Journal of
Mathematical
Inequalities
Volume 13, Number 3 (2019), 621–644
doi:10.7153/jmi-2019-13-41

NEW OSTROWSKI LIKE INEQUALITIES INVOLVING THE FUNCTIONS HAVING HARMONIC h–CONVEXITY PROPERTY AND APPLICATIONS

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(*Communicated by A. Vukelić*)

Abstract. Some new Ostrowski type inequalities are established for the class of harmonic h-convex functions. Several new and known special cases, which can be derived from our main results, are also discussed. Applications to special means of some of our main results are also discussed. Results obtained in this paper continue to hold for these special cases. Techniques of this paper may lead to further research in this dynamic field.

1. Introduction and preliminaries

Convexity plays an important role in different fields of pure and applied sciences. Due to its several important applications, theory of convexity has experienced a rapid development in recent decades. Consequently the classical concepts of convex sets and convex functions have been extended and generalized in different directions using novel and innovative ideas, see [1, 3, 4, 5, 6, 7, 9, 11, 10, 16, 15, 19, 20].

The classical convex sets and convex functions are respectively defined as:

DEFINITION 1.1. A set $C \subset \mathbb{R}$ is said to be convex, if

$$(1-t)x+ty \in C, \quad \forall x,y \in C, t \in [0,1].$$

DEFINITION 1.2. Let C be a convex set, A function $f : C \to \mathbb{R}$ is said to be convex, if

$$f((1-t)x+ty) \leqslant (1-t)f(x)+tf(y), \quad \forall x,y \in C, t \in [0,1].$$

Harmonic convex sets and harmonic convex functions are defined as:

Mathematics subject classification (2010): 26D15, 26A51.
Keywords and phrases: Convex functions, harmonic, functions, Ostrowski inequality.
Authors also extend their appreciation to the International Scientific Partnership Program ISPP at King Saud University, Riyadh, Saudi Arabia for funding this research work through ISPP-125.
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DEFINITION 1.3. ([19]) A set $K \subset \mathbb{R}_+ \setminus \{0\}$ is said to be harmonic convex, if

$$\frac{xy}{tx+(1-t)y} \in K \quad \forall x,y \in K, t \in [0,1].$$

DEFINITION 1.4. ([11]) Let K be a harmonic convex set. A function $f : K \to \mathbb{R}$ is said to be harmonic convex function, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant (1-t)f(x)+tf(y), \quad \forall x,y \in K, t \in [0,1].$$

Using the inequality $HM \leqslant AM$ it is known from [11] that a function $f : (0,\infty) \to \mathbb{R}$ defined as $f(x) = x$ is harmonic convex function.
Varošanec [20] introduced an important class of convex functions, which is called as the h-convex functions.

DEFINITION 1.5. ([20]) Let $h : J = (0,1) \subset \mathbb{R} \to \mathbb{R}$ a non-negative function. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex function $(f \in SX(h,I))$, if f is non-negative and

$$f\left(tx+(1-t)y\right) \leqslant h(t)f(x)+h(1-t)f(y), \forall x,y \in I \text{ and } t \in (0,1). \tag{1.1}$$

If inequality (1.1) is reversed, then f is said to be h-concave, i. e. $f \in SV(h,I)$.

For $h(t) = t$, $h(t) = t^s$, $h(t) = \frac{1}{t}$, $h(t) = 1$ and $h(t) = \frac{1}{t^s}$, the class of h-convex functions reduces to the class of convex functions, s-Breckner convex functions [3], Godunova-Levin functions [9], P-functions [7] and s-Godunova-Levin functions [6] respectively. This shows that the class of h-convex functions is quite general and unifying ones. Noor et al. [16] introduced and considered a new class of harmonically convex functions, which is called the harmonic h-convex function.

DEFINITION 1.6. ([16]) A function $f : K \to \mathbb{R}$ is said to be harmonic h-convex if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant h(t)f(y)+h(1-t)f(x), \forall x,y \in K \text{ and } t \in (0,1). \tag{1.2}$$

It has been shown [16] that along with harmonic convex functions the class of harmonic h-convex functions also contains some other new classes of harmonic convex functions, such as, harmonic s-convex functions, harmonic s-Godunova-Levin convex functions, harmonic Godunova-Levin functions and harmonic P-functions. Thus this class is also a unified and more generalized. For some recent developments in harmonic h-convex functions, see [13, 16] and the references therein.
It is worth to mention here the concept of harmonicity plays significant role in different fields of pure and applied sciences. For example in electric circuit theory the total resistance of a set of parallel resistors is just half of harmonic means of the total resistors. It also plays important role in Asian options of stock. For more details, see [2].
The relation between theory of convexity and theory of inequalities inspired many researchers and as a result many classical results which have been obtained for convex

functions have now been obtained for other generalizations of convex functions, see [5, 6, 7, 8, 11, 10, 13, 16, 15, 18]. One of the most intensively and extensively studied inequality via convex functions is Hermite-Hadamard inequality. An interesting problem related to the Hermite-Hadamard's inequality is its precision. Note that the left Hermite-Hadamard inequality can be estimated by the inequality of Ostrowski, which is famously known as Ostrowski's inequality, see [17]. This inequality provides us an estimate for the deviation of the values of a smooth function from its mean value.
Taking inspiration from ongoing research in this field, we again consider the class of harmonic h-convex functions. The main motivation of this article is to relate the class of harmonic h-convex functions with integral inequalities of Ostrowski type. We also obtain some Ostrowski type inequalities via other classes of harmonic convex functions, which can be viewed as special cases of the main results. Some applications to special means of our main results are also discussed. It is expected that the ideas and techniques of this paper may stimulate further research in this field.
We now recall some concepts from special functions, which are used in the development of our main results.
Gamma and Beta functions are defined respectively as:

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\mathrm{d}t, \quad \Re(x) > 0,$$

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \Re(x) > 0, \Re(y) > 0.$$

The integral form of the hypergeometric function is

$${}_2F_1(x,y;c;z) = \frac{1}{\mathrm{B}(y,c-y)} \int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}\mathrm{d}t$$

for $|z| < 1, \Re(c) > \Re(y) > 0$. For more information, see [12].

DEFINITION 1.7. ([14]) Recall the following definitions:

1. For arbitrary $a > 0, b > 0$ and $a \neq b$

$$L(b,a) = \frac{b-a}{\log b - \log a},$$

is the logarithmic mean.

2. For arbitrary $a, b \in \mathbb{R}$ and $a \neq b$

$$A(a,b) = \frac{a+b}{2},$$

is the arithmetic mean.

3. For arbitrary $a, b \in \mathbb{R}$ and $a \neq b$

$$G(a,b) = \sqrt{ab},$$

is the geometric mean.

