

L_p -DUAL AFFINE SURFACE AREAS FOR THE GENERAL L_p -CENTROID BODIES

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Abstract. Lutwak and Zhang proposed the concept of L_p -centroid bodies. Further, Haberl and Schuster extended this notion to the general L_p -centroid bodies. In this paper, associated with the L_p -dual affine surface areas, we give the extremum values of polar for the general L_p -centroid bodies. Moreover, the L_p -dual affine surface area forms of the Brunn-Minkowski type inequality and a monotone inequality are established.

1. Introduction and main results

Let \mathcal{K}^n denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space \mathbb{R}^n . For the set of star bodies (about the origin), the set of star bodies whose centroid lie at the origin and the set of origin-symmetric star bodies in \mathbb{R}^n , we write \mathcal{S}_o^n , \mathcal{S}_c^n and \mathcal{S}_{os}^n , respectively. Let S^{n-1} denote the unit sphere in \mathbb{R}^n and $V(K)$ denote the n -dimensional volume of a body K . For the standard unit ball B in \mathbb{R}^n , its volume is written by $\omega_n = V(B)$.

The notion of centroid body was introduced by Petty ([18]). For a compact set K , the centroid body, ΓK , of K is an origin-symmetric convex body whose support function is defined by (see [8])

$$h(\Gamma K, u) = \frac{1}{V(K)} \int_K |u \cdot x| dx,$$

for all $u \in S^{n-1}$. The centroid body is one of the most important notions in the Brunn-Minkowski theory. In the recent 30 years, the centroid bodies have attracted increasing attention (see [8, 21]).

In 1997, Lutwak and Zhang ([14]) introduced the notion of L_p -centroid bodies. For each compact star-shaped (about the origin) K in \mathbb{R}^n and real number $p \geq 1$,

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the L_p -centroid body, $\Gamma_p K$, of K is an origin-symmetric convex body whose support function is defined by

$$\begin{aligned} h(\Gamma_p K, u)^p &= \frac{1}{c_{n,p}V(K)} \int_K |u \cdot x|^p dx \\ &= \frac{1}{c_{n,p}(n+p)V(K)} \int_{S^{n-1}} |u \cdot v|^p \rho(K, v)^{n+p} dS(v), \end{aligned} \tag{1.1}$$

for all $u \in S^{n-1}$. Here

$$c_{n,p} = \omega_{n+p} / \omega_2 \omega_n \omega_{p-1},$$

and $dS(v)$ denotes the standard spherical Lebesgue measure on S^{n-1} . The normalization in (1.1) is chosen such that $\Gamma_p B = B$. Regarding the investigations of L_p -centroid bodies, we may refer to [1, 2, 3, 4, 15, 23, 24, 26, 27, 32].

In 2005, Ludwig ([16]) introduced a function $\varphi_\tau : \mathbb{R} \rightarrow [0, +\infty)$ by

$$\varphi_\tau(t) = |t| + \tau t, \tag{1.2}$$

with a parameter $\tau \in [-1, 1]$.

Based on L_p -centroid bodies and definition (1.2), Ludwig ([16]) defined a corresponding notion of general L_p -centroid bodies (in fact, the author defined the general L_p -moment body which is a dilatation of the general L_p -centroid body. For the definition of general L_p -centroid body, can see [6]). For $K \in \mathcal{S}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, the general L_p -centroid body, $\Gamma_p^\tau K$, of K is a convex body whose support function is defined by

$$\begin{aligned} h(\Gamma_p^\tau K, u)^p &= \frac{2}{c_{n,p}(\tau)V(K)} \int_K \varphi_\tau(u \cdot x)^p dx \\ &= \frac{2}{c_{n,p}(\tau)(n+p)V(K)} \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \rho(K, v)^{n+p} dS(v), \end{aligned} \tag{1.3}$$

where

$$c_{n,p}(\tau) = c_{n,p}[(1 + \tau)^p + (1 - \tau)^p].$$

The normalization in (1.3) is chosen such that $\Gamma_p^\tau B = B$ for every $\tau \in [-1, 1]$ and $\Gamma_p^0 K = \Gamma_p K$. For the more investigations of general L_p -centroid bodies, see [6, 19, 25].

