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EXTENSIONS OF HLAWKA'S INEQUALITY FOR FOUR VECTORS

MARIUS MUNTEANU

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Abstract. Given four real numbers a_1, a_2, a_3, a_4 we find necessary and sufficient conditions for the inequality

$$a_1\sum_{i=1}^{4}\|x_i\| + a_2\sum_{1\leqslant i<j\leqslant 4}\|x_i+x_j\| + a_3\sum_{1\leqslant i<j<k\leqslant 4}\|x_i+x_j+x_k\| + a_4\left\|\sum_{i=1}^{4}x_i\right\| \geqslant 0$$

to be satisfied for all x_1, x_2, x_3, x_4 in an inner product space, thus providing an extension of Hlawka's inequality for four vectors. As a consequence, we show that

$$\sum_{1\leqslant i\leqslant 4}\|x_i\| - \sum_{1\leqslant i<j\leqslant 4}\|x_i+x_j\| + \sum_{1\leqslant i<j<k\leqslant 4}\|x_i+x_j+x_k\| + \left\|\sum_{i=1}^{4}x_i\right\| \geqslant 0$$

and determine when equality occurs.

1. Introduction

Let E be an inner product space with a real or complex inner product $\langle\cdot,\cdot\rangle$ inducing the norm $\|\cdot\|$. Hlawka's inequality ([6]) asserts that for any $a, b, c \in E$ we have

$$\|a\| + \|b\| + \|c\| - (\|a+b\| + \|a+c\| + \|b+c\|) + \|a+b+c\| \geqslant 0.$$

Equality holds if and only if we have one of the following cases:

(i) $a = \alpha u, b = \beta u, c = \gamma u$ for some $u \in E$ and $\alpha, \beta, \gamma \geqslant 0$ or

(ii) $a = \alpha u, b = \beta u, c = \gamma u$ for some $u \in E$, two of the scalars α, β, γ are positive, and $\alpha + \beta + \gamma \leqslant 0$ or

(iii) $a + b + c = 0$.

Hlawka's proof of this result appears in [8, p.171]. Other proofs were obtained by several authors and can be found in [10] or [11]. The inequality above has been

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generalized in several directions. For example, D. Adamović showed in [1] that for any $x_i \in E, 1 \leqslant i \leqslant n$,

$$(n-2)\sum_{i=1}^{n}\|x_i\| - \sum_{1\leqslant i<j\leqslant n}\|x_i+x_j\| + \left\|\sum_{i=1}^{n}x_i\right\| \geqslant 0.$$

A further extension was obtained by S. D. Djoković in [3]:

$$\binom{n-2}{k-2}\frac{n-k}{k-1}\sum_{i=1}^{n}\|x_i\| - \sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\|x_{i_1}+x_{i_2}+\cdots+x_{i_k}\| + \binom{n-2}{k-2}\left\|\sum_{i=1}^{n}x_i\right\| \geqslant 0,$$

for all $n \geqslant 3$ and all $k, 2 \leqslant k \leqslant n-1$. For additional generalizations and a thorough survey of results connected with Hlawka's inequality, the interested reader is referred to W. Fechner's paper [4].

While significantly more general than the original inequality of Hlawka, Djoković's inequality involves only sums of the form $\sum_{I\subset\{1,2,\ldots,n\}}\|\sum_{i\in I}x_i\|$, for subsets I with $|I| = 1, k$, and n. In this paper, while we limit our investigations to the case when $n = 4$, we obtain inequalities involving sums of the aforementioned form but with the cardinality of I taking *all* values from 1 to 4.

One impediment in extending Hlawka's inequality in this direction is that if we consider

$$\sum_{1\leqslant i\leqslant 4}\|x_i\| - \sum_{1\leqslant i<j\leqslant 4}\|x_i+x_j\| + \sum_{1\leqslant i<j<k\leqslant 4}\|x_i+x_j+x_k\| - \|x_1+x_2+x_3+x_4\|, \quad (1.1)$$

one might naively expect this quantity to be nonnegative. However, as observed in [3], if we choose $x_1 = x_2 = x_3 \neq 0, x_4 = -2x_1$, then the expression above is easily seen to be negative.

Our approach will be to consider real numbers a_1, a_2, a_3, a_4 in order to find necessary and sufficient (linear) conditions involving these numbers such that

$$a_1\sum_{1\leqslant i\leqslant 4}\|x_i\| + a_2\sum_{1\leqslant i<j\leqslant 4}\|x_i+x_j\| + a_3\sum_{1\leqslant i<j<k\leqslant 4}\|x_i+x_j+x_k\| + a_4\left\|\sum_{i=1}^{4}x_i\right\| \geqslant 0,$$

for all $x_i \in E, 1 \leqslant i \leqslant 4$ (see Theorem 5.1, Section 5).