4. For arbitrary $a,b \in \mathbb{R}$ and $a \neq b$

$$H(a,b) = \frac{2ab}{a+b},$$

is the harmonic mean.

From now onward, we take the notation $\mathscr{I} = [a,b] \subset \mathbb{R}_+ \setminus \{0\}$ be the interval and $\mathscr{I}^0$ be the interior of $\mathscr{I}$ unless otherwise specified.

2. Ostrowski type inequalities

In order to obtain our main results, we need following auxiliary result.

LEMMA 2.1. ([10]) *Let $f : \mathscr{I} \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x \in (a,b)$. If $f' \in L[a,b]$, then*

$$\begin{aligned} &f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u \\ =&\frac{ab}{b-a}\left[(x-a)^2\int_0^1 \frac{t}{(ta+(1-t)x)^2} f'\left(\frac{ax}{ta+(1-t)x}\right)\mathrm{d}t\right. \\ &\left.+(b-x)^2\int_0^1 \frac{t}{(tb+(1-t)x)^2} f'\left(\frac{bx}{tb+(1-t)x}\right)\mathrm{d}t\right]. \end{aligned}$$

Proof. Integration by parts completes the proof. □

Now using Lemma 2.1, we obtain the main results.

THEOREM 2.2. *Let $f : \mathscr{I} \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x \in (a,b)$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonic h-convex function, then, for $q \geqslant 1$, we have*

$$\begin{aligned} &\left| f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u \right| \\ \leqslant&\frac{ab}{b-a}\left\{\left((x-a)^2\left[\omega_1(a,x;q;h)|f'(a)|^q + \omega_2(a,x;q;h)|f'(x)|^q\right]\right)^{\frac{1}{q}}\right. \\ &\left.+\left((b-x)^2\left[\omega_3(b,x;q;h)|f'(b)|^q + \omega_4(b,x;q;h)|f'(x)|^q\right]\right)^{\frac{1}{q}}\right\}, \end{aligned}$$

where

$$\omega_1(a,x;q;h) = \int_0^1 \frac{t^q}{(ta+(1-t)x)^{2q}} h(1-t)\mathrm{d}t \tag{2.1}$$

$$\omega_2(a,x;q;h)=\int_0^1 \frac{t^q}{(ta+(1-t)x)^{2q}}h(t)\mathrm{d}t \tag{2.2}$$

$$\omega_3(b,x;q;h)=\int_0^1 \frac{t^q}{(tb+(1-t)x)^{2q}}h(1-t)\mathrm{d}t \tag{2.3}$$

and

$$\omega_4(b,x;q;h)=\int_0^1 \frac{t^q}{(tb+(1-t)x)^{2q}}h(t)\mathrm{d}t. \tag{2.4}$$

Proof. Using Lemma 2.1, power mean inequality and the fact that $|f'|^q$ is harmonic h-convex function, then

$$\begin{aligned}
&\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right| \\
\leqslant&\frac{ab}{b-a}\left[(x-a)^2\int_0^1 \frac{t}{(ta+(1-t)x)^2}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|\mathrm{d}t\right. \\
&\left.-(b-x)^2\int_0^1 \frac{t}{(tb+(1-t)x)^2}\left|f'\left(\frac{bx}{tb+(1-t)x}\right)\right|\mathrm{d}t\right] \\
\leqslant&\frac{ab(x-a)^2}{b-a}\left(\int_0^1 1\mathrm{d}t\right)^{1-\frac{1}{q}}\times\left(\int_0^1 \frac{t^q}{(ta+(1-t)x)^{2q}}[h(1-t)|f'(a)|^q+h(t)|f'(x)|^q]\mathrm{d}t\right)^{\frac{1}{q}} \\
&+\frac{ab(b-x)^2}{b-a}\left(\int_0^1 1\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \frac{t^q}{(tb+(1-t)x)^{2q}}[h(1-t)|f'(b)|^q+h(t)|f'(x)|^q]\mathrm{d}t\right)^{\frac{1}{q}} \\
=&\frac{ab(x-a)^2}{b-a}\times\left(|f'(a)|^q\int_0^1 \frac{t^q}{(ta+(1-t)x)^{2q}}h(1-t)\mathrm{d}t\right. \\
&\left.+|f'(x)|^q\int_0^1 \frac{t^q}{(ta+(1-t)x)^{2q}}h(t)\mathrm{d}t\right)^{\frac{1}{q}}+\frac{ab(b-x)^2}{b-a}
\end{aligned}$$

$$\times\left(|f'(b)|^q\int_0^1\frac{t^q}{(tb+(1-t)x)^{2q}}h(1-t)\mathrm{d}t+|f'(x)|^q\int_0^1\frac{t^q}{(tb+(1-t)x)^{2q}}h(t)\mathrm{d}t\right)^{\frac{1}{q}}$$

$$=\frac{ab}{b-a}\left\{\left((x-a)^2\left[\omega_1(a,x;q;h)|f'(a)|^q+\omega_2(a,x;q;h)|f'(x)|^q\right]\right)^{\frac{1}{q}}\right.$$

$$\left.+\left((b-x)^2\left[\omega_3(b,x;q;h)|f'(b)|^q+\omega_4(b,x;q;h)|f'(x)|^q\right]\right)^{\frac{1}{q}}\right\}.$$

This completes the proof. □

We now discuss some special cases of Theorem 2.2.

I. If $h(t)=t$, then, we have result for harmonically convex function.

COROLLARY 2.3. *Let $f:\mathscr{I}\to\mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x\in(a,b)$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonic convex function, then, for $q\geqslant 1$, we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}\mathrm{d}u\right|$$

$$\leqslant\frac{ab}{b-a}\left\{\left((x-a)^2\left[\omega_1^+(a,x;q;t)|f'(a)|^q+\omega_2^+(a,x;q;t)|f'(x)|^q\right]\right)^{\frac{1}{q}}\right.$$

$$\left.+\left((b-x)^2\left[\omega_3^+(b,x;q;t)|f'(b)|^q+\omega_4^+(b,x;q;t)|f'(x)|^q\right]\right)^{\frac{1}{q}}\right\},$$

where

$$\begin{aligned}\omega_1^+(a,x;q;t)&=\int_0^1\frac{t^q}{(ta+(1-t)x)^{2q}}(1-t)\mathrm{d}t\\&=\frac{B(1+q,1)}{x^{2q}}\,{}_2F_1\left(2q,1+q;2+q;1-\frac{a}{x}\right)\\&\quad-\frac{B(2+q,1)}{x^{2q}}\,{}_2F_1\left(2q,2+q;3+q;1-\frac{a}{x}\right);\end{aligned}\tag{2.5}$$

$$\begin{aligned}\omega_2^+(a,x;q;t)&=\int_0^1\frac{t^{1+q}}{(ta+(1-t)x)^{2q}}\mathrm{d}t\\&=\frac{B(2+q,1)}{x^{2q}}\,{}_2F_1\left(2q,2+q;3+q;1-\frac{a}{x}\right);\end{aligned}\tag{2.6}$$