In 2009, Haberl and Schuster ([11]) introduced the notion of asymmetric L_p -centroid bodies (they actually defined the asymmetric L_p -moment body which is a dilatation of the asymmetric L_p -centroid body) as follows: For $K \in \mathcal{S}_o^n$, $p \geq 1$, the asymmetric L_p -centroid body, $\Gamma_p^+ K$, of K is the convex body whose support function is defined by

$$\begin{aligned} h(\Gamma_p^+ K, u)^p &= \frac{2}{c_{n,p}V(K)} \int_K (u \cdot x)_+^p dx \\ &= \frac{2}{c_{n,p}(n+p)V(K)} \int_{S^{n-1}} (u \cdot v)_+^p \rho(K, v)^{n+p} dS(v), \end{aligned} \tag{1.4}$$

where $(u \cdot x)_+ = \max\{u \cdot x, 0\}$ for all $u \in \mathcal{S}^{n-1}$. From (1.4) we see that $\Gamma_p^+ B = B$. In [11] Haberl and Schuster also defined

$$\Gamma_p^- K = \Gamma_p^+(-K). \tag{1.5}$$

From the definitions of $\Gamma_p^\pm K$ and (1.3), Haberl and Schuster ([11]) deduced that for $K \in \mathcal{S}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$,

$$\Gamma_p^\tau K = f_1(\tau) \cdot \Gamma_p^+ K +_p f_2(\tau) \cdot \Gamma_p^- K, \tag{1.6}$$

where ‘ $+_p$ ’ denotes the L_p -Minkowski addition, and

$$f_1(\tau) = \frac{(1 + \tau)^p}{(1 + \tau)^p + (1 - \tau)^p}, \quad f_2(\tau) = \frac{(1 - \tau)^p}{(1 + \tau)^p + (1 - \tau)^p}. \tag{1.7}$$

Obviously, by (1.7), we deduce that

$$f_1(-\tau) = f_2(\tau), \quad f_2(-\tau) = f_1(\tau), \tag{1.8}$$

$$f_1(\tau) + f_2(\tau) = 1. \tag{1.9}$$

Setting $\tau = 0$ in (1.6) and combining with (1.7), we see that

$$\Gamma_p K = \frac{1}{2} \cdot \Gamma_p^+ K +_p \frac{1}{2} \cdot \Gamma_p^- K. \tag{1.10}$$

If $\tau = \pm 1$ in (1.6), then by (1.7), $\Gamma_p^{+1} K = \Gamma_p^+ K$, $\Gamma_p^{-1} K = \Gamma_p^- K$. From (1.4), (1.6) and (1.8), we easily obtain for $\tau \in [-1, 1]$ (see [25])

$$\Gamma_p^{-\tau} K = \Gamma_p^\tau(-K) = -\Gamma_p^\tau K. \tag{1.11}$$

In 2010, Wang, Yuan and He ([30]) showed a type of L_p -dual affine surface areas. In 2015, Pei and Wang ([20]) made the following improvement: For $K \in \mathcal{S}_o^n$ and $p > 0$, the L_p -dual affine surface area, $\tilde{\Omega}_p(K)$, of K is defined by

$$n^{-\frac{p}{n}} \tilde{\Omega}_p(K)^{\frac{n+p}{n}} = \sup\{n \tilde{V}_p(K, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n\}. \tag{1.12}$$

Here the $\tilde{V}_p(M, N)$ denotes the L_p -dual mixed volume of $M, N \in \mathcal{S}_o^n$. When $Q \in \mathcal{S}_c^n$, the L_p -dual affine surface area was given by Wang and Wang (see [28]). For the studies of L_p -dual affine surface area, some results have been obtained in these articles (see [5, 22, 29]).

In this paper, associated with the L_p -dual affine surface area, we continuously study general L_p -centroid bodies. Firstly, combined with (1.12), we obtain the extremum values for the L_p -dual affine surface areas of polar of the general L_p -centroid bodies.

THEOREM 1.1. *For $K \in \mathcal{S}_o^n$, $1 \leq p < n$ and $\tau \in [-1, 1]$, then*

$$\tilde{\Omega}_p(\Gamma_p^* K) \leq \tilde{\Omega}_p(\Gamma_p^{\tau,*} K) \leq \tilde{\Omega}_p(\Gamma_p^{\pm,*} K). \tag{1.13}$$

If K is not origin-symmetric and p is not an odd integer, then there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$. Here, $\Gamma_p^ M$ denotes the polar body of $\Gamma_p M$.*

Then, we establish the following L_p -dual affine surface area version of Brunn-Minkowski inequality for the polar of the general L_p -centroid bodies.

THEOREM 1.2. For $K, L \in \mathcal{S}_o^n$, $1 \leq p < n$ and $\tau \in [-1, 1]$, then

$$\tilde{\Omega}_p(\Gamma_p^{\tau,*}(K \hat{+}_p L))^{-\frac{p(n+p)}{n(n-p)}} \geq \tilde{\Omega}_p(\Gamma_p^{\tau,*}K)^{-\frac{p(n+p)}{n(n-p)}} + \tilde{\Omega}_p(\Gamma_p^{\tau,*}L)^{-\frac{p(n+p)}{n(n-p)}}, \tag{1.14}$$

with equality if and only if $\Gamma_p^{\tau,*}K$ and $\Gamma_p^{\tau,*}L$ are dilates.

