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A NOTE ON "REMARKS ON SOME INEQUALITIES FOR POSITIVE SEMIDEFINITE MATRICES AND QUESTIONS FOR BOURIN"

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Abstract. Let A_i, $B_i \in \mathscr{M}_n$ be positive semidefinite matrices with $A_iB_i = B_iA_i$ $(i = 1,2,\cdots,m)$. Then

$$\sigma\Big(\Big(\sum_{i=1}^{m}(A_iB_i)^{\frac{1}{2}}\Big)^r\Big) \prec_{wlog} \sigma\Big(\Big(\sum_{i=1}^{m}A_i\Big)^{\frac{r}{4}}\Big(\sum_{i=1}^{m}B_i\Big)^{\frac{r}{2}}\Big(\sum_{i=1}^{m}A_i\Big)^{\frac{r}{4}}\Big),$$

where $r \geqslant 1$. This result is a refinement of M. Hayajneh, S. Hayajneh and F. Kittaneh's result.

1. Introduction

Throughout, let $\mathscr{M}_n$ be the space of $n\times n$ complex matrices. For $A \in \mathscr{M}_n$, denotes $\lambda_j(A)$ $(j = 1,2,\cdots,n)$ by the eigenvalues of A with $|\lambda_1(A)| \geqslant |\lambda_2(A)| \geqslant \cdots \geqslant |\lambda_n(A)|$. The singular values of A denoted as $\sigma_j(A)(j = 1,2,\cdots,n)$, i.e., the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$, arranged as $\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_n(A)$, where A^* is the conjugate transpose of A. Let $\lambda(A) := (\lambda_1(A), \lambda_2(A), \cdots, \lambda_n(A))$, and $\sigma(A) := (\sigma_1(A), \sigma_2(A), \cdots, \sigma_n(A))$ be the vector of eigenvalues and the singular values of A, respectively. For two Hermitian matrices A, $B \in \mathscr{M}_n$, $A \leqslant (<) B$ means $B - A$ is positive semidefinite (definite). A norm $\|\cdot\|$ on $\mathscr{M}_n$ is called a unitarily invariant norm if $\|UAV\| = \|A\|$ for A, U, $V \in \mathscr{M}_n$ with U, V are unitaries. I_n is the identity matrix of $\mathscr{M}_n$.

Let us recall some definitions of majorizations. Given a real vector $x = (x_1, x_2, \cdots, x_n) \in \mathscr{R}^n$, we rearrange its components as $x_{[1]} \geqslant x_{[2]} \geqslant \cdots \geqslant x_{[n]}$. For $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathscr{R}^n$, if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]},\ k = 1,2,\cdots,n,$$

then we say that x is weakly majorized by y and denotes by $x \prec_w y$. If $x \prec_w y$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then we say that x is majorized by y and denotes by $x \prec y$. Further, if

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$x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathscr{R}_+^n$ and

$$\prod_{i=1}^{k} x_{[i]} \leqslant \prod_{i=1}^{k} y_{[i]},\ k = 1, 2, \cdots, n,$$

then we say that x is weakly log-majorized by y and denotes by $x \prec_{wlog} y$. If $x \prec_{wlog} y$ and $\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i$, then we say that x is log-majorized by y and denotes by $x \prec_{log} y$. It is well-known that if $x \prec_{wlog} y$, then $x \prec_w y$.

For $t \in [0,1]$, the $t-$ geometric mean of A, $B \in \mathscr{M}_n$ with A, $B > 0$ is defined as:

$$A\sharp_t B := A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t A^{\frac{1}{2}}. \tag{1}$$

When $t = \frac{1}{2}$, $A\sharp_{\frac{1}{2}} B$ is the geometric mean of A and B. For convenience, we write $A\sharp B$ instead of $A\sharp_{\frac{1}{2}} B$. $A\sharp B$ has an extremal property (see, e.g., [2, Theorem 4.1.3]):

$$A\sharp B = \max\left\{X : X = X^*, \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geqslant 0\right\}. \tag{2}$$

