Journal of Mathematical Inequalities
Volume 13, Number 3 (2019), 879–890 doi:10.7153/jmi-2019-13-61

# APPROXIMATION PROPERTIES OF $(p,q)$–GAMMA OPERATORS

WENTAO CHENG*, WENHUI ZHANG AND AXIU SHU

(*Communicated by V. Gupta*)

*Abstract.* In this paper, we introduce a new analogue of Gamma operators and we call it as $(p,q)$-Gamma operators which is a generalization of $q$-Gamma operators. Moments of these operators is estimated. And some other results of these operators are studied by means of modulus of continuity and Peetre $K$-functional. Then, some theorems concerned with the rate of convergence and the weighted approximation for these operators are also obtained. Finally, a Voronovskaya asymptotic formula is also presented.

## 1. Introduction and definitions

Let $f$ be a function defined on $[0,\infty)$ and satisfy the following growth condition:

$$|f(x)| \leqslant Me^{-\beta x}\ (M > 0; \beta \geqslant 0; x \to \infty). \tag{1}$$

In [18], Zeng investigated and studied some approximation properties of the following sequence of linear positive operators (named Gamma operators)

$$G_n(f;x) = \frac{1}{x^n\Gamma(n)} \int_0^\infty f\left(\frac{t}{n}\right) t^{n-1} e^{-\frac{t}{x}} \mathrm{d}t. \tag{2}$$

During the last decade, the wide application of $q$-calculus in the field of approximation theory has led to the discovery of new generalizations of classical operators. For more comprehensive details, the readers should look through the references material [1], [14], [15]. In [3], Cai introduced and studied $q$-analogue of Gamma operators as follows:

*Mathematics subject classification* (2010): 41A36, 41A25, 41A10.
*Keywords and phrases*: $(p,q)$-Gamma operators, weighted approximation, rate of convergence, modulus of continuity.
This research is supported by the National Natural Science Foundation of China (Grant No. 11626031), the Key Natural Science Research Project in Universities of Anhui Province (Grant No. KJ2019A0572), the Philosophy and Social Sciences General Planning Project of Anhui Province of China (Grant No. AHSKYG2017D153) and the Natural Science Foundation of Anhui Province of China (Grant No. 1908085QA29).
* Corresponding author.

DEFINITION A. *For $f \in C[0,\infty)$ satisfies (1), $q \in (0,1)$ and $n \in \mathbb{N}$, the $q$-Gamma operators $G_{n,q}(f;x)$ is defined by*

$$G_{n,q}(f;x) = \frac{1}{x^n \Gamma_q(n)} \int_0^{\infty/A} f\left(\frac{t}{[n]_q}\right) t^{n-1} E_q\left(-\frac{qt}{x}\right) \mathrm{d}_q t.$$

Stancu generalization of $G_{n,q}(f;x)$ was discussed and studied in [12]. Nowadays, with the rapid development of the post-quantum calculus, which called the $(p,q)$-calculus for short, the generalizations of several classical operators have been studied intensively (Such as [4, 5, 9]) since Mursaleen et al. first defined and constructed $(p,q)$-Bernstein operators[10] and $(p,q)$-Bernstein-Stancu operators[11]. The $(p,q)$-calculus has been used in many other fields, such as Lie group theory, CAGD, physical sciences(see [7], [16]). Let us recall the basic notations of $(p,q)$-calculus which can be found in [15].

For any fixed real number $p > 0$, $q > 0$, the $(p,q)$-integers $[m]_{p,q}$ are defined by

$$[m]_{p,q} = p^{m-1} + p^{m-2}q + p^{m-3}q^2 + \cdots + pq^{m-2} + q^{m-1} = \begin{cases} \frac{p^m - q^m}{p-q}, & p \neq q \neq 1; \\ mp^{m-1}, & p = q \neq 1; \\ [m]_q, & p = 1; \\ m, & p = q = 1, \end{cases}$$

where $[m]_q$ denotes the $q$-integers and $m = 0,1,2,\cdots$. Also $(p,q)$-factorial is defined as follows:

$$[m]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q}\cdots[m]_{p,q}, & m \geqslant 1; \\ 1, & m = 0. \end{cases}$$

Now, we introduce two types of $(p,q)$-analogues of exponential function $e_{p,q}(x)$ and $E_{p,q}(x)$ (see [2]):

$$e_{p,q}(x) = \sum_{m=0}^{\infty} \frac{p^{\frac{m(m-1)}{2}} x^m}{[m]_{p,q}!}, x \in \mathbb{R}, |p| < 1 \text{ and } |q| < 1;$$

$$E_{p,q}(x) = \sum_{m=0}^{\infty} \frac{q^{\frac{m(m-1)}{2}} x^m}{[m]_{p,q}!}, x \in \mathbb{R}, |p| < 1 \text{ and } |q| < 1.$$

