Journal of
Mathematical
Inequalities
Volume 13, Number 3 (2019), 789–799 doi:10.7153/jmi-2019-13-54

LYAPUNOV–TYPE INEQUALITY FOR THE HADAMARD FRACTIONAL BOUNDARY VALUE PROBLEM ON A GENERAL INTERVAL $[a,b]$

ZAID LAADJAL, NACER ADJEROUD AND QINGHUA MA*

(*Communicated by J. Pečarić*)

Abstract. In this paper, using two different methods, we studied an open problem and obtained several results for Lyapunov-type and Hartman-Wintner-type inequalities for a Hadamard fractional differential equation on a general interval $[a,b]$, $(1 \leqslant a < b)$ with the boundary value conditions.

1. Introduction

The first result in this domain is due to Lyapunov [1], can be stated as follows: If a nontrivial continuous solution to the following boundary value problem

$$\begin{cases} u''(t)+q(t)u(t)=0, \ a<t<b, \\ \quad u(a)=u(b)=0, \end{cases} \tag{1}$$

exist, where $q:[a,b]\to\mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)|\,ds > \frac{4}{b-a}. \tag{2}$$

Lyapunov's inequality has proved useful in the study of various properties of differential and difference equations. These applications include bounds for eigenvalues, stability criteria for periodic differential equations, and estimates for intervals of disconjugacy, etc. Recently, several articles from the inequality of Lyapunov have been published about a differential equations of the integer order and fractional order, see [5, 6, 7, 8, 9, 10] and references therein. For example, The following result for the Riemann-Liouville fractional boundary value problem is found by D. O'Regan and B. Samet [4]

$$\begin{cases} {}_a^R D^{\alpha}u(t)+q(t)u(t)=0, & a<t<b,\ 3<\alpha\leqslant 4, \\ u(a)=u'(a)=u''(a)=u''(b)=0, \end{cases} \tag{3}$$

Mathematics subject classification (2010): 26D15, 26A33, 34A08, 34A40.
Keywords and phrases: Hadamard fractional derivative, boundary value problem, Green's function, Lyapunov-type inequality, Hartman-Wintner-type inequality.
* Corresponding author.

has a nontrivial continuous solution, then

$$\int_a^b |q(s)|\,ds > \frac{\Gamma(\alpha)(\alpha-2)^{\alpha-2}}{2(\alpha-3)^{\alpha-3}(b-a)^{\alpha-1}}. \tag{4}$$

In [2] Ma, Ma and Wang established a Lyapunov-type inequality for a differential equation that depends on the Hadamard fractional derivative, for the boundary value problem

$$\begin{cases} {}_1^H D^{\alpha} u(t) - q(t)u(t) = 0,\ 1 < t < e,\ \ 1 < \alpha \leqslant 2, \\ \qquad u(1) = u(e) = 0, \end{cases} \tag{5}$$

where $q : [1,e] \to \mathbb{R}$ is a continuous function. They proved that if a nontrivial continuous solution to the above problem, then

$$\int_1^e |q(s)|\,ds > \Gamma(\alpha)\lambda^{1-\lambda}(1-\alpha)exp\lambda, \tag{6}$$

where $\lambda = \frac{2\alpha-1-\sqrt{(2\alpha-2)^2+1}}{2}$. And they have presented the following open problem for readers: How to get the Lyapunov inequality for the following the Hadarmard fractional boundary value problem (HFBVP)

$$\begin{cases} {}_a^H D^{\alpha} u(t) - q(t)u(t) = 0,\ 1 \leqslant a < t < b,\ 1 < \alpha \leqslant 2, \\ \qquad u(a) = u(b) = 0, \end{cases} \tag{7}$$

where ${}_a^H D^{\alpha}$ is the Hadamard fractional derivative of order α, and $q : [a,b] \to \mathbb{R}$ is a continuous function.

In this paper we answered the previous question by using two methods, and also we get the Hartman-Wintner-type inequalities. The interest of the article does not lie only in the fact that has given the answer to the open problem, but also some mathematical analysis skills and effort for overcoming the hard obstacles to find the maximum value of the log-style Green's function. The analysis skills can be used to deal with some more complicated similar problems.

