Journal of Mathematical Inequalities
Volume 13, Number 3 (2019), 687–702 doi:10.7153/jmi-2019-13-46

APPROXIMATION PROPERTIES OF CERTAIN BERNSTEIN–STANCU TYPE OPERATORS

ANA-MARIA ACU, OGÜN DOĞRU, CARMEN VIOLETA MURARU AND VOICHIŢA ADRIANA RADU

(*Communicated by I. Raşa*)

Abstract. In this paper we introduce and investigate a new operator of Bernstein-Stancu type, based on q-polynomials. We study approximation properties for these operators based on Korovkin type approximation theorem and study some direct theorems. Also, the study contains numerical considerations regarding the constructed operators based on Maple algorithms.

1. Introduction

In 1968, Stancu [24] proposed the sequence of positive linear operators $S_n^{<\alpha>}: C[0,1]\to C[0,1]$, depending on a non-negative parameter α given by

$$S_n^{<\alpha>}(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{<\alpha>}(x), \tag{1}$$

where $p_{n,k}^{<\alpha>}(x)=\binom{n}{k}\dfrac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{j=0}^{n-k-1}(1-x+j\alpha)}{\prod_{i=0}^{n-1}(1+i\alpha)}, x\in[0,1]$.

For $\alpha=0$ these operators reduces to the classical Bernstein operators.

In the papers [11], [12] and [23] starting with the Bernstein operators, the following Stancu type operators are constructed and studied:

$$C_n: C[0,1]\to \Pi_n$$

$$C_n(f;x)=\sum_{k=0}^{n}\frac{k!}{n^k}\binom{n}{k}\mathbf{m}_{k,n}\left[0,\frac{1}{n},\dots,\frac{k}{n};f\right]x^k,\quad f\in C[0,1], \tag{2}$$

where the real numbers $\left(m_{k,n}\right)_{k=0}^{\infty}$ are selected in order to preserve some important properties of Bernstein operators and Π_n is the linear space of all real polynomials of degree $\leqslant n$.

In the following we consider that $\mathbf{m}_{0,n}=1$, $\lim\limits_{n\to\infty}\mathbf{m}_{1,n}=1$ and $\mathbf{m}_{k,n}=\dfrac{(a_n)_k}{k!}$, $a_n\in(0,1]$, where $(x)_k=x(x+1)\dots(x+k-1)$ with $(x)_0=1$ is the Pochhammer symbol.

Mathematics subject classification (2010): 41A36, 41A25.
Keywords and phrases: Bernstein-Stancu operator, q-integers, rate of convergence, moduli of continuity.

Paper JMI-13-46

For the above special case of real sequence $(m_{k,n})_{k=0}^{\infty}$, the Bernstein-Stancu operators C_n were written in the Bernstein basis as follows (see [11], Theorem 10):

$$C_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) C_{k,n}[f] \tag{3}$$

where

$$C_{k,n}[f] = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} f\left(\frac{j}{n}\right) (a_n)_j (1-a_n)_{k-j}$$

and

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

We remark that if $a_n \in (0,1]$, then C_n are linear positive operators.

The q-analog of Stancu operators have attracted much interest, and a great number of interesting results have been obtained. Recently, G. Nowak [20] introduced a q-analogue of Stancu's operators defined in (1). In 2010, O. Agratini [7] involving modulus of continuity and Lipschitz type maximal function obtained estimates for the rate of convergence of q-analog of Stancu operators. Also, a probabilistic approach is given and some approximation properties are established. The approximation properties of q-Stancu operators were studied by Acar and Aral [1], Acu [4], Agrawal et al. [8], Aral et al. [10] and Nowak and Gupta [21]. The first results in this field have been achieved by A. Lupaş [17] and G.M. Phillips [22] who consider q-analogue of Bernstein operators. In the recent years, several researchers have made significant contribution in this area of approximation theory [2, 3, 5, 6, 9, 14, 16, 19].

First of all, we recall elements of q-Calculus, see, e.g., [15] and [18]. For any fixed real number $q > 0$, the q-integer $[k]_q$, for $k \in \mathbb{N}$ is defined as

$$[k]_q = \begin{cases} \left(1-q^k\right)/(1-q), & q \neq 1 \\ k, & q = 1. \end{cases}$$

Set $[0]_q = 0$. The q-factorial $[k]_q!$ and q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are defined as follows

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \ldots [1]_q, & k = 1,2,\ldots \\ 1, & k = 0 \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \text{ for } k \in \{0,1,\ldots,n\}.$$

The q-analogue of $(x-a)^n$ is the polynomial

$$(x-a)_q^n = \begin{cases} 1, & n = 0 \\ (x-a)(x-qa)\ldots(x-q^{n-1}a), & n \geqslant 1. \end{cases}$$

For $f \in C[0,1]$, $q > 0$, $\alpha \geqslant 0$ and $n \in \mathbb{N}$, Nowak (see [20]) in 2009 defined the q-Bernstein-Stancu operators as follows

$$S_n^{<q,\alpha>}(f;x) = \sum_{k=0}^{n} p_{n,k}^{<\alpha>}(x;q) f\left(\frac{[k]_q}{[n]_q}\right), x \in [0,1],$$

with

$$p_{n,k}^{<\alpha>}(x;q) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}(x+\alpha[i]_q)\prod_{j=0}^{n-k-1}(1-q^j x+\alpha[j]_q)}{\prod_{i=0}^{n-1}(1+\alpha[i]_q)}.$$

THEOREM 1.1. [20] *Let* $0 < q < 1$, $\alpha \geqslant 0$. *Then*

$$S_n^{<q,\alpha>}(1;x) = 1;$$
$$S_n^{<q,\alpha>}(t;x) = x;$$
$$S_n^{<q,\alpha>}(t^2;x) = \frac{1}{1+\alpha}\left(x(x+\alpha) + \frac{x(1-x)}{[n]_q}\right),$$

for all $n \in N$ *and* $x \in [0,1]$.