Finally, we give a monotone inequality for the polar of the general L_p -centroid bodies.

THEOREM 1.3. For $K, L \in \mathcal{S}_o^n$, $1 \leq p < n$ and $\tau \in [-1, 1]$, if $K \subseteq L$, then

$$\frac{\tilde{\Omega}_p(\Gamma_p^{\tau,*}K)^{\frac{n+p}{n}}}{V(K)^{\frac{n-p}{p}}} \leq \frac{\tilde{\Omega}_p(\Gamma_p^{\tau,*}L)^{\frac{n+p}{n}}}{V(L)^{\frac{n-p}{p}}}, \tag{1.15}$$

equality holds when $K = L$.

2. Notation and background material

In order to complete the proofs of Theorems 1.1-1.3, we will require the following notions.

2.1. Support function, radial function and polar of convex bodies

Let \mathbb{R} be the set of real numbers. If $K \in \mathcal{K}^n$, then the support function of K , $h_K = h(K, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$, is defined by (see [8])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y in \mathbb{R}^n .

For K is a compact star shaped (about the origin) in \mathbb{R}^n , the radial function of K , $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \rightarrow [0, +\infty)$, is defined by (see [21])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (with respect to the origin).

If E is a nonempty subset and contains the origin in \mathbb{R}^n , then the polar set, E^* , of E is defined by (see [8, 21])

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in E\}.$$

Meanwhile, it is easy to get that $(K^*)^* = K$ for $K \in \mathcal{K}_o^n$. Here \mathcal{K}_o^n denotes the set of convex bodies containing the origin in their interiors in \mathbb{R}^n .

From the above definitions, we see that if $K \in \mathcal{K}_o^n$, then (see [8, 21])

$$h(K^*, \cdot) = \frac{1}{\rho(K, \cdot)}, \quad \rho(K^*, \cdot) = \frac{1}{h(K, \cdot)}. \tag{2.1}$$

Associated with (2.1), if $K, L \in \mathcal{K}_o^n$ and $K \subseteq L$, then $K^* \supseteq L^*$.

2.2. L_p -Minkowski combination and L_p -harmonic radial combination

For $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p -Minkowski combination, $\lambda \cdot K +_p \mu \cdot L \in \mathcal{K}_o^n$, of K and L is defined by (see [7, 21])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p, \tag{2.2}$$

where $\lambda \cdot K$ denotes the L_p -Minkowski scalar multiplication and we easily obtain $\lambda \cdot K = \lambda^{\frac{1}{p}} K$.

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p -harmonic radial combination, $\lambda \times K \tilde{+}_{-p} \mu \times L \in \mathcal{S}_o^n$, of K and L is defined by (see [13])

$$\rho(\lambda \times K \tilde{+}_{-p} \mu \times L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}, \tag{2.3}$$

where the operation ‘ $\tilde{+}_{-p}$ ’ is called L_p -harmonic radial addition, $\lambda \times K$ denotes the L_p -harmonic radial scalar multiplication and we easily obtain $\lambda \times K = \lambda^{-\frac{1}{p}} K$.

From (2.1), (2.2) and (2.3), we easily get that if $K, L \in \mathcal{K}_o^n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), then (see [13])

$$(\lambda \cdot K +_p \mu \cdot L)^* = \lambda \times K^* \tilde{+}_{-p} \mu \times L^*. \tag{2.4}$$

2.3. L_p -dual mixed volume

For $K, L \in \mathcal{S}_o^n$, then for $p > 0$ and $\lambda, \mu \geq 0$ (not both zero), the L_p -radial combination, $\lambda \star K \tilde{+}_p \mu \star L$, of K and L is given by (see [9, 21])

$$\rho(\lambda \star K \tilde{+}_p \mu \star L, \cdot)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p,$$

where $\lambda \star K$ denotes the L_p -radial scalar multiplication and we easily obtain $\lambda \star K = \lambda^{\frac{1}{p}} K$.

Associated with the L_p -radial combinations of star bodies, the notion of L_p -dual mixed volume as follows: For $K, L \in \mathcal{S}_o^n$, $p > 0$ and $\varepsilon > 0$, the L_p -dual mixed volume, $\tilde{V}_p(K, L)$, of K and L is given by (see [10, 31])

$$\frac{n}{p} \tilde{V}_p(K, L) = \lim_{\varepsilon \rightarrow 0^+} \frac{V(K \tilde{+}_p \varepsilon \star L) - V(K)}{\varepsilon}.$$

From above definition, the integral representation of L_p -dual mixed volume can be given by (see [10])

$$\tilde{V}_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-p} \rho(L, u)^p dS(u), \tag{2.5}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1} .