Let $f$ be an arbitrary function. The improper $(p,q)$-integral of $f(x)$ on $[0,\infty)$ is defined as (see [13])

$$\int_0^{\infty} f(x) \mathrm{d}_{p,q} x = (p-q) \sum_{i=-\infty}^{\infty} \frac{q^i}{p^{i+1}} f\left(\frac{q^i}{p^{i+1}}\right), 0 < \frac{q}{p} < 1.$$

The $(p,q)$-Gamma function of the second kind was defined in [13] as follows

$$\gamma_{p,q}(z) = \int_0^{\infty} q^{\frac{z(z-1)}{2}} t^{z-1} e_{p,q}(-pt) \mathrm{d}_{p,q} t, \Re(z) > 0.$$

Meantime, the $(p,q)$-Gamma function fulfils the following relation

$$\gamma_{p,q}(z+1) = [z]_{p,q}\gamma_{p,q}(z),$$

moreover, for any nonnegative integer $n > 0$, the following relation holds

$$\gamma_{p,q}(n+1) = [n]_{p,q}!.$$

Now, we construct $(p,q)$-Gamma operators using the $(p,q)$-Gamma function of the second kind preserving linear functions as:

DEFINITION 1. For $f \in C[0,\infty)$ satisfies (1), $0 < q < p \leqslant 1$ and $n \in \mathbb{N}$, the $(p,q)$-analogue of Gamma operators (2) are defined as

$$G_n^{p,q}(f;x) = \frac{1}{x^n \gamma_{p,q}(n)} \int_0^\infty f\left(\frac{q^n t}{[n]_{p,q}}\right) q^{\frac{n(n-1)}{2}} t^{n-1} e_{p,q}\left(-\frac{pt}{x}\right) \mathrm{d}_{p,q}t.$$

This paper is organized as follows: In Section 1, we give some basic notations and the definition of $(p,q)$-Gamma operators. In Section 2, we present basic lemmas and estimate the moments of the operators. In Section 3, we present a direct result of $(p,q)$-Gamma operators in terms of first and second order modulus of continuity. In Section 4, we deal with the rate of convergence. In Section 5, we study the weighted approximation of the $(p,q)$-Gamma operators. In Section 6, we obtain Voronovskaja type asymptotic formula.

## 2. Auxiliary results

In this section, in order to prove our main results, we first establish some useful lemmas.

LEMMA 1. *Let $0 < q < p \leqslant 1$, $x \in [0,\infty)$ and $k = 0,1,\cdots$, we have*

$$G_n^{p,q}(t^k;x) = \frac{x^k q^{-\frac{k(k-1)}{2}} [n+k-1]_{p,q}!}{[n]_{p,q}^k [n-1]_{p,q}!}.$$

*Proof.* Direct computation gives

$$\begin{aligned}
G_n^{p,q}(t^k;x) &= \frac{1}{x^n \gamma_{p,q}(n)} \int_0^\infty \left(\frac{q^n t}{[n]_{p,q}}\right)^k q^{\frac{n(n-1)}{2}} t^{n-1} e_{p,q}\left(-\frac{pt}{x}\right) \mathrm{d}_{p,q}t \\
&= \frac{x^k}{[n]_{p,q}^k \gamma_{p,q}(n)} \int_0^\infty q^{\frac{n(n-1)}{2}+nk} \left(\frac{t}{x}\right)^{n+k-1} e_{p,q}\left(-\frac{pt}{x}\right) \mathrm{d}_{p,q}\left(\frac{t}{x}\right) \\
&= \frac{x^k q^{-\frac{k(k-1)}{2}}}{[n]_{p,q}^k \gamma_{p,q}(n)} \int_0^\infty q^{\frac{(n+k)(n+k-1)}{2}} \left(\frac{t}{x}\right)^{n+k-1} e_{p,q}\left(-\frac{pt}{x}\right) \mathrm{d}_{p,q}\left(\frac{t}{x}\right) \\
&= \frac{x^k q^{-\frac{k(k-1)}{2}} [n+k-1]_{p,q}!}{[n]_{p,q}^k [n-1]_{p,q}!}.
\end{aligned}$$

Lemma 1 is proved. $\square$

LEMMA 2. *Let* $0<q<p\leqslant 1$, $x\in[0,\infty)$, *we have*

$$G_n^{p,q}(1;x)=1,\quad G_n^{p,q}(t;x)=x,\quad G_n^{p,q}(t^2;x)=\left(1+\frac{p^n}{q[n]_{p,q}}\right)x^2, \tag{3}$$

$$G_n^{p,q}(t^3;x)=\left(1+\frac{p^n\left([2]_{p,q}+q\right)}{q^2[n]_{p,q}}+\frac{p^{2n}}{q^3[n]_{p,q}^3}\right)x^3,$$

$$G_n^{p,q}(t^4;x)=\left(1+\frac{\left(q^2+q[2]_{p,q}+[3]_{p,q}\right)p^n}{q^3[n]_{p,q}}+\frac{\left([2]_{p,q}[3]_{p,q}+q[3]_{p,q}+q^2[2]_{p,q}\right)p^{2n}}{q^5[n]_{p,q}^2}\right.$$
$$\left.+\frac{[2]_{p,q}[3]_{p,q}p^{3n}}{q^6[n]_{p,q}^3}\right)x^4.$$