For $q = 1$, $S_n^{<q,\alpha>}$ turns out to be the Bernstein-Stancu operators (1). For $\alpha = 0$, $S_n^{<q,\alpha>}$ reduces to q-Bernstein operators defined by Phillips [22]:

$$B_n^{<q>}(f;x) = \sum_{k=0}^{n} b_{n,k}(x;q) f\left(\frac{[k]_q}{[n]_q}\right), x \in [0,1],$$

where $b_{n,k}(x;q) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{j=0}^{n-k-1}(1-q^j x)$.

These operators verify

$$B_n^{<q>}(1;x) = 1, \quad B_n^{<q>}(t;x) = x, \quad B_n^{<q>}(t^2;x) = x^2 + \frac{x(1-x)}{[n]_q}. \tag{4}$$

2. Construction of the generalized q-Bernstein-Stancu operators and the approximation properties

Our aim is to introduce a q-analogue of the Bernstein-Stancu operators defined in (3). A q-analogue of C_n is in fact a q-deformation of Bernstein-Stancu operator. Furthermore, in this section we study some approximation properties of these operators.

Let us define on C[0,1] the linear positive operators $C_n^{<q>}$ by

$$C_n^{<q>}(f;x) = \sum_{k=0}^{n} b_{n,k}(x;q) C_{k,n}[f;q], \text{ for all } x \in [0,1], \tag{5}$$

with

$$C_{k,n}[f;q] = \sum_{j=0}^{k} c_{k,j} f\left(\frac{[j]_q}{[n]_q}\right), \ 0 \leqslant k \leqslant n,$$

$$c_{k,j} = \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{\prod_{i=0}^{j-1}(a_n+[i]_q)\prod_{s=0}^{k-j-1}(1-q^s a_n+[s]_q)}{\prod_{i=0}^{k-1}(1+[i]_q)},$$

where $a_n \in (0,1], n \in \mathbb{N}$ and $q>0$.

We note that $c_{k,j} = p_{k,j}^{<1>}(a_n;q)$. Therefore, $C_{k,n}[f;q] = S_k^{<q,1>}\left(\tilde{f};a_n\right)$, where $\tilde{f}(t) = f\left(t\frac{[k]_q}{[n]_q}\right)$.

LEMMA 2.1. *The q-Bernstein-Stancu operators $C_n^{<q>}$ given by (5) verify the following identities*

i) $C_n^{<q>}(e_0;x) = 1$,

ii) $C_n^{<q>}(e_1;x) = a_n x$,

iii) $C_n^{<q>}(e_2;x) = \dfrac{a_n(a_n+1)}{2}\left(1-\dfrac{1}{[n]_q}\right)x^2 + \dfrac{a_n}{[n]_q}x$,

where $e_j(x) = x^j$, $j=0,1,2$ are the test functions.

Proof. From Theorem 1.1 we have

$$\sum_{j=0}^{k} c_{k,j} = 1;$$

$$\sum_{j=0}^{k} c_{k,j}\frac{[j]_q}{[k]_q} = a_n;$$

$$\sum_{j=0}^{k} c_{k,j}\frac{[j]_q^2}{[k]_q^2} = \frac{1}{2}\left(a_n(a_n+1)+\frac{a_n(1-a_n)}{[k]_q}\right).$$

Therefore

$$C_{k,n}[e_0;q] = 1; \quad C_{k,n}[e_1;q] = \frac{[k]_q}{[n]_q}a_n;$$

$$C_{k,n}[e_2;q] = \frac{1}{2}\frac{[k]_q^2}{[n]_q^2}\left(a_n(a_n+1)+\frac{a_n(1-a_n)}{[k]_q}\right).$$

Using the properties (4) of the q-Bernstein operators, the values of the operator $C_n^{<q>}$ for test functions are obtained. □

REMARK 2.1. We obtain the following values for the central moments of the q-Bernstein-Stancu operators

$$C_n^{<q>}(t-x;x) = C_n^{<q>}(e_1;x) - xC_n^{<q>}(e_0;x) = (a_n-1)x,$$

$$\begin{aligned} C_n^{<q>}\left((t-x)^2;x\right) &= C_n^{<q>}(e_2;x) - 2xC_n^{<q>}(e_1;x) + x^2C_n^{<q>}(e_0;x) \\ &= \left(\frac{a_n(a_n+1)([n]_q-1)}{2[n]_q} - 2a_n+1\right)x^2 + \frac{a_n}{[n]_q}x. \end{aligned}$$

LEMMA 2.2. *For* $f \in C[0,1]$ *we have* $||C_n^{<q>}|| \leqslant ||f||$.