From (2.5), we easily know that

$$\tilde{V}_p(K, K) = V(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n dS(u).$$

2.4. L_p -harmonic Blaschke combination

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p -harmonic Blaschke combination, $\lambda \circ K \widehat{+}_p \mu \circ L \in \mathcal{S}_o^n$, of K and L is defined by (see [17])

$$\frac{\rho(\lambda \circ K \widehat{+}_p \mu \circ L, \cdot)^{n+p}}{V(\lambda \circ K \widehat{+}_p \mu \circ L)} = \lambda \frac{\rho(K, \cdot)^{n+p}}{V(K)} + \mu \frac{\rho(L, \cdot)^{n+p}}{V(L)},$$

where the operation ‘ $\widehat{+}_p$ ’ is called L_p -harmonic Blaschke addition, $\lambda \circ K$ denotes L_p -harmonic Blaschke scalar multiplication and we easily obtain $\lambda \circ K = \lambda^{\frac{1}{p}} K$. When $\lambda = \mu = 1$, $K \widehat{+}_p L$ is called L_p -harmonic Blaschke sum.

3. Proofs of main theorems

In this section, we will prove Theorems 1.1-1.3. To complete the proof of Theorem 1.1, we require the following lemmas.

LEMMA 3.1. *If $K, L \in \mathcal{S}_o^n$, $1 \leq p < n$ and $\lambda, \mu \geq 0$ (not both zero), then for any $Q \in \mathcal{S}_o^n$,*

$$\widetilde{V}_p(\lambda \times K \widetilde{-}_p \mu \times L, Q^*)^{-\frac{p}{n-p}} \geq \lambda \widetilde{V}_p(K, Q^*)^{-\frac{p}{n-p}} + \mu \widetilde{V}_p(L, Q^*)^{-\frac{p}{n-p}},$$

with equality if and only if K and L are dilates.

Proof. For $K, L \in \mathcal{S}_o^n$ and $1 \leq p < n$, thus $-\frac{n-p}{p} < 0$. By (2.5) and the Minkowski integral inequality (see[12]), for any $Q \in \mathcal{S}_o^n$, we have

$$\begin{aligned} & \widetilde{V}_p(\lambda \times K \widetilde{-}_p \mu \times L, Q^*)^{-\frac{p}{n-p}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \rho(\lambda \times K \widetilde{-}_p \mu \times L, u)^{n-p} \rho(Q^*, u)^p dS(u) \right]^{-\frac{p}{n-p}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\rho(\lambda \times K \widetilde{-}_p \mu \times L, u)^{-p} \rho(Q^*, u)^{-\frac{p^2}{n-p}} \right)^{-\frac{n-p}{p}} dS(u) \right]^{-\frac{p}{n-p}} \\ &= \left\{ \frac{1}{n} \int_{S^{n-1}} \left[\left(\lambda \rho(K, u)^{-p} + \mu \rho(L, u)^{-p} \right) \rho(Q^*, u)^{-\frac{p^2}{n-p}} \right]^{-\frac{n-p}{p}} dS(u) \right\}^{-\frac{p}{n-p}} \\ &\geq \lambda \left[\frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-p} \rho(Q^*, u)^p dS(u) \right]^{-\frac{p}{n-p}} \\ &\quad + \mu \left[\frac{1}{n} \int_{S^{n-1}} \rho(L, u)^{n-p} \rho(Q^*, u)^p dS(u) \right]^{-\frac{p}{n-p}} \\ &= \lambda \widetilde{V}_p(K, Q^*)^{-\frac{p}{n-p}} + \mu \widetilde{V}_p(L, Q^*)^{-\frac{p}{n-p}}. \end{aligned}$$

According to the equality condition of Minkowski integral inequality, we see that equality holds if and only if K and L are dilates. \square

LEMMA 3.2. *If $K, L \in \mathcal{S}_o^n$, $1 \leq p < n$ and $\lambda, \mu \geq 0$ (not both zero), then*

$$\tilde{\Omega}_p(\lambda \times K \tilde{+}_{-p} \mu \times L)^{-\frac{p(n+p)}{n(n-p)}} \geq \lambda \tilde{\Omega}_p(K)^{-\frac{p(n+p)}{n(n-p)}} + \mu \tilde{\Omega}_p(L)^{-\frac{p(n+p)}{n(n-p)}}, \tag{3.1}$$

with equality if and only if K and L are dilates.