*Proof.* From Lemma 1, we get the first and second equalities of (3) easily. Using $[n+m]_{p,q}=q^m[n]_{p,q}+p^n[m]_{p,q}$, we have

$$G_n^{p,q}(t^2;x)=\frac{[n+1]_{p,q}}{q[n]_{p,q}}x^2=\left(1+\frac{p^n}{q[n]_{p,q}}\right)x^2.$$

Next,

$$G_n^{p,q}(t^3;x)=\frac{[n+1]_{p,q}[n+2]_{p,q}}{q^3[n]_{p,q}^2}x^3=\frac{(q[n]_{p,q}+p^n)(q^2[n]_{p,q}+p^n[2]_{p,q})}{q^3[n]_{p,q}^2}x^3$$
$$=\left(1+\frac{p^n\left([2]_{p,q}+q\right)}{q^2[n]_{p,q}}+\frac{p^{2n}}{q^3[n]_{p,q}^3}\right)x^3.$$

Finally,

$$G_n^{p,q}(t^4;x)=\frac{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}}{q^6[n]_{p,q}^3}x^4$$
$$=\frac{(q[n]_{p,q}+p^n)(q^2[n]_{p,q}+p^n[2]_{p,q})(q^3[n]_{p,q}+p^n[3]_{p,q})}{q^6[n]_{p,q}^3}x^4$$
$$=\left(1+\frac{\left(q^2+q[2]_{p,q}+[3]_{p,q}\right)p^n}{q^3[n]_{p,q}}+\frac{\left([2]_{p,q}[3]_{p,q}+q[3]_{p,q}+q^2[2]_{p,q}\right)p^{2n}}{q^5[n]_{p,q}^2}\right.$$
$$\left.+\frac{[2]_{p,q}[3]_{p,q}p^{3n}}{q^6[n]_{p,q}^3}\right)x^4.$$

Lemma 2 is proved. □

LEMMA 3. *Let* $0<q<p\leqslant 1$, $x\in[0,\infty)$, *we have*

$$G_n^{p,q}(t-x;x)=0;$$

$$A(x):=G_n^{p,q}\left((t-x)^2;x\right)=\frac{p^n}{q[n]_{p,q}}x^2; \tag{4}$$

$$G_n^{p,q}\left((t-x)^4;x\right)=\frac{p^n(p-q)^2}{q^3[n]_{p,q}}x^4+\frac{\left([2]_{p,q}[3]_{p,q}+q[3]_{p,q}+q^2[2]_{p,q}-4q^2\right)p^{2n}}{q^5[n]_{p,q}^2}x^4+\frac{[2]_{p,q}[3]_{p,q}p^{3n}}{q^6[n]_{p,q}^3}x^4.$$

*Proof.* Because $G_n^{p,q}(t-x;x)=G_n^{p,q}(t;x)-x$, $G_n^{p,q}((t-x)^2;x)=G_n^{p,q}(t^2;x)-2G_n^{p,q}(t;x)+x^2$ and $G_n^{p,q}((t-x)^4;x)=G_n^{p,q}(t^4;x)-4xG_n^{p,q}(t^3;x)+6x^2G_n^{p,q}(t^2;x)-4x^3G_n^{p,q}(t;x)+x^4$, and from Lemma 2, we obtain Lemma 3 easily. □

REMARK 1. The sequences $(p_n)$, $(q_n)$ satisfying $0<q_n<p_n<1$ such that $p_n\to 1$, $q_n\to 1$ and $p_n^n\to a$, $q_n^n\to b$, $[n]_{p_n,q_n}\to\infty$ as $n\to\infty$ where $0\leqslant a,b<1$, then:

1. $\lim_{n\to\infty}[n]_{p_n,q_n}G_n^{p_n,q_n}((t-x)^2;x)=ax^2$;
2. $\lim_{n\to\infty}[n]_{p_n,q_n}G_n^{p_n,q_n}((t-x)^4;x)=0$.