Proof. From the definition of the operator and using Lemma 2.1 we have

$$C_n^{<q>}(f;x) = \sum_{k=0}^{n} b_{n,k}(x;q) \sum_{j=0}^{k} c_{k,j} \left| f\left(\frac{[j]_q}{[n]_q}\right) \right| \leqslant ||f|| C_n^{<q>}(e_0;x) \leqslant ||f||. \quad \square$$

LEMMA 2.3. *For all* $x \in [0,1]$ *we have*

$$C_n^{<q>}\left((t-x)^2;x\right) \leqslant \frac{a_n}{[n]_q} x(1-x) + (1-a_n).$$

Proof. Using Remark 2.1, we get

$$\begin{aligned} C_n^{<q>}\left((t-x)^2;x\right) &= \left(\frac{a_n(a_n+1)([n]_q-1)}{2[n]_q} - 2a_n + 1\right)x^2 + \frac{a_n}{[n]_q}x \\ &= \frac{a_n}{[n]_q}x(1-x) + \left(\frac{a_n(a_n+1)([n]_q-1)}{2[n]_q} - 2a_n + 1 + \frac{a_n}{[n]_q}\right)x^2 \\ &= \frac{a_n}{[n]_q}x(1-x) + \frac{(a_n^2-3a_n+2)[n]_q + a_n(1-a_n)}{2[n]_q}x^2 \\ &= \frac{a_n}{[n]_q}x(1-x) + \frac{(1-a_n)}{2}\left\{(2-a_n) + \frac{a_n}{[n]_q}\right\}x^2 \\ &\leqslant \frac{a_n}{[n]_q}x(1-x) + (1-a_n). \quad \square \end{aligned}$$

LEMMA 2.4. *Let* $(q_n)_{n\geqslant 1}$, $(a_n)_{n\geqslant 1}$ *be real sequences such that* $0 < q_n < 1$, $a_n \in (0,1]$, $n \in \mathbb{N}$. *If*

$$\lim_{n\to\infty} q_n = 1, \ \lim_{n\to\infty} a_n = 1 \text{ and } \lim_{n\to\infty} [n]_{q_n}(1-a_n) = a \in \mathbb{R}, \tag{6}$$

then

$$\lim_{n\to\infty} [n]_{q_n} C_n^{<q_n>}(t-x;x) = -ax,$$

$$\lim_{n\to\infty} [n]_{q_n} C_n^{<q_n>}\left((t-x)^2;x\right) = x(1-x) + a\frac{x^2}{2}.$$

Proof. The results follow from Remark 2.1. $\square$

REMARK 2.2. The sequences $(q_n)_{n\geqslant 1}$, $(a_n)_{n\geqslant 1}$, $a_n = q_n = 1 - \frac{1}{n}$ verify the conditions from the previous Lemma, namely

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} a_n = 1 \text{ and } \lim_{n\to\infty} [n]_{q_n}(1-a_n) = 1 - \frac{1}{e}.$$

3. Direct theorems

We investigate the approximation properties of these operators and we estimate the rate of convergence by using moduli of continuity. For the classical Bernstein-Stancu operators C_n similar results was obtained in [23].

THEOREM 3.1. *Let $(q_n)_{n\geqslant 1}$, $(a_n)_{n\geqslant 1}$ be real sequences such that $q_n \in (0,1)$, $a_n \in (0,1]$. If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} q_n = 1$ and $f \in C[0,1]$, then*

$$\lim_{n\to\infty} C_n^{<q_n>}(f;x) = f(x) \text{ uniformly on } [0,1].$$

Proof. Using Lemma 2.1 follows that

$$\lim_{n\to\infty} C_n^{<q_n>}(e_k;x) = e_k(x) \text{ uniformly on } [0,1], \text{ for } k \in \{0,1,2\}.$$

Applying the Bohmann-Korovkin theorem, we get the result. □

The usual modulus of continuity for $f \in C[0,1]$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and is defined as

$$\omega(f;\delta) = \sup_{|y-x|\leqslant\delta} |f(y)-f(x)|,\ x,y \in [0,1].$$

It is known that the modulus of continuity of f has the following properties

$$\omega(f;\lambda\delta) \leqslant (1+\lambda)\omega(f;\delta)$$

and

$$|f(t)-f(x)| \leqslant \omega(f;\delta)\left(\frac{(t-x)^2}{\delta^2}+1\right).$$

Our next result is the following local theorem.

THEOREM 3.2. *Let $(q_n)_{n\geqslant 1}$, $(a_n)_{n\geqslant 1}$ be real sequences such that $q_n \in (0,1)$, $a_n \in (0,1]$. If $f \in C[0,1]$, then*

$$\left|C_n^{<q_n>}(f;x)-f(x)\right| \leqslant \frac{5}{4}\omega(f;\sqrt{\delta_n}),$$

where $\delta_n = \dfrac{a_n}{[n]_{q_n}} + 4(1-a_n)$.

Proof. We have

$$|C_n^{<q_n>}(f;x)-f(x)| \leqslant C_n^{<q_n>}(|f(t)-f(x)|;x) \leqslant \omega(f;\delta)\left(1+\frac{1}{\delta^2}C_n^{<q_n>}((t-x)^2;x)\right).$$

Using Lemma 2.3, we can write

$$|C_n^{<q_n>}(f;x)-f(x)| \leqslant \omega(f;\delta)\left[1+\frac{1}{\delta^2}\left(\frac{a_n}{[n]_{q_n}}x(1-x)+(1-a_n)\right)\right]$$
$$\leqslant \omega(f;\delta)\left[1+\frac{1}{4\delta^2}\left(\frac{a_n}{[n]_{q_n}}+4(1-a_n)\right)\right].$$