$$\omega_3^+(b,x;q;t)=\int_0^1\frac{t^q(1-t)}{(tb+(1-t)x)^{2q}}\mathrm{d}t$$

$$= \frac{B(1+q,1)}{x^{2q}}\, {}_2F_1\left(2q, 1+q; 2+q; 1-\frac{b}{x}\right) - \frac{B(2+q,1)}{x^{2q}}\, {}_2F_1\left(2q, 2+q; 3+q; 1-\frac{b}{x}\right); \tag{2.7}$$

and

$$\omega_4^{+}(b,x;q;t) = \int_0^1 \frac{t^{1+q}}{(tb+(1-t)x)^{2q}}\mathrm{d}t = \frac{B(2+q,1)}{x^{2q}}\, {}_2F_1\left(2q, 2+q; 3+q; 1-\frac{b}{x}\right). \tag{2.8}$$

II. If $h(t) = t^{-s}$, then, we have result for harmonic s-Godunova-Levin convex function, which appears to be new one.

COROLLARY 2.4. *Let $f : \mathscr{I} \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x \in (a,b)$ and $f' \in L[a,b]$. If $|f'|^q$ harmonic s-Godunova-Levin convex function where $s \in [0,1]$, then, for $q \geqslant 1$, we have*

$$\left| f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u \right| \leqslant \frac{ab}{b-a}\left\{ \left((x-a)^2\left[\omega_1^*(a,x;q;t^{-s})|f'(a)|^q + \omega_2^*(a,x;q;t^{-s})|f'(x)|^q\right]\right)^{\frac{1}{q}} + \left((b-x)^2\left[\omega_3^*(b,x;q;t^{-s})|f'(b)|^q + \omega_4^*(b,x;q;t^{-s})|f'(x)|^q\right]\right)^{\frac{1}{q}} \right\},$$

where

$$\omega_1^*(a,x;q;t^{-s}) = \int_0^1 \frac{t^q(1-t)^{-s}}{(ta+(1-t)x)^{2q}}\mathrm{d}t = \frac{B(q+1,1-s)}{x^{2q}}\, {}_2F_1\left(2q, q+1; 2+q-s; 1-\frac{a}{x}\right), \tag{2.9}$$

$$\omega_2^*(a,x;q;t^{-s}) = \int_0^1 \frac{t^{q-s}}{(ta+(1-t)x)^{2q}}\mathrm{d}t = \frac{B(q-s+1,1)}{x^{2q}}\, {}_2F_1\left(2q, q-s+1; q-s+2; 1-\frac{a}{x}\right), \tag{2.10}$$

$$\omega_3^*(b,x;q;t^{-s}) = \int_0^1 \frac{t^q(1-t)^{-s}}{(tb+(1-t)x)^{2q}}\mathrm{d}t$$

$$= \frac{B(q+1,1-s)}{x^{2q}} {}_2F_1\left(2q, q+1; 2+q-s; 1-\frac{b}{x}\right), \tag{2.11}$$

and

$$\omega_4^*(b,x;q;t^{-s}) = \int_0^1 \frac{t^{q-s}}{(tb+(1-t)x)^{2q}} \mathrm{d}t$$
$$= \frac{B(q-s+1,1)}{x^{2q}} {}_2F_1\left(2q, q-s+1; q-s+2; 1-\frac{b}{x}\right). \tag{2.12}$$

III. If $h(t) = 1$, then, we have result for harmonic P-function, which also appears to be new one.

COROLLARY 2.5. *Let $f : \mathscr{I} \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x \in (a,b)$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonic P-function, then, for $q \geqslant 1$, we have*

$$\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} \mathrm{d}u \right|$$
$$\leqslant \frac{ab}{b-a} \left\{ \left((x-a)^2 \left[\omega^*(a,x;q;1)\{|f'(a)|^q + |f'(x)|^q\}\right]\right)^{\frac{1}{q}} \right.$$
$$\left. + \left((b-x)^2 \left[\omega^{**}(b,x;q;1)\{|f'(b)|^q + |f'(x)|^q\}\right]\right)^{\frac{1}{q}} \right\},$$

where

$$\omega^*(a,x;q;1) = \int_0^1 \frac{t^q}{(ta+(1-t)x)^{2q}} \mathrm{d}t;$$
$$= \frac{B(q+1,1)}{x^{2q}} {}_2F_1\left(2q, q+1; q+2; 1-\frac{a}{x}\right), \tag{2.13}$$

and

$$\omega^{**}(b,x;q;1) = \int_0^1 \frac{t^q}{(tb+(1-t)x)^{2q}} \mathrm{d}t$$
$$= \frac{B(q+1,1)}{x^{2q}} {}_2F_1\left(2q, q+1; q+2; 1-\frac{b}{x}\right). \tag{2.14}$$

IV. If $h(t) = t^s$, then, we have result for harmonic s-convex function, see [10].

COROLLARY 2.6. *Under the assumptions of Theorem 2.2, if $|f'(.)| \leqslant M$, then*

$$\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} \mathrm{d}u \right|$$

$$\leqslant \frac{Mab}{b-a}\left\{\left((x-a)^2\left[\omega_1(a,x;q;h)+\omega_2(a,x;q;h)\right]\right)^{\frac{1}{q}} + \left((b-x)^2\left[\omega_3(b,x;q;h)+\omega_4(b,x;q;h)\right]\right)^{\frac{1}{q}}\right\},$$

where $\omega_1(a,x;q;h)$, $\omega_2(a,x;q;h)$, $\omega_3(b,x;q;h)$ *and* $\omega_4(b,x;q;h)$ *are given by (2.1), (2.2), (2.3) and (2.4) respectively.*

THEOREM 2.7. *Let* $f:\mathscr{I}\to\mathbb{R}$ *be a differentiable function on* $\mathscr{I}^0$ *with* $x\in(a,b)$ *and* $f'\in L[a,b]$. *If* $|f'|^q$ *is harmonic* h*-convex function, then, for* $q\geqslant 1$, *we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right|$$
$$\leqslant \frac{ab}{b-a}\left[\left\{\Phi^{1-\frac{1}{q}}(a,x)(x-a)^2\times\left(\omega_1(a,x;1;h)|f'(a)|^q+\omega_2(a,x;1;h)|f'(x)|^q\right)^{\frac{1}{q}}\right\} + \left\{\Phi^{1-\frac{1}{q}}(b,x)(b-x)^2\times\left(\omega_3(b,x;1;h)|f'(b)|^q+\omega_4(b,x;1;h)|f'(x)|^q\right)^{\frac{1}{q}}\right\}\right],$$

where

$$\Phi(a,x)=\frac{1}{x-a}\left\{\frac{1}{a}-\frac{\ln x-\ln a}{x-a}\right\}, \tag{2.15}$$

$$\Phi(b,x)=\frac{1}{b-x}\left\{\frac{\ln b-\ln x}{b-x}-\frac{1}{b}\right\}, \tag{2.16}$$

and $\omega_1(a,x;1;h)$, $\omega_2(a,x;1;h)$, $\omega_3(b,x;1;h)$ *and* $\omega_4(b,x;1;h)$ *can be deduced from (2.1), (2.2), (2.3) and (2.4) respectively.*

