).

Noting that

$$\lim_{x \to \pi/2^-} \sec x \left(\pi^2 - 4x^2\right) = 4\pi$$

holds, we consider the expansion of function $\sec x - \frac{4\pi}{\pi^2 - 4x^2}$, on a neighborhood of $\pi/2$. Replacing x by $\frac{\pi}{2} - t$, it is sufficient to consider the expansion of function $\csc t - \frac{4\pi}{\pi^2 - 4(\frac{\pi}{2} - t)^2}$ near the origin.

Using (2.4), we have, for $0 < t < \pi/2$,

$$\begin{aligned}
\csc t - \frac{4\pi}{\pi^2 - 4(\frac{\pi}{2} - t)^2} &= \csc t - \frac{1}{t(1 - \frac{t}{\pi})} \\
&= \frac{1}{t} + \sum_{j=1}^{\infty} \frac{(2^{2j} - 2)|B_{2j}|}{(2j)!} t^{2j-1} - \sum_{j=0}^{\infty} \left(\frac{1}{\pi}\right)^{j} t^{j-1} \\
&= \sum_{j=1}^{\infty} \frac{(2^{2j} - 2)|B_{2j}|}{(2j)!} t^{2j-1} - \sum_{j=1}^{\infty} \left(\frac{1}{\pi}\right)^{j+1} t^{j} - \frac{1}{\pi} \\
&= -\frac{1}{\pi} + \sum_{j=1}^{\infty} \alpha_j t^{2j-1} - \sum_{j=1}^{\infty} \beta_j t^j \\
&= -\frac{1}{\pi} + (\alpha_1 - \beta_1)t - \beta_2 t^2 + (\alpha_2 - \beta_3)t^3 - \beta_4 t^4 \\
&\quad + \cdots + (\alpha_j - \beta_{2j-1})t^{2j-1} - \beta_{2j} t^{2j} + \cdots,
\end{aligned} \tag{6.5}$$

where

$$\alpha_j = \frac{(2^{2j} - 2)|B_{2j}|}{(2j)!} \quad \text{and} \quad \beta_j = \left(\frac{1}{\pi}\right)^{j+1}. \tag{6.6}$$

We find by the left-hand side of (2.5) that, for $j \in \mathbb{N}$,

$$\alpha_j - \beta_{2j-1} > \frac{2(2^{2j}-2)}{(2\pi)^{2j}(1-2^{-2j})} - \left(\frac{1}{\pi}\right)^{2j} = \frac{2^{2j}-3}{2^{2j}-1}\left(\frac{2}{\pi}\right)^{2j} > 0.$$

so that (6.5) is an alternating series for $0 < x < \pi/2$.

Formula (6.5) motivated us to establish Theorem 6.2.

THEOREM 6.2. *For $0 < x < \pi/2$ and $N \in \mathbb{N}_0$,*

$$\begin{aligned} &\frac{4\pi}{\pi^2-4x^2} + \sum_{j=0}^{2N}(-1)^{j-1}d_j\left(\frac{\pi}{2}-x\right)^j < \sec x \\ &< \frac{4\pi}{\pi^2-4x^2} + \sum_{j=0}^{2N+1}(-1)^{j-1}d_j\left(\frac{\pi}{2}-x\right)^j, \end{aligned} \tag{6.7}$$

where

$$d_{2j-1} = \alpha_j - \beta_{2j-1}, \quad j \in \mathbb{N} \quad \text{and} \quad d_{2j} = \beta_{2j}, \qquad j \in \mathbb{N}_0, \tag{6.8}$$

and α_j and β_j are given in (6.6).

Proof. If we write (6.5) as

$$\begin{aligned} \csc t - \frac{4\pi}{\pi^2-4(\frac{\pi}{2}-t)^2} + \frac{1}{\pi} &= \Big((\alpha_1-\beta_1)t - \beta_2 t^2\Big) + \Big((\alpha_2-\beta_3)t^3 - \beta_4 t^4\Big) \\ &\quad + \cdots + \Big((\alpha_j-\beta_{2j-1})t^{2j-1} - \beta_{2j}t^{2j}\Big) + \cdots, \end{aligned} \tag{6.9}$$