## 3. Local approximation

Let $C_B[0,\infty)$ be the space of all real-valued continuous bounded functions $f$ on $[0,\infty)$, endowed with the norm $\|f\|=\sup_{x\in[0,\infty)}|f(x)|$. The first-order and second-order modulus of continuities, the Peetre's $K$-functional of the function $f\in C_B[0,\infty)$ are defined by for $\delta>0$

$$\omega(f;\delta)=\sup_{0<t\leqslant\delta}\sup_{x\in[0,\infty}|f(x+t)-f(x)|,$$
$$\omega_2(f;\delta)=\sup_{0<t\leqslant\delta}\sup_{x\in[0,\infty}|f(x+2t)-2f(x+h)+f(x)|,$$
$$K_2(f;\delta)=\inf_{g\in C_B^2[0,\infty)}\{\|f-g\|+\delta\|g''\|\},$$

where $C_B^2[0,\infty):=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$. By [6], there exists an absolute constant $M>0$ such that

$$K_2(f;\delta)\leqslant M\omega_2(f;\sqrt{\delta}). \tag{5}$$

LEMMA 4. *For* $f\in C_B[0,\infty)$, *we have*

$$|G_n^{p,q}(f;x)|\leqslant\|f\|.$$

*Proof.* The proof of this lemma is obvious. □

THEOREM 1. *Let $f \in C_B[0,+\infty)$, $0<q<p\leqslant 1$, then for every $x\in[0,\infty)$ and $n>2$ we have*

$$|G_n^{p,q}(f;x)-f(x)|\leqslant C\omega_2\left(f,\sqrt{A(x)}\right),$$

*where $C$ is some positive constant and $A(x)$ is given by equation (4).*

*Proof.* For all $g\in C_B^2[0,\infty)$, using the Taylor's expansion for $x\in[0,\infty)$, we have

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t (t-u)g''(u)\mathrm{d}u.$$

Applying the operators $G_n^{p,q}$ to both sides of the equality above and using Lemma 3, we get

$$|G_n^{p,q}(g;x)-g(x)|=\left|G_n^{p,q}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right)\right|\leqslant G_n^{p,q}\left(\left|\int_x^t (t-u)g''(u)\mathrm{d}u\right|;x\right)$$
$$\leqslant G_n^{p,q}\left(\|g''\|(t-x)^2;x\right)\leqslant A(x)\|g''\|.$$

By Lemma 4 , we have

$$|G_n^{p,q}(f;x)-f(x)|\leqslant|G_n^{p,q}(f-g;x)-(f-g)(x)|+|G_n^{p,q}(g;x)-g(x)|$$
$$\leqslant 2\|f-g\|+A(x)\|g''\|.$$

Lastly, taking infimum on both side of the equality above over all $g\in C_B^2[0,\infty)$

$$|G_n^{p,q}(f;x)-f(x)|\leqslant 2K_2\left(f;A(x)\right)$$

for which we have the desired result by (5). This completes the proof of Theorem 1. □

THEOREM 2. *Let $0<\gamma\leqslant 1$ and $E$ be any bounded subset of the interval $[0,\infty)$. If $f\in C_B[0,\infty)$ is locally in $\mathrm{Lip}(\gamma)$,* i.e., *the condition*

$$|f(x)-f(t)|\leqslant L|x-t|^{\gamma}, t\in E \quad \text{and} \quad x\in[0,\infty)$$

*holds, then, for each $x\in[0,\infty)$, we have*

$$|G_n^{p,q}(f;x)-f(x)|\leqslant L\left\{(A(x))^{\frac{\gamma}{2}}+2(\mathrm{d}(x;E)^{\gamma})\right\},$$

*where $L$ is a constant depending on $\gamma$ and $f$; and $\mathrm{d}(x;E)$ which is the distance between $x$ and $E$ is defined by*

$$\mathrm{d}(x;E)=\inf\{|t-x|:t\in E\}.$$

*Proof.* From the properties of infinum, at least an point $t_0$ exists in the closure of $E$, that is $t_0\in E$, such that

$$\mathrm{d}(x;E)=|t_0-x|.$$

Using the triangle inequality, we have

$$\begin{aligned}|G_n^{p,q}(f;x)-f(x)| &\leqslant G_n^{p,q}(|f(t)-f(x)|;x)\\ &\leqslant G_n^{p,q}(|f(t)-f(t_0)|;x)+G_n^{p,q}(|f(t_0)-f(x)|;x)\\ &\leqslant L\left(G_n^{p,q}(|t-t_0|^{\gamma};x)+G_n^{p,q}(|t_0-x|^{\gamma};x)\right)\\ &\leqslant L\left(G_n^{p,q}(|t-x|^{\gamma};x)+2|t_0-x|^{\gamma}\right).\end{aligned}$$

Choosing $a_1=\dfrac{2}{\gamma}$ and $a_2=\dfrac{2}{2-\gamma}$ and using the well-known Hölder inequality

$$\begin{aligned}|G_n^{p,q}(f;x)-f(x)| &\leqslant L\left\{\left(G_n^{p,q}(|t-x|^{\gamma a_1};x)\right)^{\frac{1}{a_1}}\left(G_n^{p,q}(1^{a_a};x)\right)^{\frac{1}{a_2}}+2|t_0-x|^{\gamma}\right\}\\ &\leqslant L\left\{\left(G_n^{p,q}((t-x)^2;x)\right)^{\frac{\gamma}{2}}+2|t_0-x|^{\gamma}\right\}\leqslant L\left\{A^{\frac{\gamma}{2}}(x)+2(\mathrm{d}(x;E))^{\gamma}\right\}.\end{aligned}$$