Very recently, M. Hayajneh, S. Hayajneh and F. Kittaneh [4, Theorem 2] obtained: Let A_i, $B_i \in \mathscr{M}_n$ be positive semidefinite matrices with $A_i B_i = B_i A_i$ $(i = 1, 2, \cdots, m)$. Then for all unitarily invariant norms,

$$\left\| \left(\sum_{i=1}^{m} A_i^{\frac{1}{2}} B_i^{\frac{1}{2}} \right)^2 \right\| \leqslant \left\| \left(\sum_{i=1}^{m} A_i \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} B_i \right) \left(\sum_{i=1}^{m} A_i \right)^{\frac{1}{2}} \right\|. \tag{3}$$

Inequality (3) is a refinement of the following inequality obtained by Audenaert [1]: Let A_i, $B_i \in \mathscr{M}_n$ be positive semidefinite matrices with $A_i B_i = B_i A_i$ $(i = 1, 2, \cdots, m)$. Then for all unitarily invariant norms,

$$\left\| \left(\sum_{i=1}^{m} A_i^{\frac{1}{2}} B_i^{\frac{1}{2}} \right)^2 \right\| \leqslant \left\| \left(\sum_{i=1}^{m} A_i \right) \left(\sum_{i=1}^{m} B_i \right) \right\|. \tag{4}$$

Lin [7] and Hoa [6] presented a different proof for inequality (4), respectively. Inequality (4) gave an affirmative answer to J. C. Bourin's question: Given two positive semidefinite matrices A, B and two positive real numbers p, q, is it true that

$$\|A^{p+q} + B^{p+q}\| \leqslant \|(A^p + B^p)(A^q + B^q)\| \ ? \tag{5}$$

In this short note, we will present an inequality for weak log-majorizations, which makes inequality (3) as a special case.

2. Main results

In this section, we mainly present an inequality for weak log-majorizations. To achieve our goal, we need the following lemmas. The first lemma was obtained by J. Matharu and J. Aujla [8, Theorem 2.10].

LEMMA 1. *Let* $A,\ B\in\mathscr{M}_n$ *with* $A,\ B>0$ *and* $t\in[0,1]$. *Then*

$$\lambda(A\sharp_t B)\prec_{wlog}\lambda(A^{1-t}B^t).$$

The next lemma was given by Hiai [5, Theorem 3.4].

LEMMA 2. *Let* $A,\ B\in\mathscr{M}_n$ *with* $A,\ B>0$ *and* $t\in[0,1]$. *Then*

$$\lambda\left(\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)^r\right)\prec_{wlog}\lambda\left(A^{\frac{r}{2}}B^rA^{\frac{r}{2}}\right),$$

for $r\geqslant 1$.

In the sequel, we present the famous Fan dominance theorem (see, e.g., [9, Theorem 4.24]).

LEMMA 3. *Let* $A,\ B\in\mathscr{M}_n$. *Then*

$$\sigma(A)\prec_w\sigma(B)\Leftrightarrow\|A\|\leqslant\|B\|$$

for any unitarily invariant norm $\|\cdot\|$.

The lemma 4 was due to Bourin and Uchiyama [3, Theorem 1.2].

LEMMA 4. *Let* $A,\ B\geqslant 0$ *and* $g:[0,+\infty)\to[0,+\infty)$ *be a convex function with* $g(0)=0$. *Then for every unitarily invariant norm* $\|\cdot\|$

$$\|g(A)+g(B)\|\leqslant\|g(A+B)\|.$$

It is now time to present the following theorem.

THEOREM 1. *Let* $A_i,\ B_i\in\mathscr{M}_n$ *be positive semidefinite matrices with* $A_iB_i=B_iA_i$ *(*$i=1,2,\cdots,m$*). Then*

$$\sigma\left(\left(\sum_{i=1}^m(A_iB_i)^{\frac{1}{2}}\right)^r\right)\prec_{wlog}\sigma\left(\left(\sum_{i=1}^m A_i\right)^{\frac{r}{4}}\left(\sum_{i=1}^m B_i\right)^{\frac{r}{2}}\left(\sum_{i=1}^m A_i\right)^{\frac{r}{4}}\right),\tag{6}$$

where $r\geqslant 1$.