2. Preliminaries

DEFINITION 1. [3] Let $a, b, \alpha \in \mathbb{R}^+$ where $a < b$ and $n-1 < \alpha \leqslant n$ with $n \in \mathbb{N}$, the Hadamard fractional integral of order α for a function $f(t)$ is defined by

$${}_a^H I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s)\,\frac{ds}{s},\ a \leqslant t \leqslant b, \tag{8}$$

with Γ is Euler function.

DEFINITION 2. [3] Let $a, b \in \mathbb{R}^+$ with $a < b$, the Hadamard fractional derivative of order $\alpha \in \mathbb{R}^+$ for a function $f(t)$ is defined by

$${}_a^H D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} t^n \frac{d^n}{dt^n} \int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1} f(s)\,\frac{ds}{s},\ a \leqslant t \leqslant b, \tag{9}$$

where $n-1 < \alpha \leqslant n$ with $n \in \mathbb{N}$.

LEMMA 1. [3] *Let* $0 < a < b$ *and* $\alpha > 0$ *where* $n-1 < \alpha \leqslant n$ *and* $n \in \mathbb{N}$*, the equation* ${}_a^H D^\alpha u(t) = 0$ *has this solutions*

$$u(t) = \sum_{i=1}^{i=n} c_i \left(\ln \frac{t}{a}\right)^{\alpha-i}, \ t \in [a,b], \tag{10}$$

where $c_i \in \mathbb{R}, (i = 1,...,n)$ *are constants. And moreover*

$${}_a^H I^\alpha \, {}_a^H D^\alpha u(t) = u(t) + \sum_{i=1}^{i=n} c_i \left(\ln \frac{t}{a}\right)^{\alpha-i}, \tag{11}$$

LEMMA 2. *Let* $A, B \in \mathbb{R}$*. Then*

$$AB \leqslant \frac{(A+B)^2}{4}. \tag{12}$$

3. Main results

LEMMA 3. *Let* $u \in C([a,b],\mathbb{R})$*, the following problem*

$$\begin{cases} {}_a^H D^\alpha u(t) - q(t)u(t) = 0, \ 1 \leqslant a < t < b, \ 1 < \alpha \leqslant 2, \\ u(a) = u(b) = 0, \end{cases} \tag{13}$$

has equivalent to the fractional integral equation

$$u(t) = \int_a^b G(t,s)q(s)u(s)\, ds, \tag{14}$$

where

$$G(t,s) = \begin{cases} g_1(t,s) = g_2(t,s) + \frac{1}{\Gamma(\alpha)} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{1}{s}, \ a \leqslant s \leqslant t \leqslant b, \\ \\ g_2(t,s) = -\frac{1}{\Gamma(\alpha)} \frac{\left(\ln \frac{t}{a}\right)^{\alpha-1} \left(\ln \frac{b}{s}\right)^{\alpha-1}}{\left(\ln \frac{b}{a}\right)^{\alpha-1}} \frac{1}{s}, \quad a \leqslant t \leqslant s \leqslant b, \end{cases} \tag{15}$$

with $1 \leqslant a < b$*.*

Proof. Using Lemma 1, we have

$$u(t) = c_1 \left(\ln \frac{t}{a}\right)^{\alpha-1} + c_2 \left(\ln \frac{t}{a}\right)^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} q(s)u(s) \frac{ds}{s}, \tag{16}$$

where $c_1, c_2 \in \mathbb{R}$.
Using the boundary condition $u(a) = u(b) = 0$, we get $c_2 = 0$ and

$$c_1 = -\frac{\left(\ln \frac{b}{a}\right)^{1-\alpha}}{\Gamma(\alpha)} \int_a^b \left(\ln \frac{b}{s}\right)^{\alpha-1} q(s)u(s) \frac{ds}{s}. \tag{17}$$

Substituting the values of c_1 and c_2 in (16), we obtain

$$
\begin{aligned}
u(t) = &-\frac{\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{a}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-1}q(s)u(s)\,\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}q(s)u(s)\,\frac{ds}{s}\\
=&\frac{1}{\Gamma(\alpha)}\int_a^t\left[\left(\ln\frac{t}{s}\right)^{\alpha-1}-\left(\ln\frac{b}{a}\right)^{1-\alpha}\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-1}\right]q(s)u(s)\,\frac{ds}{s}\\
&-\frac{1}{\Gamma(\alpha)}\int_t^b\left(\ln\frac{b}{a}\right)^{1-\alpha}\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-1}q(s)u(s)\,\frac{ds}{s}\\
=&\int_a^b G(t,s)q(s)u(s)\,ds.
\end{aligned}
\tag{18}
$$