we find by the left-hand side of (2.5) that, for $j \geqslant 2$,

$$\begin{aligned} &(\alpha_j-\beta_{2j-1})t^{2j-1} - \beta_{2j}t^{2j} = \Big(\alpha_j - \beta_{2j-1} - \beta_{2j}t\Big)t^{2j-1} \\ &> \left\{\alpha_j - \beta_{2j-1} - \beta_{2j}\left(\frac{\pi}{2}\right)\right\}t^{2j-1} = \left\{\frac{(2^{2j}-2)|B_{2j}|}{(2j)!} - \frac{3}{2}\left(\frac{1}{\pi}\right)^{2j}\right\}t^{2j-1} \\ &> \left\{\frac{2(2^{2j}-2)}{(2\pi)^{2j}(1-2^{-2j})} - \frac{3}{2}\left(\frac{1}{\pi}\right)^{2j}\right\}t^{2j-1} = \frac{2^{2j}-5}{2^{2j+1}-2}\left(\frac{1}{\pi}\right)^{2j}t^{2j-1} > 0. \end{aligned}$$

Noting that

$$(\alpha_1-\beta_1)t - \beta_2 t^2 = \left(\frac{1}{6} - \frac{1}{\pi^2}\right)t - \frac{1}{\pi^3}t^2 > 0$$

holds, we obtain that

$$(\alpha_j-\beta_{2j-1})t^{2j-1} - \beta_{2j}t^{2j} > 0 \quad \text{for all} \quad j \in \mathbb{N}.$$

Replacing t by $\frac{\pi}{2}-x$ in (6.9), we obtain, by truncation of (6.9), the left-hand side of (6.7).

If we write (6.5) as

$$\csc t-\frac{4\pi}{\pi^2-4(\frac{\pi}{2}-t)^2}=-\left(\frac{1}{\pi}-(\alpha_1-\beta_1)t\right)-\left(\beta_2t^2-(\alpha_2-\beta_3)t^3\right)\\ -\cdots-\left(\beta_{2j}t^{2j}-(\alpha_{j+1}-\beta_{2j+1})t^{2j+1}\right)-\cdots, \tag{6.10}$$

we find by the right-hand side of (2.5) that, for $j\in\mathbb{N}_0$,

$$\begin{aligned}\beta_{2j}t^{2j}-(\alpha_{j+1}-\beta_{2j+1})t^{2j+1}&=\left(\beta_{2j}-(\alpha_{j+1}-\beta_{2j+1})t\right)t^{2j}\\ &>\left\{\beta_{2j}-(\alpha_{j+1}-\beta_{2j+1})\left(\frac{\pi}{2}\right)\right\}t^{2j}\\ &=\left\{\frac{3}{2}\left(\frac{1}{\pi}\right)^{2j+1}-\frac{(2^{2j+2}-2)|B_{2j+2}|}{(2j+2)!}\left(\frac{\pi}{2}\right)\right\}t^{2j}\\ &\geqslant\left\{\frac{3}{2}\left(\frac{1}{\pi}\right)^{2j+1}-\frac{2(2^{2j+2}-2)}{(2\pi)^{2j+2}\left(1-2^{\beta-2(j+1)}\right)}\left(\frac{\pi}{2}\right)\right\}t^{2j}\\ &=\frac{2^{2j+2}+4-3\cdot2^{\beta}}{2\left(2^{2j+2}-2^{\beta}\right)}\left(\frac{1}{\pi}\right)^{2j+1}t^{2j}>0,\end{aligned}$$

where β is given in (2.6).

Replacing t by $\frac{\pi}{2}-x$ in (6.10), we obtain, by truncation of (6.10), the right-hand side of (6.7). The proof is complete. □

We now consider the expansion of function $\sec x\left(\pi^2-4x^2\right)$ near the origin. Using (2.3), we have

$$\begin{aligned}\sec x\left(\pi^2-4x^2\right)&=\sum_{j=0}^{\infty}\frac{\pi^2|E_{2j}|}{(2j)!}x^{2j}-\sum_{j=0}^{\infty}\frac{4|E_{2j}|}{(2j)!}x^{2j+2}\\ &=\pi^2+\sum_{j=1}^{\infty}\left\{\frac{\pi^2|E_{2j}|}{(2j)!}-\frac{4|E_{2j-2}|}{(2j-2)!}\right\}x^{2j}.\end{aligned} \tag{6.11}$$

Noting that (2.10) holds, we obtain Theorem 6.3 by truncation of (6.11).