So, if we choose $\delta=\sqrt{\delta_n}$, we have the desired result. □

THEOREM 3.3. *Let* $(q_n)_{n\geqslant 1}$, $(a_n)_{n\geqslant 1}$ *be real sequences such that* $q_n\in(0,1)$, $a_n\in(0,1]$. *If* $f\in C^1[0,1]$, *then*

$$\left|C_n^{<q_n>}(f;x)-f(x)\right| \leqslant \lambda_n(x)|f'(x)|+2\sqrt{\delta_n(x)}\,\omega(f',\sqrt{\delta_n(x)}),$$

where

$$\lambda_n(x)=(1-a_n)x,\ \delta_n(x)=\frac{a_n}{[n]_{q_n}}\phi^2(x)+(1-a_n) \text{ and } \phi^2(x)=x(1-x). \tag{7}$$

Proof. Let $f\in C^1[0,1]$. For any $x,t\in[0,1]$, we have

$$f(t)-f(x)=f'(x)(t-x)+\int_x^t\left(f'(u)-f'(x)\right)du,$$

so, we get

$$C_n^{<q_n>}(f(t)-f(x);x)=f'(x)C_n^{<q_n>}(t-x;x)+C_n^{<q_n>}\left(\int_x^t(f'(u)-f'(x))du;x\right).$$

Using the following well known property of modulus of continuity

$$|f(y)-f(x)|\leqslant\omega(f;\delta)\left(\frac{|y-x|}{\delta}+1\right),\quad \delta>0,$$

we have

$$\left|\int_x^t|f'(u)-f'(x)|du\right|\leqslant\omega(f';\delta)\left[\frac{(t-x)^2}{\delta}+|t-x|\right].$$

Therefore,

$$|C_n^{<q_n>}(f;x)-f(x)|\leqslant|f'(x)|\cdot|C_n^{<q_n>}(t-x;x)|$$
$$+\omega(f';\delta)\left\{\frac{1}{\delta}C_n^{<q_n>}\left((t-x)^2;x\right)+C_n^{<q_n>}(|t-x|;x)\right\}.$$

Using Cauchy-Schwartz inequality

$$C_n^{<q_n>}(|t-x|;x)\leqslant\sqrt{C_n^{<q_n>}(1;x)}\cdot\sqrt{C_n^{<q_n>}((t-x)^2;x)}$$

we obtain

$$|C_n^{<q_n>}(f;x)-f(x)| \leqslant |f'(x)|\cdot|C_n^{<q_n>}(t-x;x)| \\ +\omega(f',\delta)\left\{\frac{1}{\delta}\sqrt{C_n^{<q_n>}((t-x)^2;x))}+1\right\}\sqrt{C_n^{<q_n>}((t-x)^2;x)}.$$

Applying Lemma 2.3, we get

$$|C_n^{<q_n>}(f;x)-f(x)| \leqslant |f'(x)|(1-a_n)x+\omega(f',\delta)\cdot\left\{\frac{1}{\delta}\sqrt{\frac{a_n}{[n]_{q_n}}\phi^2(x)+(1-a_n)}+1\right\} \\ \cdot\sqrt{\frac{a_n}{[n]_{q_n}}\phi^2(x)+(1-a_n)}.$$

Choosing $\delta=\sqrt{\delta_n(x)}$, we find the desired inequality. □

Let

$$Lip_M(\gamma)=\{f\in C[0,1], |f(t)-f(x)|\leqslant M|t-x|^\gamma\}, 0<\gamma\leqslant 1$$

be the class of Lipschitz functions. The next result gives the rate of convergence of the operators $C_n^{<q_n>}$ in terms of the Lipschitz class.

THEOREM 3.4. *Let* $(q_n)_{n\geqslant 1}$, $(a_n)_{n\geqslant 1}$ *be real sequences such that* $q_n\in(0,1)$, $a_n\in(0,1]$. *If* $f\in Lip_M(\gamma)$, *then*

$$|C_n^{<q_n>}(f;x)-f(x)|\leqslant M(\delta_n(x))^{\gamma/2},$$

where $\delta_n(x)$ *is defined in (7).*

Proof. Since $C_n^{<q_n>}(e_0;\cdot)=e_0$ and $f\in Lip_M(\gamma)$, we have

$$|C_n^{<q_n>}(f;x)-f(x)|\leqslant\sum_{k=0}^n b_{n,k}(x;q_n)\sum_{j=0}^k c_{k,j}\left|f\left(\frac{[j]_{q_n}}{[n]_{q_n}}\right)-f(x)\right| \\ \leqslant M\sum_{k=0}^n b_{n,k}(x;q_n)\sum_{j=0}^k c_{k,j}\left|\frac{[j]_{q_n}}{[n]_{q_n}}-x\right|^\gamma.$$

Applying the Hölder's inequality with $p=\frac{2}{\gamma}$ and $q=\frac{2}{2-\gamma}$, we get