The proof is complete. □

LEMMA 4. *The Green's function G defined in Lemma 3, has the following properties*

1. $g_2(s,s)\leqslant G(t,s)\leqslant 0$, *for all* $(t,s)\in[a,b]\times[a,b]$.
2. *For any* $(t,s)\in[a,b]\times[a,b]$,

$$
|G(t,s)|\leqslant|G(s,s)|=-g_2(s,s)\leqslant\frac{1}{4^{(\alpha-1)}\Gamma(\alpha)a}\left(\ln\frac{b}{a}\right)^{(\alpha-1)}. \tag{19}
$$

Proof. We start by fixing an arbitrary $s\in[a,b]$. Differentiating $G(t,s)$ with respect to t, we get
For $1\leqslant a<t\leqslant s\leqslant b$, we have

$$
\frac{\partial}{\partial t}g_2=-\frac{(\alpha-1)(\ln\frac{t}{a})^{\alpha-2}(\ln\frac{b}{s})^{\alpha-1}}{\Gamma(\alpha)(\ln\frac{b}{a})^{\alpha-1}st}\leqslant 0. \tag{20}
$$

On the other hand, by (15) note that $g_2(t,s)\leqslant 0$, we obtain

$$
g_2(s,s)\leqslant g_2(t,s)\leqslant g_2(a,s)\leqslant 0. \tag{21}
$$

While for $1\leqslant a\leqslant s<t\leqslant b$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}g_1 &= \frac{\partial}{\partial t}g_2+\frac{(\alpha-1)}{\Gamma(\alpha)st}\left(\ln\frac{t}{s}\right)^{\alpha-2}\\
&=-\frac{(\alpha-1)}{\Gamma(\alpha)st}\frac{\left(\ln\frac{t}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}+\frac{(\alpha-1)}{\Gamma(\alpha)st}\left(\ln\frac{t}{s}\right)^{\alpha-2}\\
&=\frac{(\alpha-1)\left(\ln\frac{t}{a}\right)^{\alpha-2}}{\Gamma(\alpha)st}\left[\left(\frac{\ln\frac{t}{a}}{\ln\frac{t}{s}}\right)^{2-\alpha}-\left(\frac{\ln\frac{b}{s}}{\ln\frac{b}{a}}\right)^{\alpha-1}\right].
\end{aligned}
\tag{22}
$$

By $1 \leqslant a \leqslant s < t \leqslant b$, we get

$$\left(\frac{\ln \frac{t}{a}}{\ln \frac{t}{s}}\right)^{2-\alpha} \geqslant 1, \tag{23}$$

and

$$-\left(\frac{\ln \frac{b}{s}}{\ln \frac{b}{a}}\right)^{\alpha-1} \geqslant -1, \tag{24}$$

using (23) and (24), we obtain

$$\left[\left(\frac{\ln \frac{t}{a}}{\ln \frac{t}{s}}\right)^{2-\alpha} - \left(\frac{\ln \frac{b}{s}}{\ln \frac{b}{a}}\right)^{\alpha-1}\right] \geqslant 0, \tag{25}$$

So thus

$$\frac{\partial}{\partial t} g_1 \geqslant 0. \tag{26}$$

Using $1 \leqslant a \leqslant s < t \leqslant b$, we get

$$g_1(t,s) \leqslant g_1(b,s) = 0. \tag{27}$$

On the other hand, for all $s \in [a,t), \lim_{t \to s^+} g_1(t,s) = g_1(s,s)$, so for any $t \in [s,b]$,

$$g_1(s,s) \leqslant g_1(t,s), \tag{28}$$

and if $t = s$, then

$$g_2(s,s) = g_1(s,s), \tag{29}$$

with $g_2(a,s) = g_1(a,a) = 0$ for all $s \in [a,b]$. By (21), (27), (28) and (29), we obtain

$$g_2(s,s) = G(s,s) \leqslant G(t,s) \leqslant 0. \tag{30}$$

Now we prove that

$$|G(s,s)| \leqslant \frac{1}{4^{\alpha-1}\Gamma(\alpha)a}\left(\ln \frac{b}{a}\right)^{\alpha-1}. \tag{31}$$