THEOREM 6.3. *For* $0<x<\pi/2$ *and* $N\in\mathbb{N}$,

$$\frac{1}{\pi^2-4x^2}\left\{\pi^2+\sum_{j=1}^{N}\left(\frac{\pi^2|E_{2j}|}{(2j)!}-\frac{4|E_{2j-2}|}{(2j-2)!}\right)x^{2j}\right\}<\sec x. \tag{6.12}$$

The choice $N=2$ in (6.12) yields

$$\frac{\pi^2+\frac{\pi^2-8}{2}x^2+\frac{5\pi^2-48}{24}x^4}{\pi^2-4x^2}<\sec x. \tag{6.13}$$

There is no strict comparison between the two lower bounds in equations (1.11) and (6.13).

We now consider the expansion of function $\sec x\left(\pi^2-4x^2\right)$, on a neighborhood of $\pi/2$. Replacing x by $\frac{\pi}{2}-t$, it is sufficient to consider the expansion of function $\csc t\left(\pi^2-4\left(\frac{\pi}{2}-t\right)^2\right)$ near the origin. Using (2.4), we have

$$\begin{aligned}
&\csc t\left(\pi^2-4\left(\frac{\pi}{2}-t\right)^2\right)=\csc t\left(4\pi t-4t^2\right)\\
=&4\pi-4t+\sum_{j=1}^{\infty}\frac{4\pi(2^{2j}-2)|B_{2j}|}{(2j)!}t^{2j}-\sum_{j=1}^{\infty}\frac{4(2^{2j}-2)|B_{2j}|}{(2j)!}t^{2j+1}\\
=&4\pi-4t+\sum_{j=1}^{\infty}\lambda_j t^{2j}-\sum_{j=1}^{\infty}\mu_j t^{2j+1}\\
=&\lambda_0-\mu_0 t+\lambda_1 t^2-\mu_1 t^3+\lambda_2 t^4-\mu_2 t^5+\cdots+\lambda_j t^{2j}-\mu_j t^{2j+1}+\cdots,
\end{aligned}\tag{6.14}$$

or alternatively

$$\begin{aligned}
&\sec x\left(\pi^2-4x^2\right)\\
=&\lambda_0-\mu_0\left(\frac{\pi}{2}-x\right)+\lambda_1\left(\frac{\pi}{2}-x\right)^2-\mu_1\left(\frac{\pi}{2}-x\right)^3+\lambda_2\left(\frac{\pi}{2}-x\right)^4-\mu_2\left(\frac{\pi}{2}-x\right)^5\\
&+\cdots+\lambda_j\left(\frac{\pi}{2}-x\right)^{2j}-\mu_j\left(\frac{\pi}{2}-x\right)^{2j+1}+\cdots,
\end{aligned}\tag{6.15}$$

where

$$\begin{aligned}
&\lambda_0=4\pi,\quad \mu_0=4,\\
&\lambda_j=\frac{4\pi(2^{2j}-2)|B_{2j}|}{(2j)!}\quad\text{and}\quad \mu_j=\frac{4(2^{2j}-2)|B_{2j}|}{(2j)!}\quad\text{for}\quad j\in\mathbb{N}.
\end{aligned}\tag{6.16}$$

If we write (6.14) as

$$\begin{aligned}
&\csc t\left(\pi^2-4\left(\frac{\pi}{2}-t\right)^2\right)\\
=&\left(\lambda_0-\mu_0 t\right)+\left(\lambda_1 t^2-\mu_1 t^3\right)+\left(\lambda_2 x^4-\mu_2 t^5\right)+\cdots+\left(\lambda_j t^{2j}-\mu_j t^{2j+1}\right)+\cdots,
\end{aligned}\tag{6.17}$$

we find that, for $j\in\mathbb{N}_0$,

$$\lambda_j t^{2j}-\mu_j t^{2j+1}=\left(\lambda_j-\mu_j t\right)t^{2j}>\left\{\lambda_j-\mu_j\left(\frac{\pi}{2}\right)\right\}t^{2j}=\frac{2\pi(2^{2j}-2)|B_{2j}|}{(2j)!}t^{2j}>0.$$

We obtain by truncation of (6.17) that, for $0<t<\pi/2$ and $N\in\mathbb{N}_0$,

$$\begin{aligned}
&\csc t\left(\pi^2-4\left(\frac{\pi}{2}-t\right)^2\right)\\
>&\lambda_0-\mu_0 t+\lambda_1 t^2-\mu_1 t^3+\lambda_2 x^4-\mu_2 t^5+\cdots+\lambda_N t^{2N}-\mu_N t^{2N+1}.
\end{aligned}\tag{6.18}$$