Proof. We firstly consider the case $A_i,\ B_i>0$ ($i=1,2,\cdots,m$). Since $A_iB_i=B_iA_i$ ($i=1,2,\cdots,m$), we have $A_i\sharp B_i=A_i^{\frac{1}{2}}B_i^{\frac{1}{2}}=(A_iB_i)^{\frac{1}{2}}$. By the relation (2), we obtain

$$\begin{pmatrix} A_i & (A_iB_i)^{\frac{1}{2}} \\ (A_iB_i)^{\frac{1}{2}} & B_i \end{pmatrix}\geqslant 0,$$

for $i=1,2,\cdots,m$, which implies

$$\begin{pmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m (A_iB_i)^{\frac{1}{2}} \\ \sum_{i=1}^m (A_iB_i)^{\frac{1}{2}} & \sum_{i=1}^m B_i \end{pmatrix}\geqslant 0.$$

Using the relation (2) again, we get

$$\sum_{i=1}^{m}(A_iB_i)^{\frac{1}{2}} \leqslant (\sum_{i=1}^{m}A_i)\sharp(\sum_{i=1}^{m}B_i),$$

which implies

$$\prod_{j=1}^{k}\sigma_j\Big(\Big(\sum_{i=1}^{m}(A_iB_i)^{\frac{1}{2}}\Big)^r\Big) \leqslant \prod_{j=1}^{k}\sigma_j\Big(\Big((\sum_{i=1}^{m}A_i)\sharp(\sum_{i=1}^{m}B_i)\Big)^r\Big), \tag{7}$$

for $k=1,2,\cdots,n$.

By Lemmas 1 and 2, we have

$$\begin{aligned}\prod_{j=1}^{k}\sigma_j\Big(\Big((\sum_{i=1}^{m}A_i)\sharp(\sum_{i=1}^{m}B_i)\Big)^r\Big) &\leqslant \prod_{j=1}^{k}\lambda_j^r\Big((\sum_{i=1}^{m}A_i)^{\frac{1}{2}}(\sum_{i=1}^{m}B_i)^{\frac{1}{2}}\Big)\\ &= \prod_{j=1}^{k}\lambda_j^r\Big((\sum_{i=1}^{m}A_i)^{\frac{1}{4}}(\sum_{i=1}^{m}B_i)^{\frac{1}{2}}(\sum_{i=1}^{m}A_i)^{\frac{1}{4}}\Big)\\ &\leqslant \prod_{j=1}^{k}\lambda_j\Big((\sum_{i=1}^{m}A_i)^{\frac{r}{4}}(\sum_{i=1}^{m}B_i)^{\frac{r}{2}}(\sum_{i=1}^{m}A_i)^{\frac{r}{4}}\Big)\\ &= \prod_{j=1}^{k}\sigma_j\Big((\sum_{i=1}^{m}A_i)^{\frac{r}{4}}(\sum_{i=1}^{m}B_i)^{\frac{r}{2}}(\sum_{i=1}^{m}A_i)^{\frac{r}{4}}\Big),\end{aligned} \tag{8}$$

for $k=1,2,\cdots,n$.

Combining inequalities (7) and (8), we get the desired inequality (6).

For the general case, replacing A_i and B_i by $A_i+\varepsilon I_n$ and $B_i+\varepsilon I_n$ $(\varepsilon>0)$ for $i=1,2,\cdots,m$, respectively, and repeating the same process as above, we obtain

$$\begin{aligned}&\sigma\Big(\Big(\sum_{i=1}^{m}(A_i+\varepsilon I_n)(B_i+\varepsilon I_n)^{\frac{1}{2}}\Big)^r\Big)\\ &\prec_{wlog}\sigma\Big(\Big(\sum_{i=1}^{m}(A_i+\varepsilon I_n)\Big)^{\frac{r}{4}}\Big(\sum_{i=1}^{m}(B_i+\varepsilon I_n)\Big)^{\frac{r}{2}}\Big(\sum_{i=1}^{m}(A_i+\varepsilon I_n)\Big)^{\frac{r}{4}}\Big).\end{aligned}$$

Letting $\varepsilon\to 0^+$, by continuity, we can also get inequality (6).