Proof. Since $1 \leq p < n$, thus $-\frac{n-p}{p} < 0$. Combined with Lemma 3.1 and (1.12), we have

$$\begin{aligned} & \tilde{\Omega}_p(\lambda \times K \tilde{+}_{-p} \mu \times L)^{-\frac{p(n+p)}{n(n-p)}} \\ &= \left[\sup \left\{ n^{\frac{n+p}{n}} \tilde{V}_p(\lambda \times K \tilde{+}_{-p} \mu \times L, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n \right\} \right]^{-\frac{p}{n-p}} \\ &= \inf \left\{ n^{-\frac{p(n+p)}{n(n-p)}} \tilde{V}_p(\lambda \times K \tilde{+}_{-p} \mu \times L, Q^*)^{-\frac{p}{n-p}} V(Q)^{-\frac{p^2}{n(n-p)}} : Q \in \mathcal{S}_{os}^n \right\} \\ &\geq \inf \left\{ n^{-\frac{p(n+p)}{n(n-p)}} \left[\lambda \tilde{V}_p(K, Q^*)^{-\frac{p}{n-p}} + \mu \tilde{V}_p(L, Q^*)^{-\frac{p}{n-p}} \right] V(Q)^{-\frac{p^2}{n(n-p)}} : Q \in \mathcal{S}_{os}^n \right\} \\ &\geq \lambda \inf \left\{ \left[n^{\frac{n+p}{n}} \tilde{V}_p(K, Q^*) V(Q)^{\frac{p}{n}} \right]^{-\frac{p}{n-p}} : Q \in \mathcal{S}_{os}^n \right\} \\ &\quad + \mu \inf \left\{ \left[n^{\frac{n+p}{n}} \tilde{V}_p(L, Q^*) V(Q)^{\frac{p}{n}} \right]^{-\frac{p}{n-p}} : Q \in \mathcal{S}_{os}^n \right\} \\ &= \lambda \left[\sup \left\{ n^{\frac{n+p}{n}} \tilde{V}_p(K, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n \right\} \right]^{-\frac{p}{n-p}} \\ &\quad + \mu \left[\sup \left\{ n^{\frac{n+p}{n}} \tilde{V}_p(L, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n \right\} \right]^{-\frac{p}{n-p}} \\ &= \lambda \tilde{\Omega}_p(K)^{-\frac{p(n+p)}{n(n-p)}} + \mu \tilde{\Omega}_p(L)^{-\frac{p(n+p)}{n(n-p)}}. \end{aligned}$$

Thus

$$\tilde{\Omega}_p(\lambda \times K \tilde{+}_{-p} \mu \times L)^{-\frac{p(n+p)}{n(n-p)}} \geq \lambda \tilde{\Omega}_p(K)^{-\frac{p(n+p)}{n(n-p)}} + \mu \tilde{\Omega}_p(L)^{-\frac{p(n+p)}{n(n-p)}}.$$

This yields (3.1). According to the equality condition of Lemma 3.1, we see that equality holds in (3.1) if and only if K and L are dilates. \square

LEMMA 3.3. ([25]) *If $K \in \mathcal{S}_o^n$, $p \geq 1$ and p is not odd integer, then $\Gamma_p^+ K = \Gamma_p^- K$ if and only if K is origin-symmetric.*

LEMMA 3.4. ([25]) *If $K \in \mathcal{S}_o^n$, $p \geq 1$ and p is not odd integer, then for $\tau \in [-1, 1]$ and $\tau \neq 0$, $\Gamma_p^\tau K = \Gamma_p^{-\tau} K$ if and only if K is origin-symmetric.*

LEMMA 3.5. ([25]) *If $K \in \mathcal{S}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, then*

$$\Gamma_p K = \frac{1}{2} \cdot \Gamma_p^\tau K + \frac{1}{2} \cdot \Gamma_p^{-\tau} K.$$

Proof of Theorem 1.1. From (1.6) and (2.4), we have

$$\Gamma_p^{\tau,*}K = f_1(\tau) \times \Gamma_p^{+,*}K \tilde{+}_{-p} f_2(\tau) \times \Gamma_p^{-,*}K.$$

Combining with (3.1), we have

$$\begin{aligned} \tilde{\Omega}_p(\Gamma_p^{\tau,*}K)^{-\frac{p(n+p)}{n(n-p)}} &= \tilde{\Omega}_p(f_1(\tau) \times \Gamma_p^{+,*}K \tilde{+}_{-p} f_2(\tau) \times \Gamma_p^{-,*}K)^{-\frac{p(n+p)}{n(n-p)}} \\ &\geq f_1(\tau) \tilde{\Omega}_p(\Gamma_p^{+,*}K)^{-\frac{p(n+p)}{n(n-p)}} + f_2(\tau) \tilde{\Omega}_p(\Gamma_p^{-,*}K)^{-\frac{p(n+p)}{n(n-p)}}. \end{aligned} \tag{3.2}$$

Since $\Gamma_p^{+,*}K = -\Gamma_p^{-,*}K$ and $Q \in \mathcal{S}_{os}^n$, then $\rho(Q, u) = \rho(-Q, u) = \rho(Q, -u)$ for all $u \in S^{n-1}$, thus by (2.5) we get that

$$\tilde{V}_p(\Gamma_p^{-,*}K, Q^*) = \tilde{V}_p(-\Gamma_p^{+,*}K, Q^*) = \tilde{V}_p(\Gamma_p^{+,*}K, Q^*).$$

