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SOME INEQUALITIES FOR GENERALIZED BELL–TOUCHARD POLYNOMIALS

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Abstract. A unified generalization for the Bell-Touchard polynomials of order k and the r-Bell polynomials is established. It is shown that the generating function of the generalized Bell-Touchard polynomials is logarithmically absolutely monotonic. Applying this result we obtain some inequalities for the generalized Bell-Touchard polynomials. In particular, we obtain the logarithmic convexity of the generalized Bell-Touchard polynomials.

1. Introduction and main results

Asai et al. [1] introduced the Bell number of order k as follows. For an integer $k \geqslant 1$, define the k-times iterated exponential function denoted by $\exp_k(z)$:

$$\exp_k(z) = \underbrace{\exp(\exp\cdots(\exp(z)))}_{k-times}. \tag{1.1}$$

Let $\{B_k(n)\}_{n=0}^{\infty}$ be the sequence of numbers given in the power series of $\exp_k(z)$, namely,

$$\exp_k(z) = \sum_{n=0}^{\infty} \frac{B_k(n)}{n!} z^n. \tag{1.2}$$

The Bell numbers $\{b_k(n)\}_{n=0}^{\infty}$ of order k are defined by

$$b_k(n) = \frac{B_k(n)}{\exp_k(0)}, \; n \geqslant 0. \tag{1.3}$$

In particular, when $k = 2$, the numbers $b_2(n)$ are usually known as the Bell numbers, the first few terms of which are $1, 1, 2, 5, 15, 52, 203$. Thus, it is natural that

$$e^{e^z - 1} = \sum_{n=0}^{\infty} \frac{b_2(n)}{n!} z^n \tag{1.4}$$

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because $\exp_2(0) = e$. Qi [15] found the (logarithmically) absolute and complete monotonicity of the generating functions $e^{e^{\pm x}}$ for the Bell numbers $b_2(n)$. Based on the results, he obtain some interesting inequalities for the Bell numbers $b_2(n)$ with the aid of properties of absolutely and completely monotonic functions.

As a different generalization of the Bell numbers $b_2(n)$ for $n \geqslant 0$, the Touchard polynomials $T_n(x)$ can be defined by

$$e^{x(e^z-1)} = \sum_{n=0}^{\infty} T_n(x) \frac{z^n}{n!}.$$

It is clear $T_n(1) = b_2(n)$. It is pointed out in [16] that there have been many researches on applications of the Touchard polynomials in soliton theory [5, 6, 7]. For more details on the Touchard polynomials, see the recent book [8] and references therein.

Recently, Qi et al. [17] gave a unified generalization $T_{k,n}(\mathbf{x_k})$ so called the Bell-Touchard polynomials for the Bell numbers of order k and the Touchard polynomials:

$$\exp(x_1(\exp(\cdots x_{k-1}(\exp(x_k(\exp(z)-1))-1)\cdots))-1) = \sum_{n=0}^{\infty} T_{k,n}(\mathbf{x_k}) \frac{z^n}{n!},$$

where $\mathbf{x_k} = (x_1, x_2, \ldots, x_k)$. In their interesting paper, the explicit formula, inversion formula, and recurrence relations for the generalization in terms of the Stirling numbers of the first and second kinds were established. They also derived logarithmic convexity and logarithmic concavity for the generalization, and confirmed that the generalization satisfies conditions for sequences required in white noise distribution theory.

Let $\mathbf{r_k} = (r_1, r_2, \ldots, r_k)$. Define a sequence of bivariate functions

$$g_i(t,z) = \exp\left(x_i(\exp(t)-1) + r_i z\right), \quad 1 \leqslant i \leqslant k.$$

We construct a new function sequence $\{\mathscr{T}_i(t,z)\}_{i=0}^{k}$ recursively by

$$\begin{aligned} &\mathscr{T}_0(t,z) = t, \\ &\mathscr{T}_i(t,z) = g_{k+1-i}\left(\mathscr{T}_{i-1}(t,z), z\right), \quad 1 \leqslant i \leqslant k. \end{aligned}$$