This completes the proof. □

REMARK 1. Let A_i, $B_i\in\mathscr{M}_n$ be positive semidefinite matrices with $A_iB_i=B_iA_i$ ($i=1,2,\cdots,m$). Putting $r=2$ in inequality (6), we obtain

$$\sigma\Big(\Big(\sum_{i=1}^{m}(A_iB_i)^{\frac{1}{2}}\Big)^2\Big)\prec_{wlog}\sigma\Big(\Big(\sum_{i=1}^{m}A_i\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{m}B_i\Big)\Big(\sum_{i=1}^{m}A_i\Big)^{\frac{1}{2}}\Big),$$

which implies,

$$\sigma\Big(\Big(\sum_{i=1}^{m}(A_iB_i)^{\frac{1}{2}}\Big)^2\Big) \prec_w \sigma\Big(\Big(\sum_{i=1}^{m}A_i\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{m}B_i\Big)\Big(\sum_{i=1}^{m}A_i\Big)^{\frac{1}{2}}\Big). \tag{9}$$

By Lemma 3, inequality (9) is equivalent to inequality (3). Therefore, inequality (6) is a refinement of inequality (3).

REMARK 2. Let A_i, $B_i \in \mathscr{M}_n$ be positive semidefinite matrices with $A_iB_i = B_iA_i$ ($i = 1,2,\cdots,m$). By Lemma 4, we have (taking $g(x) = x^2$)

$$\Big\|\sum_{i=1}^{m}A_iB_i\Big\| \leqslant \Big\|\Big(\sum_{i=1}^{m}A_i^{\frac{1}{2}}B_i^{\frac{1}{2}}\Big)^2\Big\|. \tag{10}$$

Combining inequalities (9) and (10), we get

$$\Big\|\sum_{i=1}^{m}A_iB_i\Big\| \leqslant \Big\|\Big(\sum_{i=1}^{m}A_i\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{m}B_i\Big)\Big(\sum_{i=1}^{m}A_i\Big)^{\frac{1}{2}}\Big\|. \tag{11}$$

Noting the fact: If X and Y are matrices with XY is Hermitian, then $\|XY\| \leqslant \|YX\|$ for all unitarily invariant norms $\|\cdot\|$, inequality (11) gives

$$\begin{aligned}\Big\|\sum_{i=1}^{m}A_iB_i\Big\| &\leqslant \Big\|\Big(\sum_{i=1}^{m}A_i\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{m}B_i\Big)\Big(\sum_{i=1}^{m}A_i\Big)^{\frac{1}{2}}\Big\| \\ &\leqslant \Big\|\Big(\sum_{i=1}^{m}B_i\Big)\Big(\sum_{i=1}^{m}A_i\Big)\Big\| \\ &= \Big\|\Big(\sum_{i=1}^{m}A_i\Big)\Big(\sum_{i=1}^{m}B_i\Big)\Big\|. \end{aligned} \tag{12}$$

On the other hand, let A and B be two positive semidefinite matrices and p, q be two positive real numbers. Taking $m = 2$, $A_1 = A^p$, $B_1 = A^q$, $A_2 = B^p$, $B_2 = B^q$ in inequality (12), we have

$$\begin{aligned}\Big\|A^{p+q} + B^{p+q}\Big\| &\leqslant \Big\|(A^p + B^p)^{\frac{1}{2}}(A^q + B^q)(A^p + B^p)^{\frac{1}{2}}\Big\| \\ &\leqslant \Big\|(A^p + B^p)(A^q + B^q)\Big\|,\end{aligned}$$

which is just the inequality (5). So, we also get an affirmative answer to Bourin's question.

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