Therefore, from definition (1.12), it follows that

$$\tilde{\Omega}_p(\Gamma_p^{+,*}K) = \tilde{\Omega}_p(\Gamma_p^{-,*}K). \tag{3.3}$$

Combining with (3.2), (3.3) and (1.9), we can get

$$\tilde{\Omega}_p(\Gamma_p^{\tau,*}K)^{-\frac{p(n+p)}{n(n-p)}} \geq \tilde{\Omega}_p(\Gamma_p^{\pm,*}K)^{-\frac{p(n+p)}{n(n-p)}},$$

i.e.,

$$\tilde{\Omega}_p(\Gamma_p^{\tau,*}K) \leq \tilde{\Omega}_p(\Gamma_p^{\pm,*}K). \tag{3.4}$$

According to the equality condition of inequality (3.1), we know that equality holds in (3.4) if and only if $\Gamma_p^{+,*}K$ and $\Gamma_p^{-,*}K$ are dilates. Since $\Gamma_p^{+,*}K = -\Gamma_p^{-,*}K$, this means $\Gamma_p^{+,*}K = \Gamma_p^{-,*}K$. Hence, from Lemma 3.3, we see that if K is not origin-symmetric, then equality holds in (3.4) if and only if $\tau = \pm 1$.

Now, we prove the left inequality of (1.13). From Lemma 3.5 and (2.4), we have

$$\Gamma_p^*K = \frac{1}{2} \times \Gamma_p^{\tau,*}K \tilde{+}_{-p} \frac{1}{2} \times \Gamma_p^{-\tau,*}K.$$

Combining with (3.1), we have

$$\begin{aligned} \tilde{\Omega}_p(\Gamma_p^*K)^{-\frac{p(n+p)}{n(n-p)}} &= \tilde{\Omega}_p\left(\frac{1}{2} \times \Gamma_p^{\tau,*}K \tilde{+}_{-p} \frac{1}{2} \times \Gamma_p^{-\tau,*}K\right)^{-\frac{p(n+p)}{n(n-p)}} \\ &\geq \frac{1}{2} \tilde{\Omega}_p(\Gamma_p^{\tau,*}K)^{-\frac{p(n+p)}{n(n-p)}} + \frac{1}{2} \tilde{\Omega}_p(\Gamma_p^{-\tau,*}K)^{-\frac{p(n+p)}{n(n-p)}}. \end{aligned} \tag{3.5}$$

Due to $\Gamma_p^{-\tau,*}K = -\Gamma_p^{\tau,*}K$ by (1.11), similar to the proof of (3.3), we have

$$\tilde{\Omega}_p(\Gamma_p^{\tau,*}K) = \tilde{\Omega}_p(\Gamma_p^{-\tau,*}K). \tag{3.6}$$

From (3.5) and (3.6), we deduce

$$\tilde{\Omega}_p(\Gamma_p^*K) \leq \tilde{\Omega}_p(\Gamma_p^{\tau,*}K). \tag{3.7}$$

Using $\Gamma_p^{\tau,*}K = -\Gamma_p^{-\tau,*}K$ and the equality condition of inequality (3.1), we know that equality holds in (3.7) if and only if $\Gamma_p^{\tau,*}K = \Gamma_p^{-\tau,*}K$. By Lemma 3.4, we see that if K is not origin-symmetric, then equality holds in (3.7) if and only if $\tau = 0$. \square

LEMMA 3.6. ([6]) If $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, then

$$\rho(\Gamma_p^{\tau,*}(K \widehat{+}_p L), u)^{-p} = \rho(\Gamma_p^{\tau,*}K, u)^{-p} + \rho(\Gamma_p^{\tau,*}L, u)^{-p}. \tag{3.8}$$

LEMMA 3.7. If $K, L \in \mathcal{S}_o^n$, $1 \leq p < n$ and $\tau \in [-1, 1]$, then for any $Q \in \mathcal{S}_o^n$,

$$\widetilde{V}_p(\Gamma_p^{\tau,*}(K \widehat{+}_p L), Q^*)^{-\frac{p}{n-p}} \geq \widetilde{V}_p(\Gamma_p^{\tau,*}K, Q^*)^{-\frac{p}{n-p}} + \widetilde{V}_p(\Gamma_p^{\tau,*}L, Q^*)^{-\frac{p}{n-p}}, \tag{3.9}$$

with equality if and only if $\Gamma_p^{\tau,*}K$ and $\Gamma_p^{\tau,*}L$ are dilates.