We have $G(s,s) = g_2(s,s) = g_1(s,s) \leqslant 0$.
Using Lemma 2, we have

$$\begin{aligned}
|G(s,s)| &= \frac{1}{\Gamma(\alpha)\left(\ln \frac{b}{a}\right)^{\alpha-1} s}\left[\left(\ln \frac{s}{a}\right)\left(\ln \frac{b}{s}\right)\right]^{\alpha-1} \\
&\leqslant \frac{1}{4^{\alpha-1}\Gamma(\alpha)\left(\ln \frac{b}{a}\right)^{\alpha-1} s}\left[\left(\ln \frac{s}{a} + \ln \frac{b}{s}\right)^2\right]^{\alpha-1} \\
&= \frac{1}{4^{\alpha-1}\Gamma(\alpha)s}\left(\ln \frac{b}{a}\right)^{\alpha-1} \\
&\leqslant \frac{1}{4^{\alpha-1}\Gamma(\alpha)a}\left(\ln \frac{b}{a}\right)^{\alpha-1}.
\end{aligned}$$

Therefore

$$|G(t,s)| \leqslant |G(s,s)| = -g_2(s,s) \leqslant \frac{1}{4^{(\alpha-1)}\Gamma(\alpha)a}\left(\ln\frac{b}{a}\right)^{\alpha-1}. \tag{32}$$

The proof is complete. □

We have the following Hartman-Wintner-type inequality.

THEOREM 1. *If a nontrivial continuous solution to the Hadamard fractional boundary value problem (7) exist, then*

$$\int_a^b \frac{1}{s}\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}|q(s)|ds \geqslant \left(\ln\frac{b}{a}\right)^{\alpha-1}\Gamma(\alpha). \tag{33}$$

Proof. Let $E = C([a,b],\mathbb{R})$ be the Banach space endowed with the norm

$$\|u\| = \sup_{t\in[a,b]}|u(t)|.$$

We have

$$|u(t)| \leqslant \int_a^b |G(t,s)||q(s)||u(s)|\,ds,$$

which yields

$$\|u\| \leqslant \|u\|\int_a^b |g_2(s,s)||q(s)|\,ds.$$

Since u is non trivial, then $\|u\| \neq 0$, so

$$1 \leqslant \int_a^b \frac{1}{\left(\ln\frac{b}{a}\right)^{\alpha-1}\Gamma(\alpha)s}\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}|q(s)|\,ds,$$

from which the inequality in (33) follows. □

COROLLARY 1. *If a nontrivial continuous solution to the Hadamard fractional boundary value problem exist, then*

$$\int_a^b \left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}|q(s)|ds \geqslant a\left(\ln\frac{b}{a}\right)^{\alpha-1}\Gamma(\alpha). \tag{34}$$

Proof. From Theorem 1, we have

$$\int_a^b \frac{1}{s}\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}|q(s)|ds \geqslant \left(\ln\frac{b}{a}\right)^{\alpha-1}\Gamma(\alpha).$$

Next we note $\frac{1}{a} \geqslant \frac{1}{s}$, thus we get

$$\int_a^b \left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}|q(s)|ds \geqslant a\left(\ln\frac{b}{a}\right)^{\alpha-1}\Gamma(\alpha). \quad \square \tag{35}$$

We have the following Lyapunov-type inequality.