2. Preliminaries

While our goal is to derive certain inequalities in inner product spaces, as we will show below, much of the work can be reduced to the case of real numbers. More precisely, let E be an inner product and let $\|\cdot\|$ be the induced norm on E. If $L_i : \mathbb{R}^n \to \mathbb{R}, 1 \leqslant i \leqslant m$, are linear functions, by abuse of notation, we will also denote by L_i the linear function defined on E^n with values in E obtained by replacing the argument $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ of $L_i : \mathbb{R}^n \to \mathbb{R}$ by $(v_1, v_2, \ldots, v_n) \in E^n$. With this convention, we have the following:

LEMMA 2.1. *Given* $a_1, a_2, \ldots a_m \in \mathbb{R}$, *the inequality*

$$\sum_{i=1}^{m} a_i\|L_i(v)\| \geqslant 0 \quad (2.1)$$

is satisfied for all $v \in E^n$ *if and only if the inequality*

$$\sum_{i=1}^{m} a_i |L_i(x)| \geqslant 0 \tag{2.2}$$

is satisfied for all $x \in \mathbb{R}^n$.

Proof. If inequality (2.1) is satisfied, then relation (2.2) follows by choosing some $u \in E$ and letting $v = (x_1 u, x_2 u, \ldots, x_n u)$.

Conversely, if (2.2) is satisfied, then let us first show that the same inequality holds when $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is replaced by $f = (f_1, f_2, \ldots, f_n) \in (L^1(0,1))^n$ and $|\cdot|$ by the L^1-norm, where $L^1(0,1)$ denotes the space of real valued functions defined on $[0,1]$ with integrable absolute value. To see this, let us note that it is enough to consider $f_i \in L^1(0,1), 1 \leqslant i \leqslant n$, and replace x_i in (2.2) by $f_i(x)$. The inequality then follows by integration. Now, since any two dimensional normed space can be (linearly and) isometrically imbedded in $L^1(0,1)$ (see [12] or Corollary 2, Section 3, in [7]), it follows that (2.1) holds on $\mathbb{C}$ (with any norm). By repeating the same argument as before for complex valued functions this time, we may conclude that (2.1) is valid for complex valued functions with integrable modulus. (by abuse of notation, we will also denote this set of functions by $L^1(0,1)$).

Since $L^2(0,1)$ can be linearly and isometrically embedded in $L^1(0,1)$ (see [2] for real scalars and [5] for complex scalars), inequality (2.1) holds on $L^2(0,1)$. Moreover, $L^2(0,1)$ and ℓ_2 are linearly isometric, thus (2.2) holds on ℓ_2, the space of real or complex sequences $(x_i)_{1 \leqslant i \leqslant \infty}$ with $\sum |x_i|^2 < \infty$, endowed with the usual norm. Finally, we show that this implies that (2.1) is satisfied on any inner product space E. To this end, consider $v_1, v_2, \ldots, v_n \in E$ and let $V = \operatorname{span}\{v_1, v_2, \ldots, v_n\}$. V is a finite dimensional inner product space, with the inner product obtained by restricting the one on E. But then V is linearly isometric to ℓ_2^k, where $k = \dim V$, where l_2^k consists of all $k-$tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\|\alpha\| = \sqrt{|\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_k|^2}$. Since l_2^k can be linearly and isometrically embedded into l^2, the conclusion follows. □

We would like to acknowledge that the statement and proof of the lemma above were inspired by Bill Johnson's insightful response to a mathoverflow.net question related to Hlawka's inequality ([13]).

In light of the previous lemma, below we will restrict out attention to the real case.

LEMMA 2.2. *With notations as above, for* $m \geqslant 2$ *we have*

$$f(x) := \sum_{i=1}^{m} a_i |L_i(x)| \geqslant 0$$

for all $x \in \mathbb{R}^n$ *if and only if for each* $1 \leqslant i \leqslant m$ *we have* $f(x) \geqslant 0$ *for all* $x \in \ker L_i$.

Proof. Without loss of generality, we may assume that $\ker L_i \neq \ker L_j$ for all $1 \leqslant i < j \leqslant m$. The hyperplanes $\ker L_i, 1 \leqslant i \leqslant m$, divide $\mathbb{R}^n$ into a collection of nonempty, connected, convex sets and each one of these sets can be described as

$$\{x \in \mathbb{R}^n | \varepsilon_i L_i(x) > 0, i \in I \subseteq \{1,2,\dots,m\}\}, \tag{2.3}$$

for some choice of $\varepsilon_i \in \{\pm 1\}$.