Further we let $\mathfrak{T}_k(z) := \mathfrak{T}_k(z; \{x_i\}_{i=1}^{k}, \{r_i\}_{i=1}^{k}) = \lim_{t \to z} \mathscr{T}_k(t,z)$. We generalize the Bell-Touchard polynomials by the following generating function

$$\mathfrak{T}_k(z) = \sum_{n=0}^{\infty} T_{k,n}(\mathbf{x_k}; \mathbf{r_k}) \frac{z^n}{n!}. \tag{1.5}$$

For convenience, we would like to recommend $T_{k,n}(\mathbf{x_k}; \mathbf{r_k})$ the name r-Bell-Touchard polynomials of order k. It is clear that $T_{k,n}(\mathbf{x_k}; \mathbf{r_k})$ reduce to the Bell-Touchard polynomials $T_{k,n}(\mathbf{x_k})$ when all $r_i = 0$. When $k = 1$, $r_1 = r$ and $x_1 = x$, the polynomials $T_{1,n}(x; r)$ are the known r-Bell polynomials [9] usually denoted by $B_{n,r}(x)$ where r is a non-negative integer. As we know, the r-Bell polynomials are defined by

$$B_{n,r}(x) = \sum_{i=0}^{n} S_r(n+r, i+r) x^i,$$

where $S_r(n+r,i+r)$ are the r-Stirling numbers of the second kind [2]. The exponential generating function for the r-Bell polynomials $B_{n,r}(x)$ was given by [2, 9]:

$$\sum_{n=0}^{\infty} B_{n,r}(x)\frac{x^n}{n!} = e^{x(e^z-1)+rz}.$$

In particular, we call $B_{n,r}(1) := B_{n,r}$ the r-Bell numbers originally studied by Carlitz [3, 4], and they were systematically treated in [2, 9]. For divisibility properties of the r-Bell numbers one can refer to the recent work [10].

The first purpose of this paper is to verify that the functions $\mathfrak{T}_k(-z)$ are logarithmically completely monotonic for all $k \geqslant 1$.

THEOREM 1. *If $r_i \geqslant 0$ and $x_i > 0$ for $1 \leqslant i \leqslant k$, then the functions $\mathfrak{T}_k(-z)$ are logarithmically completely monotonic on $(-\infty,\infty)$ for all $k \geqslant 1$.*

REMARK 1. *According to Theorem 1, it is equivalent that $\mathfrak{T}_k(z)$ is a logarithmically absolutely monotonic function on $(-\infty,\infty)$ if $r_i \geqslant 0$ and $x_i > 0$ for $1 \leqslant i \leqslant k$.*

By using the above theorem, we establish the following inequalities for the r-Bell-Touchard polynomials $\left\{T_{k,n}(\mathbf{x_k};\mathbf{r_k})\right\}_{n=0}^{\infty}$ by Qi's technique used in [15, 17, 18].

THEOREM 2. *Let $r_i \geqslant 0$ and $x_i > 0$ for $1 \leqslant i \leqslant k$. Let $q \geqslant 1$ be a positive integer and let $|a_{ij}|_q$ denote a determinant of order q with elements a_{ij}. If a_i for $1 \leqslant i \leqslant q$ are non-negative integers, then*

$$|T_{k,a_i+a_j}(\mathbf{x_k};\mathbf{r_k})|_q \geqslant 0 \tag{1.6}$$

and

$$|(-1)^{a_i+a_j} T_{k,a_i+a_j}(\mathbf{x_k};\mathbf{r_k})|_q \geqslant 0. \tag{1.7}$$

THEOREM 3. *Let $r_i \geqslant 0$ and $x_i > 0$ for $1 \leqslant i \leqslant k$. If $a = (a_1,a_2,\cdots,a_n)$ and $c = (c_1,c_2,\cdots,c_n)$ are non-increasing n-tuples of non-negative integers such that $\sum_{i=1}^{j} a_i \geqslant \sum_{i=1}^{j} c_i$ for $1 \leqslant j \leqslant n-1$ and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} c_i$, then*

$$\prod_{i=1}^{n} T_{k,a_i}(\mathbf{x_k};\mathbf{r_k}) \geqslant \prod_{i=1}^{n} T_{k,c_i}(\mathbf{x_k};\mathbf{r_k}). \tag{1.8}$$

Taking $a_1 = a_2 = \cdots = a_q = n+l$, $a_{q+1} = a_{q+2} = \cdots = a_n = l$ and $c_1 = c_2 = \cdots = c_n = q+l$ in (1.8), we immediately have the following corollary.