Proof. Since $1 \leq p < n$, thus $-\frac{n-p}{p} < 0$. Hence by (2.5), (3.8) and the Minkowski integral inequality (see[12]), for any $Q \in \mathcal{S}_o^n$, we have that

$$\begin{aligned} & \widetilde{V}_p(\Gamma_p^{\tau,*}(K \widehat{+}_p L), Q^*)^{-\frac{p}{n-p}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \rho(\Gamma_p^{\tau,*}(K \widehat{+}_p L), u)^{n-p} \rho(Q^*, u)^p dS(u) \right]^{-\frac{p}{n-p}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\rho(\Gamma_p^{\tau,*}(K \widehat{+}_p L), u)^{-p} \rho(Q^*, u)^{-\frac{p^2}{n-p}} \right)^{-\frac{n-p}{p}} dS(u) \right]^{-\frac{p}{n-p}} \\ &= \left\{ \frac{1}{n} \int_{S^{n-1}} \left[\left(\rho(\Gamma_p^{\tau,*}K, u)^{-p} + \rho(\Gamma_p^{\tau,*}L, u)^{-p} \right) \rho(Q^*, u)^{-\frac{p^2}{n-p}} \right]^{-\frac{n-p}{p}} dS(u) \right\}^{-\frac{p}{n-p}} \\ &\geq \left[\frac{1}{n} \int_{S^{n-1}} \rho(\Gamma_p^{\tau,*}K, u)^{n-p} \rho(Q^*, u)^p dS(u) \right]^{-\frac{p}{n-p}} \\ &\quad + \left[\frac{1}{n} \int_{S^{n-1}} \rho(\Gamma_p^{\tau,*}L, u)^{n-p} \rho(Q^*, u)^p dS(u) \right]^{-\frac{p}{n-p}} \\ &= \widetilde{V}_p(\Gamma_p^{\tau,*}K, Q^*)^{-\frac{p}{n-p}} + \widetilde{V}_p(\Gamma_p^{\tau,*}L, Q^*)^{-\frac{p}{n-p}}. \end{aligned}$$

According to the equality condition of Minkowski integral inequality, we see that equality holds in (3.9) if and only if $\Gamma_p^{\tau,*}K$ and $\Gamma_p^{\tau,*}L$ are dilates. \square

Proof of Theorem 1.2. Since $-\frac{n-p}{p} < 0$, thus by definition (1.12) and inequality (3.9), we obtain

$$\begin{aligned} & \widetilde{\Omega}_p(\Gamma_p^{\tau,*}(K \widehat{+}_p L))^{-\frac{p(n+p)}{n(n-p)}} \\ &= \left[\sup \left\{ n^{\frac{n+p}{n}} \widetilde{V}_p(\Gamma_p^{\tau,*}(K \widehat{+}_p L), Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{OS}^n \right\} \right]^{-\frac{p}{n-p}} \\ &= \inf \left\{ n^{-\frac{p(n+p)}{n(n-p)}} \widetilde{V}_p(\Gamma_p^{\tau,*}(K \widehat{+}_p L), Q^*)^{-\frac{p}{n-p}} V(Q)^{-\frac{p^2}{n(n-p)}} : Q \in \mathcal{S}_{OS}^n \right\} \\ &\geq \inf \left\{ n^{-\frac{p(n+p)}{n(n-p)}} \left[\widetilde{V}_p(\Gamma_p^{\tau,*}K, Q^*)^{-\frac{p}{n-p}} + \widetilde{V}_p(\Gamma_p^{\tau,*}L, Q^*)^{-\frac{p}{n-p}} \right] V(Q)^{-\frac{p^2}{n(n-p)}} : Q \in \mathcal{S}_{OS}^n \right\} \\ &\geq \inf \left\{ n^{\frac{n+p}{n}} \widetilde{V}_p(\Gamma_p^{\tau,*}K, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{OS}^n \right\}^{-\frac{p}{n-p}} \end{aligned}$$

$$\begin{aligned}
 & + \inf \left\{ n^{\frac{n+p}{n}} \tilde{V}_p(\Gamma_p^{\tau,*} L, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{OS}^n \right\}^{-\frac{p}{n-p}} \\
 = & \left[\sup \left\{ n^{\frac{n+p}{n}} \tilde{V}_p(\Gamma_p^{\tau,*} K, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{OS}^n \right\} \right]^{-\frac{p}{n-p}} \\
 & + \left[\sup \left\{ n^{\frac{n+p}{n}} \tilde{V}_p(\Gamma_p^{\tau,*} L, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{OS}^n \right\} \right]^{-\frac{p}{n-p}} \\
 = & \tilde{\Omega}_p(\Gamma_p^{\tau,*} K)^{-\frac{p(n+p)}{n(n-p)}} + \tilde{\Omega}_p(\Gamma_p^{\tau,*} L)^{-\frac{p(n+p)}{n(n-p)}}.
 \end{aligned}$$

This yields inequality (1.14).