Proof. Using Lemma 2.1, power mean inequality and the fact that $|f'|^q$ is harmonic h-convex function, then

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right|$$
$$\leqslant \frac{ab(x-a)^2}{b-a}\left(\int_0^1 \frac{t}{(ta+(1-t)x)^2}\mathrm{d}t\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \frac{t}{(ta+(1-t)x)^2}\left[h(1-t)|f'(a)|^q+h(t)|f'(x)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}$$
$$+\frac{ab(b-x)^2}{b-a}\left(\int_0^1 \frac{t}{(tb+(1-t)x)^2}\mathrm{d}t\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \frac{t}{(tb+(1-t)x)^2}\left[h(1-t)|f'(b)|^q+h(t)|f'(x)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}$$

$$=\frac{ab(x-a)^2}{b-a}\left(\frac{1}{x-a}\left\{\frac{1}{a}-\frac{\ln x-\ln a}{x-a}\right\}\right)^{1-\frac{1}{q}}$$

$$\times\left(|f'(a)|^q\int_0^1 \frac{t}{(ta+(1-t)x)^2}h(1-t)\mathrm{d}t+|f'(x)|^q\int_0^1 \frac{t}{(ta+(1-t)x)^2}h(t)\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\frac{ab(b-x)^2}{b-a}\left(\frac{1}{b-x}\left\{\frac{\ln b-\ln x}{b-x}-\frac{1}{b}\right\}\right)^{1-\frac{1}{q}}$$

$$\times\left(|f'(b)|^q\int_0^1 \frac{t}{(tb+(1-t)x)^2}h(1-t)\mathrm{d}t+|f'(x)|^q\int_0^1 \frac{t}{(tb+(1-t)x)^2}h(t)\mathrm{d}t\right)^{\frac{1}{q}}.$$

This completes the proof. □

Now we discuss some special cases of Theorem 2.7.

I. If $h(t)=t$, then, we have result for harmonic convex functions.

COROLLARY 2.8. *Let $f:\mathscr{I}\to\mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x\in(a,b)$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonic convex function, then, for $q\geqslant 1$, we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right|$$

$$\leqslant\frac{ab}{b-a}\left[\left\{\Phi^{1-\frac{1}{q}}(a,x)(x-a)^2\times\left(\omega_1^+(a,x;1;t)|f'(a)|^q+\omega_2^+(a,x;1;t)|f'(x)|^q\right)^{\frac{1}{q}}\right\}\right.$$

$$\left.+\left\{\Phi^{1-\frac{1}{q}}(b,x)(b-x)^2\times\left(\omega_3^+(b,x;1;t)|f'(b)|^q+\omega_4^+(b,x;1;t)|f'(x)|^q\right)^{\frac{1}{q}}\right\}\right],$$

where $\Phi(a,x)$ and $\Phi(b,x)$ are given by (2.15) and (2.16) respectively. Also one can deduce $\omega_1^+(a,x;1;t)$, $\omega_2^+(a,x;1;t)$, $\omega_3^+(a,x;1;t)$ and $\omega_4^+(a,x;1;t)$ from (2.5), (2.6), (2.7) and (2.8) respectively.

II. If $h(t)=t^{-s}$, then, we have result for harmonic s-Godunova-Levin convex functions, which appears to be new result.

COROLLARY 2.9. *Let $f:\mathscr{I}\to\mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x\in(a,b)$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonic s-Godunova-Levin convex function, then, for $q\geqslant 1$, we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right|$$

$$\leqslant \frac{ab}{b-a}\left[\left\{\Phi^{1-\frac{1}{q}}(a,x)(x-a)^2 \times \left(\omega_1^*(a,x;1;t^{-s})|f'(a)|^q + \omega_2^*(a,x;1;t^{-s})|f'(x)|^q\right)^{\frac{1}{q}}\right\}\right.$$
$$\left. + \left\{\Phi^{1-\frac{1}{q}}(b,x)(b-x)^2 \times \left(\omega_3^*(b,x;1;t^{-s})|f'(b)|^q + \omega_4^*(b,x;1;t^{-s})|f'(x)|^q\right)^{\frac{1}{q}}\right\}\right],$$

where $\Phi(a,x)$ *and* $\Phi(b,x)$ *are given by* (2.15) *and* (2.16) *respectively and* $\omega_1^*(a,x;1;t^{-s})$, $\omega_2^*(a,x;1;t^{-s})$, $\omega_3^*(b,x;1;t^{-s})$ *and* $\omega_4^*(b,x;1;t^{-s})$ *can be deduced from* (2.9), (2.10), (2.11) *and* (2.12) *respectively.*

III. If $h(t)=1$, then, we have result for harmonic P-function, which appears to be new result.

COROLLARY 2.10. *Let* $f:\mathscr{I}\to\mathbb{R}$ *be a differentiable function on* $\mathscr{I}^0$ *with* $x\in(a,b)$ *and* $f'\in L[a,b]$. *If* $|f'|^q$ *is harmonic* P*-function, then, for* $q\geqslant 1$, *we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right|$$
$$\leqslant \frac{ab(x-a)^2}{b-a}\Phi^{1-\frac{1}{q}}(a,x)\left(\omega^*(a,x;1;1)\left\{|f'(a)|^q+|f'(x)|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{ab(b-x)^2}{b-a}\Phi^{1-\frac{1}{q}}(b,x)\left(\omega^{**}(b,x;1;1)\left\{|f'(b)|^q+|f'(x)|^q\right\}\right)^{\frac{1}{q}},$$

where $\omega^*(a,x;1;1)$ *and* $\omega^{**}(a,x;1;1)$ *can be deduced from* (2.13) *and* (2.14) *respectively.*

IV. If $h(t)=t^s$, then, we have result for harmonic s-convex functions, see [10].

COROLLARY 2.11. *Under the assumptions of Theorem* 2.7, *if* $|f'(.)|\leqslant M$, *then, we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right|$$
$$\leqslant \frac{Mab}{b-a}\left[\left\{\Phi^{1-\frac{1}{q}}(a,x)(x-a)^2\left(\omega_1(a,x;1;h)+\omega_2(a,x;1;h)\right)^{\frac{1}{q}}\right\}\right.$$
$$\left.+\left\{\Phi^{1-\frac{1}{q}}(b,x)(b-x)^2\left(\omega_3(b,x;1;h)+\omega_4(b,x;1;h)\right)^{\frac{1}{q}}\right\}\right].$$