This completes the proof. □

## 4. Rate of convergence

Let $B_{x^2}[0,\infty)$ be the set of all functions $f$ defined on $(0,\infty)$ satisfying the condition $|f(x)|\leqslant M_f(1+x^2)$, where $M_f>0$ is a constant depending only on $f$. Let $C_{x^2}[0,\infty)$ denote the subset of all continuous functions in $B_{x^2}[0,\infty)$. Let $C^*_{x^2}[0,\infty)$ be the subset of all functions $f\in C_{x^2}[0,\infty)$ with the norm $\|f\|_{x^2}=\sup\limits_{x\in[0,\infty)}\dfrac{|f(x)|}{1+x^2}$ and $C^*_{x^2}[0,\infty)=\left\{f\in C_{x^2}[0,\infty):\lim\limits_{x\to\infty}\dfrac{|f(x)|}{1+x^2}<\infty\right\}$. Meantime, we denote the modulus of continuity on $f$ on the closed interval $[0,a]$, $a>0$ by

$$\omega_a(f,\delta)=\sup_{|t-x|\leqslant\delta}\ \sup_{x,t\in[0,a]}|f(t)-f(x)|.$$

Obviously, for the function $f\in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero as $\delta\to 0^+$. Now we give the theorem about the rate of convergence theorem for the operators $G_n^{p,q}(f;x)$.

THEOREM 3. *Let $f\in C_{x^2}[0,\infty)$, $0<q<p\leqslant 1$ and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1]\subset[0,\infty)$, where $a>0$. Then, for every $n>2$,*

$$\|G_n^{p,q}(f;x)-f(x)\|_{C[0,a]}\leqslant 4M_f(1+a^2)A(a)+2\omega_{a+1}\left(f,\sqrt{A(a)}\right).$$

*Proof.* For all $x\in[0,a]$ and $t>a+1$, we easily have $(t-x)^2\geqslant(t-a)^2\geqslant 1$, therefore,

$$\begin{aligned}|f(t)-f(x)| &\leqslant |f(t)|+|f(x)|\leqslant M_f(2+x^2+t^2)=M_f\left(2+x^2+(x-t-x)^2\right)\\ &\leqslant M_f\left(2+3x^2+2(x-t)^2\right)\leqslant M_f(4+3x^2)(t-x)^2\leqslant 4M_f(1+a^2)(t-x)^2,\end{aligned}\tag{6}$$

and for all $x \in [0,a]$, $t \in [0,a+1]$ and $\delta > 0$, we have

$$|f(t)-f(x)| \leqslant \omega_{a+1}(f,|t-x|) \leqslant \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta). \tag{7}$$

From (6) and (7) , we get

$$|f(t)-f(x)| \leqslant 4M_f(1+a^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta).$$

By Schwarz's inequality and Lemma 3, we have

$$\begin{aligned}
|G_n^{p,q}(f;x)-f(x)| \leqslant & G_n^{p,q}(|f(t)-f(x)|;x) \\
\leqslant & 4M_f(1+a^2)G_n^{p,q}((t-x)^2;x) + G_n^{p,q}\left(\left(1+\frac{|t-x|}{\delta}\right);x\right)\omega_{a+1}(f,\delta) \\
\leqslant & 4M_f(1+a^2)G_n^{p,q}((t-x)^2;x) + \omega_{a+1}(f,\delta) \\
& \times\left(1+\frac{1}{\delta}\sqrt{G_n^{p,q}((t-x)^2;x)}\right) \\
\leqslant & 4M_f(1+a^2)A(x) + \omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{A(x)}\right) \\
\leqslant & 4M_f(1+a^2)A(a) + \omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{A(a)}\right).
\end{aligned}$$

By taking $\delta = \sqrt{A(a)}$, we get the proof of Theorem 3. □

As is known, if $f$ is not uniformly continuous on the interval $(0,\infty)$, the usual first modulus of continuity $\omega(f;\delta)$ does not tend to zero as $\delta \to 0$. For every $f \in C_{x^2}^0[0,\infty)$, we would like to take a weighted modulus of continuity $\Omega(f;\delta)$ which tends to zero as $\delta \to 0$.

Let

$$\Omega(f;\delta) = \sup_{0<h\leqslant\delta, x\geqslant 0} \frac{|f(x+h)-f(x)|}{1+(x+h)^2}, \quad \text{for every } f \in C_{x^2}^0[0,\infty).$$

The weighted modulus of continuity $\Omega(f;\delta)$ was defined by Yuksel and Ispir in [17]. It is known that $\Omega(f;\delta)$ has the following properties:

(i) $\Omega(f;\delta)$ is a monotone increasing function of $\delta$;

(ii) For each $f \in C_{x^2}^0[0,\infty)$, $\lim\limits_{\delta\to 0^+}\Omega(f;\delta) = 0$;

(iii) For each $m \in \mathbb{N}$, $\Omega(f;m\delta) \leqslant m\Omega(f;\delta)$;

(iv) For each $\lambda \in \mathbb{R}^+$, $\Omega(f;\lambda\delta) \leqslant (1+\lambda)\Omega(f;\delta)$.