By the equality condition of (3.9), we see that equality holds in (1.14) if and only if $\Gamma_p^{\tau,*} K$ and $\Gamma_p^{\tau,*} L$ are dilates. \square

Letting $\tau = 0$ in Theorem 1.2, we get another Brunn-Minkowski type inequality with respect to L_p -harmonic Blaschke combination:

COROLLARY 3.1. *For $K, L \in \mathcal{S}_O^n$ and $1 \leq p < n$, then*

$$\tilde{\Omega}_p(\Gamma_p^*(K \hat{+}_p L))^{-\frac{p(n+p)}{n(n-p)}} \geq \tilde{\Omega}_p(\Gamma_p^* K)^{-\frac{p(n+p)}{n(n-p)}} + \tilde{\Omega}_p(\Gamma_p^* L)^{-\frac{p(n+p)}{n(n-p)}},$$

with equality if and only if $\Gamma_p^* K$ and $\Gamma_p^* L$ are dilates.

Proof of Theorem 1.3. For $K, L \in \mathcal{S}_O^n$, $1 \leq p < n$ and $\tau \in [-1, 1]$. If $K \subseteq L$, then

$$\rho(K, \cdot) \leq \rho(L, \cdot), \tag{3.10}$$

with equality if and only if $K = L$.

From (1.3) and (2.1), we have

$$\rho(\Gamma_p^{\tau,*} K, u)^{-p} = \frac{2}{c_{n,p}(\tau)(n+p)V(K)} \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \rho(K, v)^{n+p} dS(v). \tag{3.11}$$

By (2.5), (3.10) and (3.11), we can get

$$\begin{aligned}
 & V(K)^{-\frac{n-p}{p}} \tilde{V}_p(\Gamma_p^{\tau,*} K, Q^*) \\
 = & V(K)^{-\frac{n-p}{p}} \frac{1}{n} \int_{S^{n-1}} \rho(\Gamma_p^{\tau,*} K, u)^{n-p} \rho(Q^*, u)^p dS(u) \\
 = & V(K)^{-\frac{n-p}{p}} \frac{1}{n} \int_{S^{n-1}} \left[\rho(\Gamma_p^{\tau,*} K, u)^{-p} \rho(Q^*, u)^{-\frac{p^2}{n-p}} \right]^{-\frac{n-p}{p}} dS(u) \\
 = & \frac{1}{n} \int_{S^{n-1}} \left[\frac{2}{c_{n,p}(\tau)(n+p)} \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \rho(K, v)^{n+p} dS(v) \rho(Q^*, u)^{-\frac{p^2}{n-p}} \right]^{-\frac{n-p}{p}} dS(u) \\
 \leq & \frac{1}{n} \int_{S^{n-1}} \left[\frac{2}{c_{n,p}(\tau)(n+p)} \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \rho(L, v)^{n+p} dS(v) \rho(Q^*, u)^{-\frac{p^2}{n-p}} \right]^{-\frac{n-p}{p}} dS(u) \\
 = & V(L)^{-\frac{n-p}{p}} \tilde{V}_p(\Gamma_p^{\tau,*} L, Q^*).
 \end{aligned}$$

Combined with (1.12), we easily get

$$\begin{aligned} \frac{\tilde{\Omega}_p(\Gamma_p^{\tau,*}K)^{\frac{n+p}{n}}}{V(K)^{\frac{n-p}{p}}} &= V(K)^{-\frac{n-p}{p}} \sup \left\{ n^{\frac{n+p}{n}} \tilde{V}_p(\Gamma_p^{\tau,*}K, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n \right\} \\ &\leq V(L)^{-\frac{n-p}{p}} \sup \left\{ n^{\frac{n+p}{n}} \tilde{V}_p(\Gamma_p^{\tau,*}L, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n \right\} \\ &= \frac{\tilde{\Omega}_p(\Gamma_p^{\tau,*}L)^{\frac{n+p}{n}}}{V(L)^{\frac{n-p}{p}}}. \end{aligned}$$

According to the equality condition of (3.10), we see that equality holds in (1.15) when $K = L$. \square

Letting $\tau = 0$ in Theorem 1.3, we get another monotone inequality for the polar of the L_p -centroid bodies:

COROLLARY 3.2. *For $K, L \in \mathcal{S}_o^n$ and $1 \leq p < n$, if $K \subseteq L$, then*

$$\frac{\tilde{\Omega}_p(\Gamma_p^*K)^{\frac{n+p}{n}}}{V(K)^{\frac{n-p}{p}}} \leq \frac{\tilde{\Omega}_p(\Gamma_p^*L)^{\frac{n+p}{n}}}{V(L)^{\frac{n-p}{p}}},$$

equality holds when $K = L$.

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