THEOREM 2.12. *Let* $f:\mathscr{I}\to\mathbb{R}$ *be a differentiable function on* $\mathscr{I}^0$ *with* $x\in(a,b)$ *and* $f'\in L[a,b]$. *If* $|f'|^q$ *is harmonic* h*-convex function, then, for* $q\geqslant 1$, *we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right|$$

$$\leqslant \frac{ab}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \times \Bigg[\left((x-a)^2\left\{\psi_1(a,x;q;h)|f'(a)|^q+\psi_2(a,x;q;h)|f'(x)|^q\right\}\right)^{\frac{1}{q}}$$
$$+\left((b-x)^2\left\{\psi_3(b,x;q;h)|f'(b)|^q+\psi_4(b,x;q;h)|f'(x)|^q\right\}\right)^{\frac{1}{q}}\Bigg],$$

where

$$\psi_1(a,x;q;h)=\int_0^1 \frac{t}{(ta+(1-t)x)^{2q}}h(1-t)\mathrm{d}t, \tag{2.17}$$

$$\psi_2(a,x;q;h)=\int_0^1 \frac{t}{(ta+(1-t)x)^{2q}}h(t)\mathrm{d}t, \tag{2.18}$$

$$\psi_3(b,x;q;h)=\int_0^1 \frac{t}{(tb+(1-t)x)^{2q}}h(1-t)\mathrm{d}t, \tag{2.19}$$

and

$$\psi_4(b,x;q;h)=\int_0^1 \frac{t}{(tb+(1-t)x)^{2q}}h(t)\mathrm{d}t, \tag{2.20}$$

respectively.

Proof. Using Lemma 2.1 and the given hypothesis, we have

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right|$$
$$\leqslant \frac{ab(x-a)^2}{b-a}\left(\int_0^1 t\mathrm{d}t\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \frac{t}{(ta+(1-t)x)^{2q}}\left[h(1-t)|f'(a)|^q+h(t)|f'(x)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}$$
$$+\frac{ab(b-x)^2}{b-a}\left(\int_0^1 t\mathrm{d}t\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \frac{t}{(tb+(1-t)x)^{2q}}\left[h(1-t)|f'(b)|^q+h(t)|f'(x)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}$$
$$=\frac{ab(x-a)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\times\left(|f'(a)|^q\int_0^1\frac{t}{(ta+(1-t)x)^{2q}}h(1-t)\mathrm{d}t+|f'(x)|^q\int_0^1\frac{t}{(ta+(1-t)x)^{2q}}h(t)\mathrm{d}t\right)^{\frac{1}{q}}$$
$$+\frac{ab(b-x)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left(|f'(b)|^q\int_0^1\frac{t}{(tb+(1-t)x)^{2q}}h(1-t)\mathrm{d}t+|f'(x)|^q\int_0^1\frac{t}{(tb+(1-t)x)^{2q}}h(t)\mathrm{d}t\right)^{\frac{1}{q}}$$
$$=\frac{ab}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left((x-a)^2\left\{\psi_1(a,x;q;h)|f'(a)|^q+\psi_2(a,x;q;h)|f'(x)|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left((b-x)^2\left\{\psi_3(b,x;q;h)|f'(b)|^q+\psi_4(b,x;q;h)|f'(x)|^q\right\}\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Now we discuss some special cases of Theorem 2.12.

I. If $h(t)=t$, then, we have result for harmonic convex functions.

COROLLARY 2.13. *Let $f:\mathscr{I}\to\mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x\in(a,b)$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonic convex function, then, for $q\geqslant 1$, we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}\mathrm{d}u\right|$$
$$\leqslant\frac{ab}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left((x-a)^2\left\{\psi_1^*(a,x;q;t)|f'(a)|^q+\psi_2^*(a,x;q;t)|f'(x)|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left((b-x)^2\left\{\psi_3^*(b,x;q;t)|f'(b)|^q+\psi_4^*(b,x;q;t)|f'(x)|^q\right\}\right)^{\frac{1}{q}}\right],$$

where

$$\psi_1^*(a,x;q;t)=\int_0^1\frac{t}{(ta+(1-t)x)^{2q}}h(1-t)\mathrm{d}t$$
$$=\frac{1}{2x^{2q}}\,{}_2F_1\left(2q,2;3;1-\frac{a}{x}\right)-\frac{1}{12x^{2q}}\,{}_2F_1\left(2q,3;4;1-\frac{a}{x}\right);\qquad(2.21)$$

$$\psi_2^*(a,x;q;t)=\int_0^1 \frac{t}{(ta+(1-t)x)^{2q}}h(t)\mathrm{d}t=\frac{1}{12x^{2q}}\,{}_2F_1\left(2q,3;4;1-\frac{a}{x}\right); \tag{2.22}$$

$$\begin{aligned}\psi_3^*(b,x;q;t)&=\int_0^1 \frac{t}{(tb+(1-t)x)^{2q}}h(1-t)\mathrm{d}t\\ &=\frac{1}{2x^{2q}}\,{}_2F_1\left(2q,2;3;1-\frac{b}{x}\right)-\frac{1}{12x^{2q}}\,{}_2F_1\left(2q,3;4;1-\frac{b}{x}\right);\end{aligned} \tag{2.23}$$

and

$$\psi_4^*(b,x;q;t)=\int_0^1 \frac{t}{(tb+(1-t)x)^{2q}}h(t)\mathrm{d}t=\frac{1}{12x^{2q}}\,{}_2F_1\left(2q,3;4;1-\frac{b}{x}\right). \tag{2.24}$$

respectively.

II. If $h(t)=t^{-s}$, then, we have result for harmonic s-Godunova-Levin convex functions.

COROLLARY 2.14. *Let $f:\mathscr{I}\to\mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x\in(a,b)$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonic s-Goduova-Levin convex function, where $s\in[0,1]$, then, for $q\geqslant 1$, we have*

$$\begin{aligned}&\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right|\\ \leqslant&\frac{ab(x-a)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\\ &\times\left[\left((x-a)^2\left\{\psi_1(a,x;q;t^{-s})|f'(a)|^q+\psi_2(a,x;q;t^{-s})|f'(x)|^q\right\}\right)^{\frac{1}{q}}\right.\\ &\left.+\left((b-x)^2\left\{\psi_3(b,x;q;t^{-s})|f'(b)|^q+\psi_4(b,x;q;t^{-s})|f'(x)|^q\right\}\right)^{\frac{1}{q}}\right],\end{aligned}$$