$$\begin{aligned}
|C_n^{<q_n>}(f;x)-f(x)| &\leqslant M\sum_{k=0}^{n} b_{n,k}(x;q_n)\left(\sum_{j=0}^{k} c_{k,j}\right)^{\frac{2-\gamma}{2}}\left[\sum_{j=0}^{k} c_{k,j}\left(\frac{[j]_{q_n}}{[n]_{q_n}}-x\right)^2\right]^{\frac{\gamma}{2}}\\
&= M\sum_{k=0}^{n} b_{n,k}(x;q_n)\left[\sum_{j=0}^{k} c_{k,j}\left(\frac{[j]_{q_n}}{[n]_{q_n}}-x\right)^2\right]^{\frac{\gamma}{2}}\\
&= M\sum_{k=0}^{n} \left(b_{n,k}(x;q_n)\right)^{\frac{2-\gamma}{2}}\left[b_{n,k}(x;q_n)\sum_{j=0}^{k} c_{k,j}\left(\frac{[j]_{q_n}}{[n]_{q_n}}-x\right)^2\right]^{\frac{\gamma}{2}}\\
&\leqslant M\left(\sum_{k=0}^{n} b_{n,k}(x;q_n)\right)^{\frac{2-\gamma}{2}}\left[\sum_{k=0}^{n} b_{n,k}(x;q_n)\sum_{j=0}^{k} c_{k,j}\left(\frac{[j]_{q_n}}{[n]_{q_n}}-x\right)^2\right]^{\frac{\gamma}{2}}\\
&= M\left\{C_n^{<q_n>}\left((t-x)^2;x\right)\right\}^{\frac{\gamma}{2}}\leqslant M(\delta_n(x))^{\gamma/2}. \quad \square
\end{aligned}$$

In order to give the next result we recall the definition of K-functional:

$$K_2(f,\delta):=\inf\left\{\|f-g\|+\delta\left\|g''\right\|: g\in W^2\right\},$$

where

$$W^2=\left\{g\in C[0,1]: g''\in C[0,1]\right\},$$

$\delta\geqslant 0$ and $\|\cdot\|$ is the uniform norm on $C[0,1]$. The second order modulus of continuity is defined as follows

$$\omega_2\left(f,\sqrt{\delta}\right)=\sup_{0<h\leqslant\sqrt{\delta}}\ \sup_{x,x+2h\in[0,1]}\left\{|f(x+2h)-2f(x+h)+f(x)|\right\}.$$

It is well known that K-functional and the second order modulus of continuity $\omega_2\left(f,\sqrt{\delta}\right)$ are equivalent, namely

$$K_2(f,\delta)\leqslant C\omega_2\left(f,\sqrt{\delta}\right), \tag{8}$$

where $\delta\geqslant 0$ and $C>0$.

THEOREM 3.5. *If $f\in C[0,1]$ and $(q_n)_{n\geqslant 1}$, $(a_n)_{n\geqslant 1}$ are real sequences such that $q_n\in(0,1)$, $a_n\in(0,1]$, then*

$$\left|C_n^{<q_n>}(f;x)-f(x)\right|\leqslant C\omega_2\left(f,\sqrt{\delta_n(x)}\right)+\omega\left(f,\lambda_n(x)\right),$$

where $\delta_n(x)$ and $\lambda_n(x)$ are defined in (7).

Proof. In order to prove the inequality, we construct the following auxiliary operators:

$$\tilde{C}_n^{<q_n>}(f;x) = C_n^{<q_n>}(f;x) + f(x) - f(a_n x). \tag{9}$$

It is not difficult to see that

$$\tilde{C}_n^{<q_n>}(e_0;x) = C_n^{<q_n>}(e_0;x) = 1$$
$$\tilde{C}_n^{<q_n>}(e_1;x) = C_n^{<q_n>}(e_1;x) + x - a_n x = x.$$

From Taylor's formula we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \quad g \in W^2.$$

Applying $\tilde{C}_n^{<q_n>}$ to above relation, it follows

$$\tilde{C}_n^{<q_n>}(g;x) = g(x) + \tilde{C}_n^{<q_n>}\left(\int_x^t (t-u)g''(u)du;x\right).$$

Therefore

$$\tilde{C}_n^{<q_n>}(g;x) = g(x) + C_n^{<q_n>}\left(\int_x^t (t-u)g''(u)du;x\right) - \int_x^{a_n x} (a_n x - u)\, g''(u)du.$$

This implies that

$$\begin{aligned} |\tilde{C}_n^{<q_n>}(g;x) - g(x)| &\leqslant \left|C_n^{<q_n>}\left(\int_x^t (t-u)g''(u)du;x\right)\right| + \left|\int_x^{a_n x} (a_n x - u)\, g''(u)du\right| \\ &\leqslant C_n^{<q_n>}((t-x)^2;x)\left\|g''\right\| + (a_n x - x)^2\left\|g''\right\|. \end{aligned}$$

From Lemma 2.3 it follows

$$|\tilde{C}_n^{<q_n>}(g;x) - g(x)| \leqslant \delta_n(x)\left\|g''\right\| + \lambda_n^2(x)\left\|g''\right\|.$$

Since $(1-a_n)\delta_n(x) = \dfrac{(1-a_n)a_n}{[n]_q}\phi^2(x) + (1-a_n)^2 \geqslant (1-a_n)^2 \geqslant \lambda_n^2(x)$, we get

$$|\tilde{C}_n^{<q_n>}(g;x) - g(x)| \leqslant (2-a_n)\delta_n(x)\left\|g''\right\| \leqslant 2\delta_n(x)\|g''\|. \tag{10}$$

In view of (9) and Lemma 2.2 we obtain

$$|\tilde{C}_n^{<q_n>}(f;x)| \leqslant |C_n^{<q_n>}(f;x)| + |f(x)| + |f(a_n x)| \leqslant 3\|f\|. \tag{11}$$