THEOREM 2. *If a nontrivial continuous solution to the Hadamard fractional boundary value problem (7) exist, then*

$$\int_a^b |q(s)|\,ds \geqslant 4^{(\alpha-1)}\Gamma(\alpha)a\left(\ln\frac{b}{a}\right)^{1-\alpha}. \tag{36}$$

Proof. From the Corollary 1, we have

$$\int_a^b |q(s)|\,ds \geqslant a\left(\ln\frac{b}{a}\right)^{\alpha-1}\frac{\Gamma(\alpha)}{\max\limits_{s\in[a,b]} h(s)}, \tag{37}$$

where

$$h(s) = \left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}. \tag{38}$$

If $s=a$ or $s=b$, then $h(s)=0$. Else if $s\in]a,b[$, we differentiate $h(s)$,

$$\begin{aligned} h'(s) &= \frac{(\alpha-1)}{s\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{2-\alpha}}\left(\ln\frac{b}{s}-\ln\frac{s}{a}\right)\\ &= \frac{(\alpha-1)\left(\ln\frac{ab}{s^2}\right)}{s\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{2-\alpha}}, \end{aligned}$$

we have only one solution $s_0=\sqrt{ab}$ of the equation $h'(s)=0$ on $]a,b[$. We obtain

$$\max_{s\in[a,b]} h(s) = h(s_0) = \left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right)^{\alpha-1}. \tag{39}$$

We have

$$ab=\sqrt{ab}\sqrt{ab}\Leftrightarrow \ln\frac{\sqrt{ab}}{a}=\ln\frac{b}{\sqrt{ab}}\Leftrightarrow\left(\ln\frac{\sqrt{ab}}{a}-\ln\frac{b}{\sqrt{ab}}\right)^2=0$$

$$\begin{aligned} &\Leftrightarrow 4\left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right)=\left(\ln\frac{\sqrt{ab}}{a}\right)^2+\left(\ln\frac{b}{\sqrt{ab}}\right)^2+2\left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right)\\ &\Leftrightarrow\left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right)=\frac{1}{4}\left(\ln\frac{\sqrt{ab}}{a}+\ln\frac{b}{\sqrt{ab}}\right)^2=\frac{1}{4}\left(\ln\frac{b}{a}\right)^2\\ &\Leftrightarrow\left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right)^{\alpha-1}=\frac{1}{4^{(\alpha-1)}}\left(\ln\frac{b}{a}\right)^{2(\alpha-1)}, \end{aligned} \tag{40}$$

by (39) and (40),

$$\max_{s\in[a,b]} h(s) = h(s_0) = \frac{1}{4^{(\alpha-1)}}\left(\ln\frac{b}{a}\right)^{2(\alpha-1)}, \tag{41}$$

substituting (41) into (37), we obtain

$$\int_a^b |q(s)|\,ds \geqslant 4^{(\alpha-1)}\Gamma(\alpha)a\left(\ln\frac{b}{a}\right)^{1-\alpha}.$$

The proof is complete. □

We define the constants:

$$\xi_1 = \exp\left(\frac{1}{2}\left[[2(\alpha-1)+\ln ba]-\sqrt{4(\alpha-1)^2+ln^2\frac{b}{a}}\right]\right), \tag{42}$$

and

$$\xi_2 = \exp\left(\frac{1}{2}\left[[2(\alpha-1)+\ln ba]+\sqrt{4(\alpha-1)^2+ln^2\frac{b}{a}}\right]\right), \tag{43}$$

LEMMA 5. *The function G defined in Lemma 3, satisfies the following property*

$$\max_{t,s\in[a,b]}|G(t,s)| = \frac{1}{\Gamma(\alpha)\xi_1}\left(\frac{\ln\frac{\xi_1}{a}\ln\frac{b}{\xi_1}}{\ln\frac{b}{a}}\right)^{\alpha-1}. \tag{44}$$

Proof. We observe that $g_2([a,b]\times[a,b])\subset G([a,b]\times[a,b])$, and by the first property in the Lemma 4, we get

$$\max_{t,s\in[a,b]}|G(t,s)| = \max_{s\in[a,b]}|g_2(s,s)|, \tag{45}$$

where $g_2(s,s) = -\dfrac{1}{\Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-1}}\dfrac{\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}}{s}$.