where

$$\begin{aligned}\psi_1(a,x;q;t^{-s})&=\int_0^1 \frac{t(1-t)^{-s}}{(ta+(1-t)x)^{2q}}\mathrm{d}t\\ &=\frac{B(2,1-s)}{x^{2q}}\,{}_2F_1\left(2q,2;3-s;1-\frac{a}{x}\right),\end{aligned} \tag{2.25}$$

$$\psi_2(a,x;q;t^{-s})=\int_0^1 \frac{t^{1-s}}{(ta+(1-t)x)^{2q}}\mathrm{d}t$$

$$= \frac{B(2-s,1)}{x^{2q}} {}_2F_1\left(2q, 2-s; 3-s; 1-\frac{a}{x}\right), \tag{2.26}$$

$$\begin{aligned}\psi_3(b,x;q;t^{-s}) &= \int_0^1 \frac{t(1-t)^{-s}}{(tb+(1-t)x)^{2q}}\mathrm{d}t \\ &= \frac{B(2,1-s)}{x^{2q}} {}_2F_1\left(2q, 2; 3-q; 1-\frac{b}{x}\right),\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}\psi_4(b,x;q;t^{-s}) &= \int_0^1 \frac{t^{1-s}}{(tb+(1-t)x)^{2q}}\mathrm{d}t \\ &= \frac{B(2-s,1)}{x^{2q}} {}_2F_1\left(2q, 2-s; 3-s; 1-\frac{b}{x}\right),\end{aligned} \tag{2.28}$$

respectively.

III. If $h(t) = t^s$, then, we have result for harmonic s-convex functions, see [10].

COROLLARY 2.15. *Under the assumptions of Theorem 2.12, if $|f'(x)| \leqslant M$, then, we have*

$$\begin{aligned}&\left| f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u \right| \\ \leqslant &\frac{Mab}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \\ &\times \left[\left((x-a)^2\left\{\psi_1(a,x;q;h) + \psi_2(a,x;q;h)\right\}\right)^{\frac{1}{q}} \right. \\ &\left. + \left((b-x)^2\left\{\psi_3(b,x;q;h) + \psi_4(b,x;q;h)\right\}\right)^{\frac{1}{q}} \right].\end{aligned}$$

THEOREM 2.16. *Let $f : \mathscr{I} \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x \in (a,b)$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonic h-convex function, then, for $q \geqslant 1$, we have*

$$\begin{aligned}&\left| f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u \right| \\ \leqslant &\frac{ab}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\ &\times \left[(x-a)^2 \left\{ \left(\theta_1(a,x;q;h)|f'(a)|^q + \theta_2(a,x;q;h)|f'(x)|^q\right)^{\frac{1}{q}} \right\} \right.\end{aligned}$$

$$+(b-x)^2\left\{\left(\theta_3(b,x;q;h)|f'(b)|^q+\theta_4(b,x;q;h)|f'(x)|^q\right)^{\frac{1}{q}}\right\}\Bigg],$$

where

$$\theta_1(a,x;q;h)=\int_0^1\frac{1}{(ta+(1-t)x)^{2q}}h(1-t)\mathrm{d}t, \tag{2.29}$$

$$\theta_2(a,x;q;h)=\int_0^1\frac{1}{(ta+(1-t)x)^{2q}}h(t)\mathrm{d}t, \tag{2.30}$$

$$\theta_3(b,x;q;h)=\int_0^1\frac{1}{(tb+(1-t)x)^{2q}}h(1-t)\mathrm{d}t, \tag{2.31}$$

and

$$\theta_4(b,x;q;h)=\int_0^1\frac{1}{(tb+(1-t)x)^{2q}}h(t)\mathrm{d}t, \tag{2.32}$$

respectively.

Proof. Using Lemma 2.1, Holder's inequality and the fact that $|f'|^q$, is harmonic h-convex function, then, we have

$$\begin{aligned}
&\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}\mathrm{d}u\right|\\
\leqslant&\frac{ab(x-a)^2}{b-a}\left(\int_0^1 t^p\mathrm{d}t\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\frac{1}{(ta+(1-t)x)^{2q}}\left[h(1-t)|f'(a)|^q+h(t)|f'(x)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\\
&+\frac{ab(b-x)^2}{b-a}\left(\int_0^1 t^p\mathrm{d}t\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\frac{1}{(tb+(1-t)x)^{2q}}\left[h(1-t)|f'(b)|^q+h(t)|f'(x)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\\
=&\frac{ab}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left[(x-a)^2\left\{\left(|f'(a)|^q\int_0^1\frac{1}{(ta+(1-t)x)^{2q}}h(1-t)\mathrm{d}t\right.\right.\right.$$

$$\left.\left.+|f'(x)|^q\int_0^1\frac{1}{(ta+(1-t)x)^{2q}}h(t)\mathrm{d}t\right)^{\frac{1}{q}}\right\}$$

$$+(b-x)^2\left\{\left(|f'(b)|^q\int_0^1\frac{1}{(tb+(1-t)x)^{2q}}h(1-t)\mathrm{d}t\right.\right.$$

$$\left.\left.\left.+|f'(x)|^q\int_0^1\frac{1}{(tb+(1-t)x)^{2q}}h(t)\mathrm{d}t\right)^{\frac{1}{q}}\right\}\right]$$

$$=\frac{ab}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\times\left[(x-a)^2\left\{\left(\theta_1(a,x;q;h)|f'(a)|^q+\theta_2(a,x;q;h)|f'(x)|^q\right)^{\frac{1}{q}}\right\}\right.$$

$$\left.+(b-x)^2\left\{\left(\theta_3(b,x;q;h)|f'(b)|^q+\theta_4(b,x;q;h)|f'(x)|^q\right)^{\frac{1}{q}}\right\}\right].$$

This completes the proof. □

Now we discuss some special cases of Theorem 2.16.

I. If $h(t)=t$, then, we have result for harmonic convex functions.

COROLLARY 2.17. *Let $f:\mathscr{I}\to\mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x\in(a,b)$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonic convex function, then, for $q\geqslant 1$, we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}\mathrm{d}u\right|$$

$$\leqslant\frac{ab}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\times\left[(x-a)^2\left\{\left(\theta_1^*(a,x;q;t)|f'(a)|^q+\theta_2^*(a,x;q;t)|f'(x)|^q\right)^{\frac{1}{q}}\right\}\right.$$

$$\left.+(b-x)^2\left\{\left(\theta_3^*(b,x;q;t)|f'(b)|^q+\theta_4^*(b,x;q;t)|f'(x)|^q\right)^{\frac{1}{q}}\right\}\right],$$

where

$$\theta_1^*(a,x;q;t)=\int_0^1\frac{1}{(ta+(1-t)x)^{2q}}h(1-t)\mathrm{d}t$$

$$= \frac{1}{x^{2q}}\,{}_2F_1\left(2q,1;2;1-\frac{a}{x}\right) - \frac{1}{2x^{2q}}\,{}_2F_1\left(2q,2;3;1-\frac{a}{x}\right);$$