THEOREM 4. *Let $f \in C_{x^2}^0[0,\infty)$ and the sequences $(p_n)$, $(q_n)$ satisfying $0 < q_n < p_n < 1$ such that $p_n \to 1$, $q_n \to 1$, $[n]_{p_n,q_n} \to \infty$ as $n \to \infty$, then there exists a positive integer $N \in \mathbb{N}_+$ such that for all $n > N$, the inequality*

$$\sup_{x\in[0,\infty)} \frac{|G_n^{p_n,q_n}(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}} \leqslant 10\Omega\left(f;\frac{1}{\sqrt{[n]_{p_n,q_n}}}\right)$$

*holds.*

*Proof.* For $t>0$, $x\in(0,\infty)$ and $\delta>0$, by the definition of $\Omega(f;\delta)$ and the above property (iv), we get

$$\begin{aligned}|f(t)-f(x)| &\leqslant (1+(x+|x-t|))^2\,\Omega(f;|t-x|)\\ &\leqslant 2(1+x^2)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta).\end{aligned}$$

Using $\lim_{n\to\infty} p_n=\lim_{n\to\infty} q_n=1$, $\lim_{n\to\infty}[n]_{p_n,q_n}=\infty$ and Lemma 3, there exists a positive integer $N\in\mathbb{N}_+$ such that for all $n>N$,

$$G_n^{p_n,q_n}((t-x)^2;x)\leqslant \frac{2(1+x^2)}{[n]_{p_n,q_n}}, \tag{8}$$

$$G_n^{p_n,q_n}((t-x)^4;x)\leqslant x^4. \tag{9}$$

Since $G_n^{p_n,q_n}$ is linear and positive, we have

$$\begin{aligned}|G_n^{p_n,q_n}(f;x)-f(x)|\leqslant 2(1+x^2)\Omega(f;\delta)\Big\{&1+G_n^{p_n,q_n}\left((t-x)^2;x\right)\\ &+G_n^{p_n,q_n}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right)\Big\}.\end{aligned} \tag{10}$$

To estimate the second term of (10), applying the Cauchy-Schwartz inequality and $(x+y)^2\leqslant 2(x^2+y^2)$, we have

$$G_n^{p_n,q_n}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right)\leqslant\sqrt{2}\left(G_n^{p_n,q_n}\left(1+(t-x)^4;x\right)\right)^{\frac12}\left(G_n^{p_n,q_n}\left(\frac{(t-x)^2}{\delta^2};x\right)\right)^{\frac12}.$$

By (8) and (9),

$$G_n^{p_n,q_n}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right)\leqslant\frac{2(1+x^2)^{\frac32}}{\delta[n]_{p_n,q_n}}.$$

Taking $\delta=\frac{1}{\sqrt{[n]_{p_n,q_n}}}$, we can obtain

$$|G_n^{p_n,q_n}(f;x)-f(x)|\leqslant 10(1+x^2)^{\frac52}\,\Omega\left(f;\frac{1}{\sqrt{[n]_{p_n,q_n}}}\right).$$

The proof is completed. □

## 5. Weighted approximation

Now, we obtain the weighted approximation theorem as follows:

THEOREM 5. *Let the sequences $(p_n)$, $(q_n)$ satisfying $0<q_n<p_n\leqslant 1$ such that $p_n\to 1$, $q_n\to 1$, $[n]_{p_n,q_n}\to\infty$ as $n\to\infty$. Then for $f\in C^0_{x^2}[0,\infty)$, we have*

$$\lim_{n\to\infty}\|G_n^{p_n,q_n}(f)-f\|_{x^2}=0.$$

*Proof.* Using Korovkin's theorem (see[8]), it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty} \|G_n^{p_n,q_n}(t^k)-x^k\|_{x^2}=0, k=0,1,2. \tag{11}$$

Since $G_n^{p_n,q_n}(1;x)=1$, $G_n^{p_n,q_n}(t;x)=x$, (11) holds for $k=0,1$.

By (3), we have,

$$\begin{aligned}\left\|G_n^{p_n,q_n}(t^2;x)-x^2\right\|_{x^2} &= \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|G_n^{p_n,q_n}(t^2;x)-x^2\right| = \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2}\left|\frac{p_n^n}{q_n[n]_{p_n,q_n}}\right| \\ &= \frac{p_n^n}{q_n[n]_{p_n,q_n}} = 0, n\to\infty,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\left\|G_n^{p_n,q_n}(t^2;x)-x^2\right\|_{x^2}=0.$$

Thus the proof is completed. □

Now, we present a weighted approximation theorem for function in $C_{x^2}[0,\infty)$.