It follows that we only need to get the maximum value of the function

$$f(s) = \frac{\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}}{s}. \tag{46}$$

We observe that $f(a)=f(b)=0$. If $s\in]a,b[$, differentiate $f(s)$.

$$f'(s) = \left[(\alpha-1)\frac{\ln\frac{b}{s}-\ln\frac{s}{a}}{\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)}-1\right]\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}\frac{1}{s^2}.$$

we have

$$\begin{aligned}
f'(s)=0 &\Leftrightarrow (\alpha-1)\left(\ln\tfrac{b}{s}-\ln\tfrac{s}{a}\right) = \ln\tfrac{s}{a}\ln\tfrac{b}{s}\\
&\Leftrightarrow [2(\alpha-1)+\ln b+\ln a]\ln s-[(\alpha-1)+\ln b]\ln a-\ln^2 s-(\alpha-1)\ln b=0\\
&\Leftrightarrow \ln^2 s-[2(\alpha-1)+\ln ba]\ln s+[(\alpha-1)\ln ba+\ln b\ln a]=0\\
&\Leftrightarrow x^2-[2(\alpha-1)+\ln ba]x+[(\alpha-1)\ln ba+\ln b\ln a]=0,
\end{aligned}$$

where $x = \ln s$.
We get

$$\begin{cases} x_1 = \frac{[2(\alpha-1)+\ln ba]-\sqrt{\Delta}}{2} = \ln \xi_1, \\ x_2 = \frac{[2(\alpha-1)+\ln ba]+\sqrt{\Delta}}{2} = \ln \xi_2, \end{cases} \tag{47}$$

where $\Delta = 4(\alpha-1)^2 + \ln^2 \frac{b}{a}$.
We have

$$x_2 > \frac{\ln ba + \sqrt{\left(\ln \frac{b}{a}\right)^2}}{2} = \ln b,$$

we obtain $\xi_2 \notin]a,b[$. Also we have

$$\begin{aligned} x_1 &= \frac{1}{2}\left(2(\alpha-1) + \ln ba - \sqrt{\left(2(\alpha-1) + \ln \frac{b}{a}\right)^2 - 4(\alpha-1)\left(\ln \frac{b}{a}\right)}\right) \\ &> \frac{1}{2}\left(2(\alpha-1) + \ln ba - \sqrt{\left(2(\alpha-1) + \ln \frac{b}{a}\right)^2}\right) \\ &= \frac{1}{2}\left(2(\alpha-1) + \ln ba - 2(\alpha-1) - \ln \frac{b}{a}\right) = \ln a \\ &\Rightarrow \xi_1 > a, \end{aligned}$$

and

$$\begin{aligned} x_1 &= \frac{1}{2}\left(2(\alpha-1) + \ln ba - \sqrt{\left(2(\alpha-1) - \ln \frac{b}{a}\right)^2 + 4(\alpha-1)\ln \frac{b}{a}}\right) \\ &< \frac{1}{2}\left(2(\alpha-1) + \ln ba - \sqrt{\left(2(\alpha-1) - \ln \frac{b}{a}\right)^2}\right) \\ &= \frac{1}{2}\left(2(\alpha-1) + \ln ba - \left|2(\alpha-1) - \ln \frac{b}{a}\right|\right) \\ &\leqslant \frac{1}{2}\left(2(\alpha-1) + \ln ba - \left(|2(\alpha-1)| - \left|\ln \frac{b}{a}\right|\right)\right) \\ &= \frac{1}{2}\left(2(\alpha-1) + \ln ba - \left(2(\alpha-1) - \ln \frac{b}{a}\right)\right) = \ln b \\ &\Rightarrow \xi_1 < b, \end{aligned}$$

we obtaint $\xi_1 \in]a,b[$.
Hence

$$\max_{s\in[a,b]} |f(s)| = \frac{1}{\xi_1}\left(\ln \frac{\xi_1}{a} \ln \frac{b}{\xi_1}\right)^{\alpha-1}. \tag{48}$$

Therefore

$$\max_{t,s\in[a,b]} |G(t,s)| = \frac{1}{\Gamma(\alpha)\xi_1}\left(\frac{\ln\frac{\xi_1}{a}\ln\frac{b}{\xi_1}}{\ln\frac{b}{a}}\right)^{\alpha-1}. \tag{49}$$

The proof is complete. □

We have the following Lyapunov-type inequality.