$$\theta_2^*(a,x;q;t) = \int_0^1 \frac{1}{(ta+(1-t)x)^{2q}} h(t)\mathrm{d}t = \frac{1}{2x^{2q}}\,{}_2F_1\left(2q,2;3;1-\frac{a}{x}\right);$$

$$\theta_3^*(b,x;q;t) = \int_0^1 \frac{1}{(tb+(1-t)x)^{2q}} h(1-t)\mathrm{d}t$$
$$= \frac{1}{x^{2q}}\,{}_2F_1\left(2q,1;2;1-\frac{b}{x}\right) - \frac{1}{2x^{2q}}\,{}_2F_1\left(2q,2;3;1-\frac{b}{x}\right);$$

and

$$\theta_4^*(b,x;q;t) = \int_0^1 \frac{1}{(tb+(1-t)x)^{2q}} h(t)\mathrm{d}t = \frac{1}{2x^{2q}}\,{}_2F_1\left(2q,2;3;1-\frac{b}{x}\right).$$

respectively.

II. If $h(t)=t^{-s}$, then, we have result for harmonic s-Godunova-Levin convex function, which appears to be new result in the literature.

COROLLARY 2.18. *Let $f:\mathscr{I}\to\mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x\in(a,b)$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonic s-Godunova-Levin convex function where $s\in[0,1]$, then, for $q\geqslant 1$, we have*

$$\left| f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u \right|$$
$$\leqslant \frac{ab}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times \left[(x-a)^2 \left\{ \left(\theta_1(a,x;q;t^{-s})|f'(a)|^q + \theta_2(a,x;q;t^{-s})|f'(x)|^q\right)^{\frac{1}{q}} \right\} \right.$$
$$\left. +(b-x)^2 \left\{ \left(\theta_3(b,x;q;t^{-s})|f'(b)|^q + \theta_4(b,x;q;t^{-s})|f'(x)|^q\right)^{\frac{1}{q}} \right\} \right],$$

where

$$\theta_1(a,x;q;t^{-s}) = \int_0^1 \frac{(1-t)^{-s}}{(ta+(1-t)x)^{2q}}\mathrm{d}t = \frac{B(1,1-s)}{x^{2q}}\,{}_2F_1\left(2q,1;2-s;1-\frac{a}{x}\right) \quad (2.33)$$

$$\theta_2(a,x;q;t^{-s}) = \int_0^1 \frac{t^{-s}}{(ta+(1-t)x)^{2q}}\mathrm{d}t$$
$$= \frac{B(1-s,1)}{x^{2q}}\,{}_2F_1\left(2q,1-s;2-s;1-\frac{a}{x}\right) \tag{2.34}$$

$$\theta_3(b,x;q;t^{-s}) = \int_0^1 \frac{(1-t)^{-s}}{(tb+(1-t)x)^{2q}}\mathrm{d}t = \frac{B(1,1-s)}{x^{2q}}\,{}_2F_1\left(2q,1;2-s;1-\frac{b}{x}\right) \tag{2.35}$$

and

$$\theta_4(b,x;q;t^{-s}) = \int_0^1 \frac{t^{-s}}{(tb+(1-t)x)^{2q}}\mathrm{d}t$$
$$= \frac{B(1-s,1)}{x^{2q}}\,{}_2F_1\left(2q,1-s;2-s;1-\frac{b}{x}\right), \tag{2.36}$$

respectively.

III. If $h(t)=t^s$, then, we have result for harmonic s-convex functions, see [10].

COROLLARY 2.19. *Under the assumptions of Theorem 2.16, if $|f'(.)| \leqslant M$, then, we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right|$$
$$\leqslant \frac{Mab}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times\left[(x-a)^2\left\{(\theta_1(a,x;q;h)+\theta_2(a,x;q;h))^{\frac{1}{q}}\right\}\right.$$
$$\left.+(b-x)^2\left\{(\theta_3(b,x;q;h)+\theta_4(b,x;q;h))^{\frac{1}{q}}\right\}\right],$$

THEOREM 2.20. *Let $f:\mathscr{I}\to\mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x\in(a,b)$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonic h-convex function, then, for $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right|$$
$$\leqslant \frac{ab}{b-a}\Theta^{\frac{1}{p}}(a,x;p)\left\{(x-a)^2\left([|f'(a)|^q+|f'(x)|^q]\int_0^1 h(t)\mathrm{d}t\right)^{\frac{1}{q}}\right.$$

$$+(b-x)^2\left(\left[|f'(b)|^q+|f'(x)|^q\right]\int_0^1 h(t)\mathrm{d}t\right)^{\frac{1}{q}}\Bigg\},$$

where

$$\Theta(a,x;p)=\int_0^1 \frac{t^p}{(ta+(1-t)x)^{2p}}\mathrm{d}t=\frac{B(p+1,1)}{x^{2p}}\,{}_2F_1\left(2p,p+1;p+2;1-\frac{a}{x}\right). \tag{2.37}$$

Proof. Using Lemma 2.1, Holder's inequality and the fact that $|f'|^q$ is harmonic h-convex functions, we have

$$\begin{aligned}
&\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right|\\
\leqslant&\frac{ab(x-a)^2}{b-a}\left(\int_0^1 \frac{t^p}{(ta+(1-t)x)^{2p}}\mathrm{d}t\right)^{\frac{1}{p}}\times\left(\int_0^1 \left[h(1-t)|f'(a)|^q+h(t)|f'(x)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\\
&+\frac{ab(b-x)^2}{b-a}\left(\int_0^1 \frac{t^p}{(tb+(1-t)x)^{2p}}\mathrm{d}t\right)^{\frac{1}{p}}\times\left(\int_0^1 \left[h(1-t)|f'(b)|^q+h(t)|f'(x)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\\
=&\frac{ab}{b-a}\Theta^{\frac{1}{p}}(a,x;p)\times\left\{(x-a)^2\left(\left[|f'(a)|^q+|f'(x)|^q\right]\int_0^1 h(t)\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\left.+(b-x)^2\left(\left[|f'(b)|^q+|f'(x)|^q\right]\int_0^1 h(t)\mathrm{d}t\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

This completes the proof. □

Now we discuss some special cases of Theorem 2.20.

I. If $h(t)=t$, then, we have a result for harmonic convex functions.

COROLLARY 2.21. *Let $f:\mathscr{I}\to\mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ with $x\in(a,b)$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonic convex function, then, for $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u\right|$$

$$\leqslant \frac{ab}{b-a}\Theta^{\frac{1}{p}}(a,x;p)$$
$$\times\left\{(x-a)^2\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}+(b-x)^2\left(\frac{|f'(b)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right\},$$

where $\Theta(a,x;p)$ *is given by* 2.37.