THEOREM 6. *Let the sequences $(p_n)$, $(q_n)$ satisfying $0<q_n<p_n\leqslant 1$ such that $p_n\to 1$, $q_n\to 1$, $[n]_{p_n,q_n}\to\infty$ as $n\to\infty$. For every $f\in C_{x^2}[0,\infty)$ and $\alpha>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|G_n^{p_n,q_n}(f;x)-f(x)|}{(1+x^2)^{1+\alpha}}=0.$$

*Proof.* Let $x_0\in[0,\infty)$ be arbitrary but fixed. Then

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|G_n^{p_n,q_n}(f;x)-f(x)|}{(1+x^2)^{1+\alpha}} &\leqslant \sup_{x\in[0,x_0]}\frac{|G_n^{p_n,q_n}(f;x)-f(x)|}{(1+x^2)^{1+\alpha}}+\sup_{x\in(x_0,\infty)}\frac{|G_n^{p_n,q_n}(f;x)-f(x)|}{(1+x^2)^{1+\alpha}} \\ &\leqslant \|G_n^{p_n,q_n}(f;x)-f(x)\|_{C[0,x_0]}+M_f\sup_{x\in(x_0,\infty)}\frac{|G_n^{p_n,q_n}((1+t^2);x)|}{(1+x^2)^{1+\alpha}} \\ &\quad+\sup_{x\in(x_0,\infty)}\frac{|f(x)|}{(1+x^2)^{1+\alpha}}.\end{aligned} \tag{12}$$

Since $|f(x)|\leqslant M_f(1+x^2)$, we have $\sup_{x\in(x_0,\infty)}\frac{|f(x)|}{(1+x^2)^{1+\alpha}}\leqslant\frac{M_f}{(1+x_0^2)^{\alpha}}$. Let $\varepsilon>0$ be arbitrary. We can choose $x_0$ to be so large that

$$\frac{M_f}{(1+x_0^2)^{\alpha}}<\varepsilon. \tag{13}$$

In view of Lemma 2, while $x\in(x_0,\infty)$, we obtain

$$M_f\lim_{n\to\infty}\frac{|G_n^{p_n,q_n}((1+t^2);x)|}{(1+x^2)^{1+\alpha}}=M_f\frac{(1+x^2)}{(1+x^2)^{1+\alpha}}=\frac{M_f}{(1+x^2)^{\alpha}}\leqslant\frac{M_f}{(1+x_0^2)^{\alpha}}<\varepsilon.$$

Using Theorem 3, we can see that the first term of the inequality (12), implies that

$$\|G_n^{p_n,q_n}(f;x)-f(x)\|_{C[0,x_0]}<\varepsilon,\ \text{as}\ n\to\infty. \tag{14}$$

Combining (12)-(14), we get the desired result. □

## 6. Voronovskaja type theorem

In this section, we give a Voronovskaja-type asymptotic formula for $G_n^{p_n,q_n}(f;x)$ by means of the second and fourth central moments.

THEOREM 7. *The sequences* $(p_n)$, $(q_n)$ *satisfying* $0<q_n<p_n\leqslant 1$ *such that* $p_n\to 1$, $q_n\to 1$ *and* $p_n^n\to a$, $q_n^n\to b$, $[n]_{p_n,q_n}\to\infty$ *as* $n\to\infty$ *where* $0\leqslant a,b<1$. *For* $f\in C_B^2[0,\infty)$, *the following equality holds*

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(G_n^{p_n,q_n}(f;x)-f(x)\right)=\frac{a}{2}f''(x)x^2,$$

*for every* $x\in[0,A]$, $A>0$.

*Proof.* Let $x\in[0,\infty)$ be fixed. In order to prove this identity, we use Taylor's expansion

$$f(t)-f(x)=(t-x)f'(x)+(t-x)^2\left(\frac{f''(x)}{2}+\theta(t,x)\right),$$

where $\theta(t,x)$ is bounded and $\lim_{t\to x}\theta(t,x)=0$. By applying the operator $G_{n,q_n}(f;x)$ to the above relation, we obtain

$$\begin{aligned}G_n^{p_n,q_n}(f;x)-f(x)=&f'(x)G_n^{p_n,q_n}\left((t-x);x\right)+\frac{1}{2}f''(x)G_n^{p_n,q_n}\left((t-x)^2;x\right)\\&+G_n^{p_n,q_n}\left(\theta(t,x)(t-x)^2;x\right)\\=&\frac{1}{2}f''(x)G_n^{p_n,q_n}\left((t-x)^2;x\right)+G_n^{p_n,q_n}\left(\theta(t,x)(t-x)^2;x\right).\end{aligned}$$