Now, for $f \in C[0,1]$ and $g \in W^2$, using (9), (10) and (11) we obtain

$$\begin{aligned} &|C_n^{<q_n>}(f;x) - f(x)| = \left|\tilde{C}_n^{<q_n>}(f;x) - f(x) + f(a_n x) - f(x)\right| \\ &\leqslant \left|\tilde{C}_n^{<q_n>}(f-g;x)\right| + \left|\tilde{C}_n^{<q_n>}(g;x) - g(x)\right| + |g(x) - f(x)| + |f(a_n x) - f(x)| \\ &\leqslant 4\|f-g\| + 2\delta_n(x)\left\|g''\right\| + \omega\left(f, |(1-a_n)x|\right). \end{aligned}$$

Taking the infimum on the right side over all $g \in W^2$, we have

$$|C_n^{<q_n>}(f;x) - f(x)| \leqslant 4K_2(f, \delta_n(x)) + \omega(f, |(1-a_n)x|).$$

Finally, using the equivalence between K-functional and the second order modulus of continuity (8), we have

$$|C_n^{<q_n>}(f;x) - f(x)| \leqslant C\omega_2\left(f, \sqrt{\delta_n(x)}\right) + \omega(f, (1-a_n)x),$$

which completes the proof. □

Let $\phi(x) = \sqrt{x(1-x)}$ and $f \in C[0,1]$. The Ditzian-Totik first order modulus of smoothness is given by (see [13])

$$\omega_\phi(f;t) = \sup_{0<h\leqslant t}\left\{\left|f\left(x + \frac{h\phi(x)}{2}\right) - f\left(x - \frac{h\phi(x)}{2}\right)\right|, x \pm \frac{h\phi(x)}{2} \in [0,1]\right\}. \quad (12)$$

The corresponding K-functional to (12) is defined by

$$K_\phi(f;t) = \inf_{g\in W_\phi[0,1]}\{||f-g|| + t||\phi g'||\}\ (t>0), \quad (13)$$

where $W_\phi[0,1] = \{g : g \in AC[0,1], ||\phi g'|| < \infty\}$ and $AC[0,1]$ is the class of all absolutely continuous functions on $[0,1]$. It is well known ([13], p.11) that there exists a constant $C > 0$ such that

$$K_\phi(f;t) \leqslant C\omega_\phi(f;t). \quad (14)$$

A direct approximation theorem by means of Ditzian-Totik modulus of smoothness is given in the next result.

THEOREM 3.6. *If $f \in C[0,1]$ and $(q_n)_{n\geqslant 1}$, $(a_n)_{n\geqslant 1}$ are real sequences such that $q_n \in (0,1)$, $a_n \in (0,1]$, then*

$$|\, C_n^{<q_n>}(f;x) - f(x)\,| < C\omega_\phi\left(f; \frac{\sqrt{\delta_n(x)}}{\phi(x)}\right), \quad (15)$$

where $\delta_n(x)$ is defined in (7) and C is a constant independent of n and x.

Proof. Using the relation

$$g(t) = g(x) + \int_x^t g'(u)du,$$

we get

$$\left|C_n^{<q_n>}(g;x) - g(x)\right| = \left|C_n^{<q_n>}\left(\int_x^t g'(u)du; x\right)\right|. \quad (16)$$

For any $x,t \in (0,1)$, we find that

$$\left| \int_x^t g'(u)du \right| \leqslant \|\phi g'\| \left| \int_x^t \frac{1}{\phi(u)} du \right| \leqslant \|\phi g'\| \left| \int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}} \right) du \right| \tag{17}$$
$$\leqslant 2\|\phi g'\| \left(|\sqrt{t} - \sqrt{x}| + |\sqrt{1-t} - \sqrt{1-x}| \right)$$
$$= 2\|\phi g'\| |t-x| \left(\frac{1}{\sqrt{t}+\sqrt{x}} + \frac{1}{\sqrt{1-t}+\sqrt{1-x}} \right)$$
$$< 2\|\phi g'\| |t-x| \left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}} \right) \leqslant \frac{2\sqrt{2}\|\phi g'\| \, |t-x|}{\phi(x)}.$$

Using Cauchy-Schwarz inequality, we obtain

$$|C_n^{<q_n>}(g;x) - g(x)| < 2\sqrt{2}||\phi g'||\phi^{-1}(x) C_n^{<q_n>}(|t-x|;x)$$
$$\leqslant 2\sqrt{2}||\phi g'||\phi^{-1}(x) \left(C_n^{<q_n>}((t-x)^2;x) \right)^{1/2}.$$

Applying Lemma 2.3, we get

$$|C_n^{<q_n>}(g;x) - g(x)| < 2\sqrt{2} \frac{\|\phi g'\|}{\phi(x)} \sqrt{\delta(x)}. \tag{18}$$

Therefore we can write

$$| C_n^{<q_n>}(f;x) - f(x) | \leqslant | C_n^{<q_n>}(f-g;x) | + |f(x) - g(x)| + | C_n^{<q_n>}(g;x) - g(x) |$$
$$\leqslant 2\|f-g\| + \frac{2\sqrt{2}\|\phi g'\|}{\phi(x)} \sqrt{\delta_n(x)}$$
$$\leqslant 2\sqrt{2} \left\{ \|f-g\| + \frac{\|\phi g'\|}{\phi(x)} \sqrt{\delta_n(x)} \right\}. \tag{19}$$

Taking infimum on the right hand side of the above inequality over all $g \in W_\phi[0,1]$, we get

$$| C_n^{<q_n>}(f;x) - f(x) | < C K_\phi \left(f; \frac{\sqrt{\delta_n(x)}}{\phi(x)} \right).$$

Using the relation (14) this theorem is proven. □

4. Numerical examples

In order to show the relevance of the operators $C_n^{<q>}$, in this section are given some numerical examples regarding the approximation properties. Furthermore, we compare the convergence of the operators $C_n^{<q>}$ with the q-Bernstein-Stancu operators proposed by Nowak [20]. From these results follows that for certain functions, the operators introduced in this paper $C_n^{<q>}$ converge faster than the q-Bernstein-Stancu operators $S_n^{<q,1>}$.