II. If $h(t)=t^{-s}$, then, we have result for harmonic s-Godunova-Levin convex function.

COROLLARY 2.22. *Let* $f:\mathscr{I}\to\mathbb{R}$ *be a differentiable function on* $\mathscr{I}^0$ *with* $x\in(a,b)$ *and* $f'\in L[a,b]$. *If* $|f'|^q$ *is harmonic* s*-Godunova-levin convex function, then, for* $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}\mathrm{d}u\right|$$
$$\leqslant \frac{ab}{b-a}\Theta^{\frac{1}{p}}(a,x;p)$$
$$\times\left\{(x-a)^2\left(\frac{|f'(a)|^q+|f'(x)|^q}{1-s}\right)^{\frac{1}{q}}+(b-x)^2\left(\frac{|f'(b)|^q+|f'(x)|^q}{1-s}\right)^{\frac{1}{q}}\right\},$$

where $\Theta(a,x;p)$ *is given by* 2.37.

III. If $h(t)=1$, then, we have result for harmonic P-function.

COROLLARY 2.23. *Let* $f:\mathscr{I}\to\mathbb{R}$ *be a differentiable function on* $\mathscr{I}^0$ *with* $x\in(a,b)$ *and* $f'\in L[a,b]$. *If* $|f'|^q$ *is harmonic* P*-function, then, for* $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}\mathrm{d}u\right|$$
$$\leqslant \frac{ab}{b-a}\Theta^{\frac{1}{p}}(a,x;p)$$
$$\times\left\{(x-a)^2\left(|f'(a)|^q+|f'(x)|^q\right)^{\frac{1}{q}}+(b-x)^2\left(|f'(b)|^q+|f'(x)|^q\right)^{\frac{1}{q}}\right\},$$

where $\Theta(a,x;p)$ *is given by* 2.37.

IV. If $h(t)=t^s$, then, we have result for harmonic s-convex function, see [10].

COROLLARY 2.24. *Under the assumptions of Theorem* 2.20, *if* $|f'(.)|\leqslant M$, *then, we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}\mathrm{d}u\right|\leqslant\frac{abM^{\frac{1}{q}}}{b-a}\Theta^{\frac{1}{p}}(a,x;p)\left\{(x-a)^2+(b-x)^2\right\}\left(\int_0^1 h(t)\mathrm{d}t\right)^{\frac{1}{q}}.$$

3. Applications

In this section, we discuss applications of some of our results derived in the previous section.

PROPOSITION 3.1. *If $x = \frac{a+b}{2}$ and $f(x) = x$ in Corollary 2.3, then for $0 < a < b$, we have*

$$\begin{aligned}
&\left|A(a,b) - G^2(a,b)L^{-1}(a,b)\right| \\
\leqslant &\frac{ab(b-a)^{\frac{2-q}{q}}}{2^{\frac{2}{q}-2}(b+a)^2} \\
&\times \left\{ \left[\mathrm{B}(1+q,1)\,{}_2F_1\left(2q,1+q;2+q;\frac{b-a}{b+a}\right) - \mathrm{B}(2+q,1)\,{}_2F_1\left(2q,2+q;3+q;\frac{b-a}{b+a}\right)\right]^{\frac{1}{q}} \right. \\
&\left. + \left[\mathrm{B}(1+q,1)\,{}_2F_1\left(2q,1+q;2+q;\frac{a-b}{b+a}\right) - \mathrm{B}(2+q,1)\,{}_2F_1\left(2q,2+q;3+q;\frac{a-b}{b+a}\right)\right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

PROPOSITION 3.2. *If $x = \frac{a+b}{2}$ and $f(x) = x$ in Corollary 2.8, then for $0 < a < b$, we have*

$$\begin{aligned}
&\left|A(a,b) - G^2(a,b)L^{-1}(a,b)\right| \\
\leqslant &\frac{ab(b-a)^{\frac{3-2q}{q}}}{2^{\frac{2}{q}-1}(b+a)^{\frac{2}{q}}} \left\{ \left(\frac{1}{a} - \frac{2}{b-a}\ln\frac{a+b}{2a}\right)^{1-\frac{1}{q}} \right. \\
&\times \left[{}_2F_1\left(2q,1+q;2+q;\frac{b-a}{b+a}\right) + \frac{1}{3}\,{}_2F_1\left(2q,2+q;3+q;\frac{b-a}{b+a}\right)\right]^{\frac{1}{q}} \\
&\left. + \left(\frac{1}{b-a}\ln\frac{2b}{a+b} - \frac{1}{b}\right)^{1-\frac{1}{q}} \left[{}_2F_1\left(2q,1+q;2+q;\frac{a-b}{a+b}\right)\right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

PROPOSITION 3.3. *If $x = \frac{a+b}{2}$ and $f(x) = x$ in Corollary 2.13, then for $0 < a < b$, we have*

$$\begin{aligned}
&\left|A(a,b) - G^2(a,b)L^{-1}(a,b)\right| \\
\leqslant &\frac{ab(b-a)^{\frac{2-q}{q}}}{2^{\frac{2-q}{q}}(a+b)^2} \left\{ \left[{}_2F_1\left(2q,2;3;\frac{b-a}{b+a}\right)\right]^{\frac{1}{q}} + \left[{}_2F_1\left(2q,2;3;\frac{a-b}{a+b}\right)\right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

PROPOSITION 3.4. *If $x = \frac{a+b}{2}$ and $f(x) = x$ in Corollary 2.17, then for $0 < a < b$, we have*

$$
\begin{aligned}
&\left|A(a,b)-G^2(a,b)L^{-1}(a,b)\right| \\
\leqslant &\frac{ab(b-a)^{\frac{2-q}{q}}}{2^{\frac{2}{q}-2}(a+b)^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left[{}_2F_1\left(2q,1;2;\frac{b-a}{b+a}\right)\right]^{\frac{1}{q}}+\left[{}_2F_1\left(2q,1;2;\frac{a-b}{a+b}\right)\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

PROPOSITION 3.5. *If* $x=\frac{a+b}{2}$ *and* $f(x)=x$ *in Corollary 2.21, then for* $0<a<b$*, we have*

$$
\left|A(a,b)-G^2(a,b)L^{-1}(a,b)\right| \leqslant \frac{ab(b-a)^{\frac{2-q}{q}}}{2^{\frac{2}{q}-2q}(a+b)^{2q-2}}\left[{}_2F_1\left(2p,p+1;p+2;\frac{b-a}{b+a}\right)\right]^{\frac{1}{p}}.
$$

Acknowledgement. Authors are thankful to editor and anonymous referee for their valuable comments and suggestions. Authors also extend their appreciation to the International Scientific Partnership Program ISPP at King Saud University, Riyadh, Saudi Arabia for funding this research work through ISPP.

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(Received April 4, 2018)

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