If we write (6.14) as

$$\begin{aligned}&\csc t\left(\pi^2-4\left(\frac{\pi}{2}-t\right)^2\right)\\ =&\lambda_0-\left(\mu_0 t-\lambda_1 t^2\right)-\left(\mu_1 t^3-\lambda_2 x^4\right)-\cdots-\left(\mu_{j-1}t^{2j-1}-\lambda_j t^{2j}\right)-\cdots,\end{aligned}\tag{6.19}$$

we find by (2.8) that for $j \geqslant 2$,

$$\begin{aligned}&\mu_{j-1}t^{2j-1}-\lambda_j t^{2j}=\left(\mu_{j-1}-\lambda_j t\right)t^{2j-1}>\left\{\mu_{j-1}-\lambda_j\left(\frac{\pi}{2}\right)\right\}t^{2j-1}\\ =&\frac{4(2^{2j-2}-2)|B_{2j}|}{(2j-2)!}\left\{\frac{|B_{2j-2}|}{|B_{2j}|}-\frac{\pi^2(2^{2j}-2)(2j-2)!}{2(2^{2j-2}-2)(2j)!}\right\}t^{2j-1}\\ >&\frac{4(2^{2j-2}-2)|B_{2j}|}{(2j-2)!}\left\{\frac{\pi^2(2^{2j}-1)(2j-2)!}{(2^{2j-2}-1)(2j)!}-\frac{\pi^2(2^{2j}-2)(2j-2)!}{2(2^{2j-2}-2)(2j)!}\right\}t^{2j-1}\\ =&\frac{2\pi^2(2^{2j-2}-2)|B_{2j}|}{(2j)!}\left\{\frac{16^j-12\cdot 4^j+8}{(2^{2j-2}-1)(4^j-8)}\right\}t^{2j-1}>0.\end{aligned}$$

Noting that

$$\mu_0 t-\lambda_1 t^2=4t-\frac{2\pi}{3}t^2>0,\quad 0<t<\frac{\pi}{2}$$

holds, we obtain by truncation of (6.19) that, for $0<t<\pi/2$ and $N\in\mathbb{N}_0$,

$$\begin{aligned}&\csc t\left(\pi^2-4\left(\frac{\pi}{2}-t\right)^2\right)\\ <&\lambda_0-\mu_0 t+\lambda_1 t^2-\mu_1 t^3+\lambda_2 x^4-\cdots-\mu_{N-1}t^{2N-1}+\lambda_N t^{2N}.\end{aligned}\tag{6.20}$$

Replacing t by $\frac{\pi}{2}-x$ in (6.18) and (6.20), we obtain Theorem 6.4.

THEOREM 6.4. *For $0<x<\pi/2$ and $N\in\mathbb{N}_0$,*

$$\frac{P_{2N+1}(x)}{\pi^2-4x^2}<\sec x<\frac{P_{2N}(x)}{\pi^2-4x^2},\tag{6.21}$$

with

$$\begin{aligned}P_{2N}(x)=\lambda_0-\mu_0\left(\frac{\pi}{2}-x\right)+\lambda_1\left(\frac{\pi}{2}-x\right)^2-\mu_1\left(\frac{\pi}{2}-x\right)^3+\lambda_2\left(\frac{\pi}{2}-x\right)^4\\ -\cdots-\mu_{N-1}\left(\frac{\pi}{2}-x\right)^{2N-1}+\lambda_N\left(\frac{\pi}{2}-x\right)^{2N}\end{aligned}$$

and

$$P_{2N+1}(x)=P_{2N}(x)-\mu_N\left(\frac{\pi}{2}-x\right)^{2N+1},$$

where λ_j and μ_j are given in (6.16).

The choice $N = 1$ in (6.21) yields

$$\frac{1}{\pi^2-4x^2}\left\{4\pi-4\left(\frac{\pi}{2}-x\right)+\frac{2\pi}{3}\left(\frac{\pi}{2}-x\right)^2-\frac{2}{3}\left(\frac{\pi}{2}-x\right)^3\right\}$$
$$<\sec x<\frac{1}{\pi^2-4x^2}\left\{4\pi-4\left(\frac{\pi}{2}-x\right)+\frac{2\pi}{3}\left(\frac{\pi}{2}-x\right)^2\right\}. \tag{6.22}$$

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