Since $\lim_{t\to x}\theta(t,x)=0$, then for all $\varepsilon>0$, there exists $\delta>0$ such that $|t-x|<\delta$ implies $|\theta(t,x)|<\varepsilon$ for all fixed $x\in[0,\infty)$ where $n$ is large enough. While if $|t-x|\geqslant\delta$, then $|\theta(t,x)|\leqslant\frac{M}{\delta^2}(t-x)^2$, where $M>0$ is a constant. Using Remark 1, we have

$$\lim_{n\to\infty}[n]_{p_n,q_n}G_n^{p_n,q_n}((t-x)^2;x)=ax^2$$

and

$$\begin{aligned}[n]_{p_n,q_n}\left|G_n^{p_n,q_n}\left(\theta(t,x)(t-x)^2;x\right)\right|\leqslant&\varepsilon[n]_{p_n,q_n}G_n^{p_n,q_n}\left((t-x)^2;x\right)\\&+\frac{M}{\delta^2}[n]_{p_n,q_n}G_n^{p_n,q_n}\left((t-x)^4;x\right)\to 0\ (n\to\infty).\end{aligned}$$

The proof is completed. □

## REFERENCES

[1] ARAL A., GUPTA V., AGARWAL R. P., *Application of $q$-Calculus in Operator Theory*, Springer, Berlin,(2013).
[2] ARAL A., GUPTA V., *Applications of $(p,q)$-Gamma functions to Szász-Durrmeyer operators*, Publ. Inst. Math. **102**, 116(2017), 211–220.
[3] Q. B. CAI, *Properties of Convergence for a Class of Generalized $q$-Gamma Operators*, J. Math. Study. **47**, 4(2014), 388–395.
[4] W. T. CHENG, W. H. ZHANG, *On $(p,q)$-Analogue of Gamma operators*, J. Funct. Space., (2019)[Article ID 9607517].
[5] W. T. CHENG, W. H. ZHANG, Q. B. CAI, *$(p,q)$-gamma operators which preserve $x^2$*, J. Inequal. Appl., **108** (2019).
[6] R. A. DEVORE, G. G. LORENTZ, *Constructive Approximation*, Springer, Berlin,(1993).
[7] BURBAN I., *Two-parameter deformation of the oscillator algebra and $(p,q)$-analog of two-dimensional conformal field theory*, J. Nonlinear Math. Phys., **2**, 3-4(1995), 384–391.
[8] P. P. KOROVKIN, *Linear Operators and Approximation Theory*, Hindustan Publ. Corp., India, 1960.
[9] S. A. MOHIUDDINE, A. ALOTAIBI, T. ACAR, *Durrmeyer type (p,q)-Baskakov operators preserving linear functions*, J. Math. Inequal., **12**, 4(2018), 961–973.
[10] M. MURSALEEN, F. KHAN, A. KHAN, *On $(p,q)$-analogue of Bernstein operators*, Appl. Math. Comput., **266**, (2015), 874–882.
[11] M. MURSALEEN, F. KHAN, A. KHAN, *Some approximation results by $(p,q)$-analogue of Bernstein-Stancu operators*, Appl. Math. Comput., **264**, (2015), 392–402.
[12] D. ÖZGE, M. ÖRKCÜ, *On convergence properties of Gamma-Stancu operators based on $q$-integers*, Math. Inequal. Appl., **19**, 4(2016), 1335–1348.
[13] P. N. SADJANG, *On two $(p,q)$-analogues of the Laplace transform*, J. Differ. Equ. Appl., **23**, 9(2017), 1562–1583.
[14] GUPTA V., AGARWAL R. P., *Convergence Estimates in Approximation Theory*, Springer, New York,(2014).
[15] GUPTA V., RASSIAS T. M., AGRAWAL P. N., ACU A. M., *Recent Advances in Constructive Approximation Theory*, Springer, New York,(2018).
[16] SAHAI V., YADAV S., *Representations of two parameter quantum algebras and $p,q$-special functions*, J. Math. Anal. Appl., **335**, (2007),268–279.
[17] I. YUKSEL, N. ISPIR, *Weighted approximation by a certain family of summation integral-type operators*, Comput. Math. Appl, **52**, 10-11,(2006), 1463–1470.
[18] X. M. ZENG, *Approximation properties of Gamma operators*, J. Math. Anal. Appl. **311**, (2005), 389–401.

(Received November 17, 2018)

*Wentao Cheng*
*School of Mathematics and Computation Sciences*
*Anqing Normal University*
*Anhui 246133 P.R. China*
*e-mail:* chengwentao_0517@163.com

*Wenhui Zhang*
*Qingdao hengxing university of science and technology*
*Shangdong 266041 P.R. China*
*e-mail:* zhwnhui@163.com

*Axiu Shu*
*School of Mathematics and Computation Sciences*
*Anqing Normal University*
*Anhui 246133 P.R. China*
*e-mail:* saxshu2004@163.com