COROLLARY 1. *Let $r_i \geqslant 0$ and $x_i > 0$ for $1 \leqslant i \leqslant k$. If $l \geqslant 0$ and $n \geqslant q \geqslant 0$, then*

$$\left(T_{k,n+l}(\mathbf{x_k};\mathbf{r_k})\right)^q \left(T_{k,l}(\mathbf{x_k};\mathbf{r_k})\right)^{n-q} \geqslant \left(T_{k,q+l}(\mathbf{x_k};\mathbf{r_k})\right)^n. \tag{1.9}$$

Taking $a_1 = q+l$, $a_2 = n-q+l$, $c_1 = m+l$ and $c_2 = n-m+l$ in (1.8), we have another inequality for the r-Bell-Touchard polynomials.

COROLLARY 2. *Let $r_i \geqslant 0$ and $x_i > 0$ for $1 \leqslant i \leqslant k$. If $l \geqslant 0$, $n \geqslant q \geqslant m$, $2q \geqslant n$ and $2m \geqslant n$, then*

$$T_{k,q+l}(\mathbf{x_k};\mathbf{r_k})T_{k,n-q+l}(\mathbf{x_k};\mathbf{r_k}) \geqslant T_{k,m+l}(\mathbf{x_k};\mathbf{r_k})T_{k,n-m+l}(\mathbf{x_k};\mathbf{r_k}). \tag{1.10}$$

In particular, when $n = 2$ and $q = 1$ in (1.9), it is easy to see

$$T_{k,l}(\mathbf{x_k};\mathbf{r_k})T_{k,l+2}(\mathbf{x_k};\mathbf{r_k}) \geqslant \left(T_{k,l+1}(\mathbf{x_k};\mathbf{r_k})\right)^2, \tag{1.11}$$

which means that the sequence $\{T_{k,l}(\mathbf{x_k};\mathbf{r_k})\}_{l=0}^{\infty}$ is logarithmically convex. Note that (1.11) is also a special case of (1.10) when $n = q = 2$ and $m = 1$.

THEOREM 4. *Let $r_i \geqslant 0$ and $x_i > 0$ for $1 \leqslant i \leqslant k$. For $l \geqslant 0$ and $m,n \in \mathbb{N}$, let*

$$\begin{aligned}
\mathscr{G}_{l,m,n} =& T_{k,l+2m+n}(\mathbf{x_k};\mathbf{r_k})(T_{k,l}(\mathbf{x_k};\mathbf{r_k}))^2 - T_{k,l+m+n}(\mathbf{x_k};\mathbf{r_k})T_{k,l+m}(\mathbf{x_k};\mathbf{r_k})T_{k,l}(\mathbf{x_k};\mathbf{r_k})\\
&- T_{k,l+n}(\mathbf{x_k};\mathbf{r_k})T_{k,l+2m}(\mathbf{x_k};\mathbf{r_k})T_{k,l}(\mathbf{x_k};\mathbf{r_k}) + T_{k,l+n}(\mathbf{x_k};\mathbf{r_k})(T_{k,l+m}(\mathbf{x_k};\mathbf{r_k}))^2,\\
\mathscr{H}_{l,m,n} =& T_{k,l+2m+n}(\mathbf{x_k};\mathbf{r_k})(T_{k,l}(\mathbf{x_k};\mathbf{r_k}))^2 - 2T_{k,l+m+n}(\mathbf{x_k};\mathbf{r_k})T_{k,l+m}(\mathbf{x_k};\mathbf{r_k})T_{k,l}(\mathbf{x_k};\mathbf{r_k})\\
&+ T_{k,l+n}(\mathbf{x_k};\mathbf{r_k})(T_{k,l+m}(\mathbf{x_k};\mathbf{r_k}))^2,\\
\mathscr{I}_{l,m,n} =& T_{k,l+2m+n}(\mathbf{x_k};\mathbf{r_k})(T_{k,l}(\mathbf{x_k};\mathbf{r_k}))^2 - 2T_{k,l+n}(\mathbf{x_k};\mathbf{r_k})T_{k,l+2m}(\mathbf{x_k};\mathbf{r_k})T_{k,l}(\mathbf{x_k};\mathbf{r_k})\\
&+ T_{k,l+n}(\mathbf{x_k};\mathbf{r_k})(T_{k,l+m}(\mathbf{x_k};\mathbf{r_k}))^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathscr{G}_{l,m,n} \geqslant 0, \quad \mathscr{H}_{l,m,n} \geqslant 0,\\
&\mathscr{H}_{l,m,n} \lesseqgtr \mathscr{G}_{l,m,n}, \text{ when } \quad m \lesseqgtr n,\\
&\mathscr{I}_{l,m,n} \geqslant \mathscr{G}_{l,m,n} \geqslant 0, \text{ when } \quad n \geqslant m.
\end{aligned} \tag{1.12}$$