THEOREM 3. *If a nontrivial continuous solution to the HFBVP (7) exist, then*

$$\int_a^b |q(s)|\,ds \geqslant \Gamma(\alpha)\xi_1\left(\frac{\ln\frac{\xi_1}{a}\ln\frac{b}{\xi_1}}{\ln\frac{b}{a}}\right)^{1-\alpha}, \tag{50}$$

where ξ_1 *defined as in (42).*

Proof. By Lemma 3, the solution of the HFBVP can be written as

$$u(t) = \int_a^b G(t,s)\,q(s)u(s)\;ds.$$

Thus for all $t \in [a,b]$, we have

$$\begin{aligned}|u(t)| &\leqslant \int_a^b |G(t,s)|\,|q(s)|\,|u(s)|\;ds\\ &\leqslant \|u\| \int_a^b |G(t,s)|\,|q(s)|\,ds.\end{aligned}$$

which yields

$$\|u\| \leqslant \|u\| \int_a^b |G(t,s)|\,|q(s)|\,ds.$$

Since u is non trivial, then $\|u\| \neq 0$, so

$$1 \leqslant \int_a^b |G(t,s)|\,|q(s)|\,ds.$$

New, an application of Lemma 5, we obtain

$$\int_a^b |q(s)|\,ds \geqslant \Gamma(\alpha)\xi_1\left(\frac{\ln\frac{\xi_1}{a}\ln\frac{b}{\xi_1}}{\ln\frac{b}{a}}\right)^{1-\alpha}.$$

The proof is complete. □

REMARK 1. Let a=1, b=e, from Theorem 3, we can conclude the main result in [2], and we have solved the open problem in [2] using the directly analysis method. As well as there have been an answer to the open problem with different method in the paper [11].

Acknowledgement. The corresponding author's research is partially supported by the NNSF of China (Grant No. 11771102), the key NSF of Guangdong Province of China (Grant No. 2018B030311004) and NSF of Guangdong Province of China (Grant No. 2017A030313397).

REFERENCES

[1] A. M. LYAPUNOV, *Problème générale de la stabilité du mouvement*, Ann. of Math. Stud., Vol. **17**, Princeton University, 1949.

[2] QINGHUA MA, CHAO MA AND JINXUN WANG, *A Lyapunov-type inequality for a fractional differential equation with Hadamard derivative*, Journal of Mathematical Inequalities 11, 1, 2017, 135–141.

[3] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies **204**, Elsevier Science B.V, Amsterdam, 2006.

[4] D. O'REGAN, B. SAMET, *Lyapunov-type inequalities for a class of fractional differential equations*, J. Inequal. Appl., (2015), 2015:249.

[5] A. TIRYAKI, *Recent developments of Lyapunov-type inequalities*, Adv. Dyn. Syst. Appl. 5, 2, 2010, 231–248.

[6] X. YANG, Y. KIM, K. LO, *Lyapunov-type inequality for a class of linear differential systems*, Appl. Math. Comput., **219**, (2012), 1805–1812.

[7] M. JLELI, B. SAMET, *Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions*, Math. Inequal. Appl. 18, 2, 2015, 433–451.

[8] N. ARIFI, I. ALTUN, M. JLELI, A. LASHIN, B. SAMET, *Lyapunov-type inequalities for a fractional p-Lapacian equation*, J. Inequal. Appl., **189**, (2016).

[9] R. A. C. FERREIRA, *On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function*, J. Math. Anal. Appl., **412**, (2014), 185–1063.

[10] R. A. C. FERREIRA, *A Lyapunov-type inequality for a fractional boundary value problem*, Fract. Calc. Appl. Anal. 16, 4, 2013, 978–984.

[11] M. JLELI, M. KIRANE AND B. SAMET, *Hartman-Wintner-Type Inequality for a Fractional Boundary Value Problem via a Fractional Derivative with respect to Another Function*, Discrete Dynamics in Nature and Society, Volume 2017, Article ID 5123240, 8 pages.

(Received August 25, 2018)

Zaid Laadjal
Department of Mathematics and Computer Sciences
ICOSI Laboratory, University of Abbes Laghrour
Khenchela, 40000, Algeria.
e-mail: zaid.laadjal@yahoo.com

Nacer Adjeroud
Department of Mathematics and Computer Sciences
ICOSI Laboratory, University of Abbes Laghrour
Khenchela, 40000, Algeria.
e-mail: adjeroud_n@yahoo.fr

Qinghua Ma
Applied Mathematics Department, School of Mathematics and Statistics
Guangdong University of Foreign Studies
Guangzhou 510420, P. R. China.
e-mail: gdqhma@21cn.com