EXAMPLE 4.1. Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\dfrac{x^4}{12}-\dfrac{x^3}{6}+\dfrac{3x^2}{32}$. For $q=0.8$, $a_n=1-\dfrac{1}{n^2}$ the convergence of the operators $C_n^{<q>}$ to the function f is illustrated in Figure 1. We note that if the sequence (a_n) converges to 1 the operators $C_n^{<q>}$ are going to the graph of the function f.

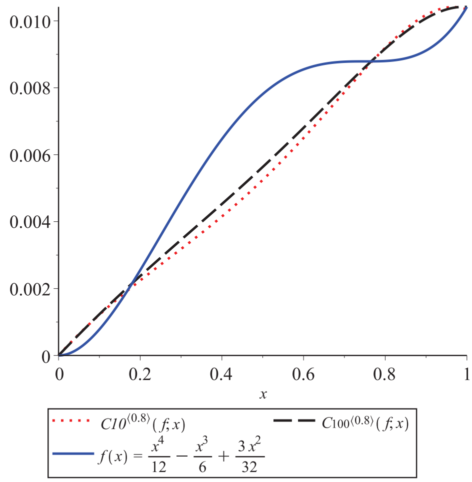

Figure 1: Approximation process by $C_n^{<q>}$

EXAMPLE 4.2. We consider $f:\mathbb{R}\to\mathbb{R}$, $f(x)=-\dfrac{4x^4}{3}+\dfrac{5x^3}{3}-\dfrac{x^2}{2}$. For $n=100$, $a_n=1-\dfrac{1}{n^2}$ and $q\in\{0.7;0.8;0.9\}$ the convergence of the operators $C_n^{<q>}$ to the function f is illustrated in Figure 2. We note that if the sequence (q_n) converges to 1 the operators $C_n^{<q>}$ are going to the graph of the function f.

EXAMPLE 4.3. We consider $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sin(2\pi x)$. For $q=0.8$, $n=10$ and $a_n=1-\dfrac{1}{n^2}$ the convergence of the operators $S_n^{<q,1>}$, $C_n^{<q>}$ to the function f is illustrated in Figure 3. From this graph follows that the operator $C_n^{<q>}$ introduced in this paper converges faster than the Stancu operator $S_n^{<q,1>}$ for this particular choice of function f. A similar example is given in Figure 4 for the function $f(x)=x^3e^{x+1}$.

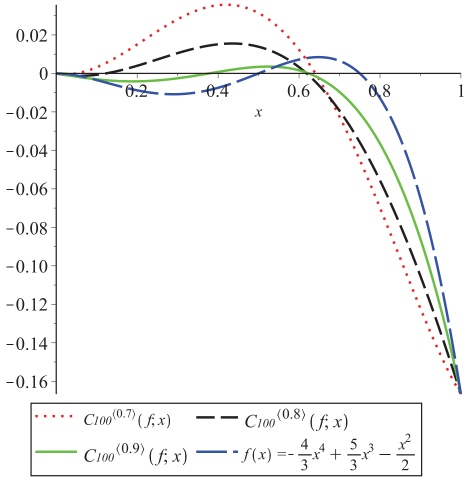

Figure 2: Approximation process by $C_n^{<q>}$

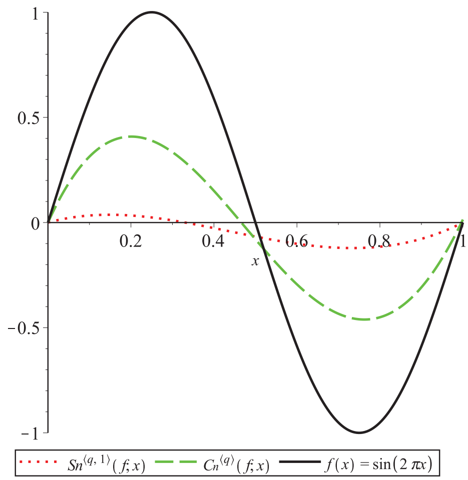

Figure 3: Approximation process by $S_n^{<q,1>}$, $C_n^{<q>}$

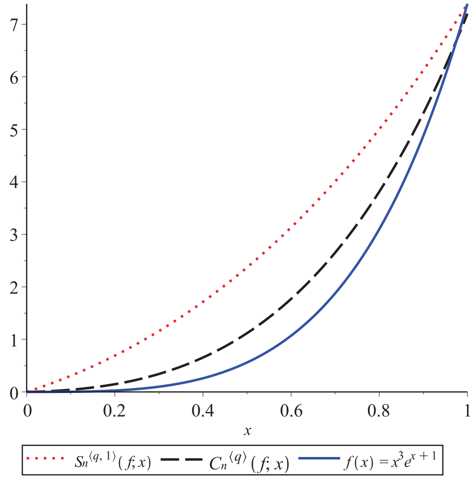

Figure 4: Approximation process by $S_n^{<q,1>}$, $C_n^{<q>}$

Acknowledgement. Project financed from Lucian Blaga University of Sibiu research grants LBUS-IRG-2018-04.