THEOREM 5. *Let $r_i \geqslant 0$ and $x_i > 0$ for $1 \leqslant i \leqslant k$. For $q,n \in \mathbb{N}$, we have*

$$\left(\prod_{l=0}^{n} T_{k,q+2l}(\mathbf{x_k};\mathbf{r_k})\right)^{\frac{1}{n+1}} \geqslant \left(\prod_{l=0}^{n-1} T_{k,q+2l+1}(\mathbf{x_k};\mathbf{r_k})\right)^{\frac{1}{n}}. \tag{1.13}$$

As consequences, it is worth noting that the r-Bell polynomials $B_{n,r}(x)$ have the same properties when $x > 0$ because $B_{n,r}(x) = T_{1,n}(x;r)$. In particular, for fixed $x > 0$, the sequence of the r-Bell polynomials $\{B_{n,r}(x)\}_{n=0}^{\infty}$ is logarithmically convex.

2. Proofs of main theorems

It was introduced in [15] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $(-1)^m f^{(m)}(z) \geqslant 0$ on I for all $m \geqslant 0$. An infinitely differentiable function f is said to be logrithmically completely monotonic on an interval I if $(-1)^m(\ln f(z))^{(m)} \geqslant 0$ on I for all $m \geqslant 1$. For more information, see [13, 19, 21]. As pointed out in [15], a logrithmically completely monotonic function on an interval I is also completely monotonic on the interval I, but not conversely. Based on these facts, we can give a proof on the complete monotonicity for the functions $\mathfrak{T}_k(-z)$ for all $k \geqslant 1$.

Proof of Theorem 1. We prove this theorem by induction on k. For $k=1$, it is obvious that

$$(-1)^m (\ln(\mathfrak{T}_1(-z)))^{(m)} = \begin{cases} x_1 e^{-z} + r_1, & m=1, \\ x_1 e^{-z}, & m \geqslant 2, \end{cases}$$

which implies that, for $z \in (-\infty, \infty)$,

$$(-1)^m (\ln(\mathfrak{T}_1(-z)))^{(m)} \geqslant 0,$$

because $x_1 > 0$ and $r_1 \geqslant 0$. Thus, $\mathfrak{T}_1(-z)$ is a logarithmically completely monotonic function. We assume that $\mathfrak{T}_k(-z)$ is logarithmically completely monotonic for all $k \leqslant K$ with $K \geqslant 1$. By the assumption, $\mathfrak{T}_K(-z) := \mathfrak{T}_K\left(-z; \{x_i\}_{i=1}^K, \{r_i\}_{i=1}^K\right)$ is completely monotonic. It implies that $\mathfrak{T}_K\left(-z; \{x_i\}_{i=2}^{K+1}, \{r_i\}_{i=2}^{K+1}\right)$ is also completely monotonic, which is equivalent to

$$(-1)^m \left(\mathfrak{T}_K(-z; \{x_i\}_{i=2}^{K+1}, \{r_i\}_{i=2}^{K+1})\right)^{(m)} \geqslant 0, \quad m \geqslant 0.$$

Since $\mathfrak{T}_{K+1}(-z) = \exp\left\{x_1 \mathfrak{T}_K(-z; \{x_i\}_{i=2}^{K+1}, \{r_i\}_{i=2}^{K+1}) - r_1 z\right\}$, we have

$$(-1)^m (\ln(\mathfrak{T}_{K+1}(-z)))^{(m)} = \begin{cases} -x_1 \left(\mathfrak{T}_K(-z; \{x_i\}_{i=2}^{K+1}, \{r_i\}_{i=2}^{K+1})\right)' + r_1, & m=1, \\ (-1)^m x_1 \left(\mathfrak{T}_K(-z; \{x_i\}_{i=2}^{K+1}, \{r_i\}_{i=2}^{K+1})\right)^{(m)}, & m \geqslant 2. \end{cases}$$