REFERENCES

[1] T. ACAR, A. ARAL, *On Pointwise Convergence of q-Bernstein Operators and Their q-Derivatives*, Numerical Functional Analysis and Optimization, 36(3), 2015, 287–304.

[2] A.M. ACU, C.V. MURARU, V.A. RADU, F.D. SOFONEA, *Some approximation properties of a Durrmeyer variant of q-Bernstein-Schurer operators*, Mathematical Methods in Applied Science, 39(18), 2016, 5636–5650.

[3] A.M. ACU, C.V. MURARU, V.A. RADU, *On the monotonicity of q-Schurer-Stancu type polynomials*, Miskolc Mathematical Notes, 19 (1), 2018, 19–28.

[4] A.M. ACU, *Stancu-Schurer-Kantorovich operators based on q-integers*, Applied Mathematics and Computation, 259, 2015, 896–907.

[5] A.M. ACU, D. BĂRBOSU, D.F. SOFONEA, *Note on a q-analogue of Stancu-Kantorovich operators*, Miskolc Mathematical Notes, 16 (1), 2015, 3–15.

[6] A.M. ACU, P.N. AGRAWAL, TRAPTI NEER, *Approximation properties of the modified Stancu operators*, Numerical Functional Analysis and Optimization, 38 (3), 2017, 279–292.

[7] O. AGRATINI, *On a q-analogue of Stancu operators*, Cent. Eur. J. Math, 8(1), 2010, 191–198.

[8] R.P. AGARWAL, V. GUPTA, A.S. KUMAR, *On q-analogue of Bernstein-Schurer-Stancu operators*, Applied Mathematics and Computation, 219, 2013, 7754–7764.

[9] P.N. AGRAWAL, A.M. ACU, M. SIDHARTH, *Approximation degree of a Kantorovich variant of Stancu operators based on Polya-Eggenberger distribution*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matemáticas, 2017, https://doi.org/10.1007/s13398-017-0461-0

[10] A. ARAL, V. GUPTA, R.P. AGARWAL, *Applications of q Calculus in Operator Theory*, Springer 2013, XII, 262 p.

[11] V.A. CLECIU, *Bernstein-Stancu operators*, Studia Univ.Babes-Bolyai, Mathematica, 52 (4), 2007, 53–65.

[12] V.A. CLECIU, *Approximation properties of a class of Bernstein-Stancu type operators*, Proceedings of the international conference NAAT, ed. O. Agratini, 2006, 171–178.

[13] Z. DITZIAN, TOTIK, V., *Moduli of Smoothness*, Springer, New York, (1987).

[14] O. DOĞRU, C. MURARU, *Statistical approximation by Stancu type bivariate generalization of Meyer-König and Zeller type operators*, Mathematical and Computer Modelling 48 (5), 2008, 961–968.

[15] V. KAC, P. CHEUNG, *Quantum Calculus*, Universitext, Springer, 2002, 112 p.

[16] A. KAJLA, N. ISPIR, P.N. AGRAWAL, M. GOYALA, *q-Bernstein -Schurer-Durrmeyer type operators for functions of one and two variables*, Applied Mathematics and Computation, 275, 2016, 372–385.

[17] A. LUPAŞ, *A q-analogue of the Bernstein operator*, University of Cluj-Napoca, Seminar on Numerical and Statistical Calculus, Preprint 9, 1987, 85–92.

[18] A. LUPAŞ, *q-analogue of Stancu operator*, Mathematical Analysis and Approximation Theory, 2002, 145–154.

[19] T. NEER, A.M. ACU, P.N. AGRAWAL, *Bezier variant of genuine-Durrmeyer type operators based on Polya distribution*, Carpathian Journal of Mathematics, Vol. 33 (1), 2017, 73–86.

[20] G. NOWAK, *Approximation properties for generalized q-Bernstein polynomials*, J. Math. Anal. Appl., 350, 2009, 50–55.

[21] G. NOWAK, V. GUPTA, *The rate of pointwise approximation of positive linear operators based on q-integer*, Ukrainian Math. Journal, 63 (3), 2011, 403–415.

[22] G.M. PHILLIPS, *On generalized Bernstein polynomials*, in D. F. Griffits, G. A. Watson (Eds.), 1996, 263–269.

[23] V.A. RADU, *Quantitative Estimates for Some Modified Bernstein-Stancu Operators*, Miskolc Math. Notes 19 (1), 2018, 517–525.

[24] D.D. STANCU, *Approximation of functions by a new class of linear polynomial operators*, rev. Roum. Math. Pures Appl., 13, 1968, 1173–1194.

(Received August 6, 2018)

Ana-Maria Acu
Lucian Blaga University of Sibiu
Department of Mathematics and Informatics
Str. Dr. I. Ratiu, No.5-7, RO-550012 Sibiu, Romania
e-mail: acuana77@yahoo.com

Ogün Doğru
Gazi University, Faculty of Arts and Sciences
Department of Mathematics
Teknikokullar, 06500 Ankara, Turkey
e-mail: ogun.dogru@gazi.edu.tr

Carmen Violeta Muraru
Vasile Alecsandri University of Bacău
Department of Mathematics, Informatics and Educational Sciences
Bacău, Romania
e-mail: carmen_7419@yahoo.com, *e-mail:* cmuraru@ub.ro

Voichiţa Adriana Radu
Babes-Bolyai University, FSEGA
Department of Statistics-Forecasts-Mathematics
Cluj-Napoca, Romania
e-mail: voichita.radu@econ.ubbcluj.ro, *e-mail:* voichita.radu@gmail.com