Therefore, we can conclude that the function $\mathfrak{T}_{K+1}(-z)$ is logarithmically completely monotonic because $(-1)^m (\ln(\mathfrak{T}_{K+1}(-z)))^{(m)} \geqslant 0$, and the proof is complete. □

Clearly, $\mathfrak{T}_k(-z)$ is completely monotonic according to Theorem 1. We are now in a position to give the proofs of theorems 2-5 by Qi's technique used in [15, 17, 18].

Proof of Theorem 2. According to [12] and [13, p. 367], we obtain that if f is completely monotonic on $[0, \infty)$, then

$$|f^{(a_i+a_j)}(z)|_q \geqslant 0 \tag{2.1}$$

and

$$|(-1)^{a_i+a_j} f^{(a_i+a_j)}(z)|_q \geqslant 0. \tag{2.2}$$

By replacing $f(z)$ by the function $\mathfrak{T}_k(-z)$ in (2.1) and (2.2) and taking the limit $z \to 0^+$, we have

$$\lim_{z \to 0^+} |(\mathfrak{T}_k(-z))^{(a_i+a_j)}|_q = |(-1)^{a_i+a_j} T_{k,a_i+a_j}(\mathbf{x_k}; \mathbf{r_k})|_q \geqslant 0$$

and

$$\lim_{z\to 0^+} |(-1)^{a_i+a_j}(\mathfrak{T}_k(-z))^{(a_i+a_j)}|_q = |T_{k,a_i+a_j}(\mathbf{x_k};\mathbf{r_k})|_q \geqslant 0.$$

Thus, the desired determinant inequalities (1.6) and (1.7) are derived. □

Proof of Theorem 3. According to [13, p. 367, Theorem 2], we obtain that if f is completely monotonic on $[0,\infty)$, then

$$\prod_{i=1}^{n}(-1)^{a_i} f^{(a_i)}(z) \geqslant \prod_{i=1}^{n}(-1)^{c_i} f^{(c_i)}(z).$$

If we replace $f(z)$ by the function $\mathfrak{T}_k(-z)$, we have

$$\prod_{i=1}^{n}(-1)^{a_i}(\mathfrak{T}_k(-z))^{(a_i)} \geqslant \prod_{i=1}^{n}(-1)^{c_i}(\mathfrak{T}_k(-z))^{(c_i)}.$$

Taking the limit $z \to 0^+$ gives

$$\prod_{i=1}^{n} T_{k,a_i}(\mathbf{x_k};\mathbf{r_k}) \geqslant \prod_{i=1}^{n} T_{k,c_i}(\mathbf{x_k};\mathbf{r_k}),$$

Thus, the proof of Theorem 3 is complete. □

Proof of Theorem 4. In [20, Theorem 1 and Remark 2], it was obtained that if f is completely monotonic on $(0,\infty)$ and

$$G_{m,n} = (-1)^n \left\{ f^{(n+2m)} f^2 - f^{(n+m)} f^{(m)} f - f^{(n)} f^{(2m)} f + f^{(n)} [f^{(m)}]^2 \right\},$$
$$H_{m,n} = (-1)^n \left\{ f^{(n+2m)} f^2 - 2f^{(n+m)} f^{(m)} f + f^{(n)} [f^{(m)}]^2 \right\},$$
$$I_{m,n} = (-1)^n \left\{ f^{(n+2m)} f^2 - 2f^{(n)} f^{(2m)} f + f^{(n)} [f^{(m)}]^2 \right\},$$

for $n,m \in \mathbb{N}$, then

$$\begin{aligned} &G_{m,n} \geqslant 0,\ H_{m,n} \geqslant 0,\\ &H_{m,n} \lesseqgtr G_{m,n}, \quad \textit{when } m \lesseqgtr n,\\ &I_{m,n} \geqslant G_{m,n} \geqslant 0, \quad \textit{when } m \geqslant n. \end{aligned} \tag{2.3}$$

Replacing $f(z)$ by $(-1)^l(\mathfrak{T}_k(-z))^{(l)}$ in $G_{m,n}$, $H_{m,n}$ and $I_{m,n}$, and simplifying give

$$\begin{aligned} G_{m,n} = (-1)^{l+n} \Big\{ &(\mathfrak{T}_k(-z))^{(l+2m+n)}[(\mathfrak{T}_k(-z))^{(l)}]^2\\ &- (\mathfrak{T}_k(-z))^{(l+m+n)}(\mathfrak{T}_k(-z))^{(l+m)}(\mathfrak{T}_k(-z))^{(l)}\\ &- (\mathfrak{T}_k(-z))^{(l+n)}(\mathfrak{T}_k(-z))^{(l+2m)}(\mathfrak{T}_k(-z))^{(l)}\\ &+ (\mathfrak{T}_k(-z))^{(l+n)}[(\mathfrak{T}_k(-z))^{(l+m)}]^2 \Big\}, \end{aligned}$$

$$H_{m,n} = (-1)^{l+n}\Big\{(\mathfrak{T}_k(-z))^{(l+2m+n)}[(\mathfrak{T}_k(-z))^{(l)}]^2 - 2(\mathfrak{T}_k(-z))^{(l+m+n)}(\mathfrak{T}_k(-z))^{(l+m)}(\mathfrak{T}_k(-z))^{(l)} + (\mathfrak{T}_k(-z))^{(l+n)}[(\mathfrak{T}_k(-z))^{(l+m)}]^2\Big\},$$

and

$$I_{m,n} = (-1)^{l+n}\Big\{(\mathfrak{T}_k(-z))^{(l+2m+n)}[(\mathfrak{T}_k(-z))^{(l)}]^2 - 2(\mathfrak{T}_k(-z))^{(l+n)}(\mathfrak{T}_k(-z))^{(l+2m)}(\mathfrak{T}_k(-z))^{(l)} + (\mathfrak{T}_k(-z))^{(l+n)}[(\mathfrak{T}_k(-z))^{(l+m)}]^2\Big\}.$$

Further taking $z \to 0^+$ gives

$$\lim_{x\to 0^+} G_{m,n} = \mathscr{G}_{l,m,n},$$
$$\lim_{x\to 0^+} H_{m,n} = \mathscr{H}_{l,m,n},$$
$$\lim_{x\to 0^+} I_{m,n} = \mathscr{I}_{l,m,n}.$$

Substituting these into (2.3) and simplifying we obtain the inequalities in (1.12). The proof of Theorem 4 is complete. □

Proof of Theorem 5. In [13, p. 369] and [14, p. 429, remark], it was stated that if $f(z)$ is a completely monotonic function such that $f^{(k)}(z) \neq 0$ for $k \geqslant 0$, then the sequence $\ln[(-1)^{k-1}f^{(k-1)}(z)]$, $k \geqslant 1$, is convex. Combining with Nanson's inequality listed in [11, p. 205, 3.2.27], we have

$$\left[\prod_{l=0}^{n}(-1)^{q+2l+1}f^{(q+2l+1)}(z)\right]^{\frac{1}{n+1}} \geqslant \left[\prod_{l=1}^{n}(-1)^{q+2l}f^{(q+2l)}(z)\right]^{\frac{1}{n}}, \quad q \geqslant 0.$$

Replacing $f(z)$ by $\mathfrak{T}_k(-z)$ in the above inequality gives

$$\left[\prod_{l=0}^{n}(-1)^{q+2l+1}(\mathfrak{T}_k(-z))^{(q+2l+1)}\right]^{\frac{1}{n+1}} \geqslant \left[\prod_{l=1}^{n}(-1)^{q+2l}(\mathfrak{T}_k(-z))^{(q+2l)}\right]^{\frac{1}{n}}, \quad q \geqslant 0.$$

Letting $z \to 0^+$ in the above inequality leads to (1.13). The proof of Theorem 5 is complete. □

3. Conclusions

In this paper, we have established a unified generalization for the Bell-Touchard polynomials of order k and the r-Bell polynomials, and have further shown that the generating function of the generalized Bell-Touchard polynomials is logarithmically absolutely monotonic. Making using of the result we have obtained some inequalities for the generalized Bell-Touchard polynomials. In particular, the logarithmic convexity of the generalized Bell-Touchard polynomials has been derived.

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