There are two reasons why we may not have $I = \{1,2,\dots,m\}$. To see this, consider the following example. Let $L_1, L_2, L_3 : \mathbb{R}^2 \to \mathbb{R}$ given by $L_1(x_1,x_2) = x_1, L_2(x_1,x_2) = x_2, L_3(x_1,x_2) = x_1 + x_2$. On one hand we have

$$\{x \in \mathbb{R}^n | L_1(x) > 0, L_2(x) > 0, L_3(x) < 0\} = \emptyset.$$

On the other hand,

$$\{x \in \mathbb{R}^n | L_1(x) > 0, L_2(x) > 0, L_3(x) > 0\} = \{x \in \mathbb{R}^2 | L_1(x) > 0, L_2(x) > 0\}.$$

Thus, each (nonempty) set defined by relation (2.3) is fully determined by a minimal set $I = \{i_1, i_2 \dots, i_r\} \subseteq \{1,2,\dots,m\}$ and a set of $\Omega_I = \{\varepsilon_{i_j} \in \{\pm 1\}, 1 \leqslant j \leqslant r\}$. Consequently, we make the following notation:

$$C_{I,\Omega_I} = \{x \in \mathbb{R}^n | \varepsilon_i L_i(x) > 0, \text{ for all } i \in I\}.$$

Note that for any $j \notin I$, we have that either $\varepsilon_j L_j$ is redundant, i.e., it is positive throughout C_{I,Ω_I} or it is incompatible, i.e., it is negative throughout C_{I,Ω_I}. Thus, for any $i \in \{1,2,\dots,m\}$ the sign of L_i is constant on C_{I,Ω_I}.

Considering the closure of C_{I,Ω_I}, we have

$$\bar{C}_{I,\Omega_I} = \{x \in \mathbb{R}^n | \varepsilon_i L_i(x) \geqslant 0, \text{ for all } i \in I\}$$

and

$$\mathbb{R}^n = \bigcup_I C_{I,\Omega_I} \cup \bigcup_{1 \leqslant i \leqslant m} \ker(L_i) = \bigcup_I \bar{C}_{I,\Omega_I}. \tag{2.4}$$

Thus, we may conclude that f is nonnegative on $\mathbb{R}^n$ if and only if the restriction of f to each $\bar{C}_{I,\Omega_I}$ is nonnegative. As we will show next, f is nonnegative on $\bar{C}_{I,\Omega_I}$ iff it is nonnegative on the boundary of C_{I,Ω_I}. Since the boundary of each C_{I,Ω_I} is contained in the union of $\ker L_i$ for all $i, 1 \leqslant i \leqslant m$ (as shown below), the lemma follows.

Note that the boundary of C_{I,Ω_I} is given by

$$\partial C_{I,\Omega_I} = \bigcup_{i \in I} W_{I,i} \subset \bigcup_{i \in I} \ker L_i,$$

where for each $i \in I$,

$$W_{I,i} = \{x \in \mathbb{R}^n | L_i(x) = 0, \varepsilon_j L_j(x) \geqslant 0, j \in I - \{i\}\} \subset \ker L_i.$$

To prove that f is nonnegative on $\bar{C}_{I,\Omega_I}$ iff it is nonnegative on the boundary of C_{I,Ω_i}, observe that the restriction of f to $\bar{C}_{I,\Omega_I}$ is the restriction to $\bar{C}_{I,\Omega_I}$ of a linear combination of linear functions (due to the fact that the sign of L_i is constant on C_{I,Ω_I}), hence the restriction of a linear function. Thus, it will be enough to show that any

$x \in C_{I,\Omega_I}$ can be written as a sum of two vectors in $\partial C_{I,\Omega_I}$. In order to accomplish this, let us define the set

$$\tilde{W}_{I,i} = \{x \in \mathbb{R}^n | L_i(x) = 0, \varepsilon_j L_j(x) > 0, j \in I - \{i\}\}.$$

Observe that

$$\partial C_{I,\Omega_i} \supseteq W_{I,i} \supseteq \tilde{W}_{I,i} \neq \emptyset.$$

As the inclusions are obvious, let us show that $\tilde{W}_{I,i}$ is nonempty. By contradiction, assume $\tilde{W}_{I,i}$ is empty. Consider the set

$$S_{I,i} = \{x \in \mathbb{R}^n | \varepsilon_j L_j(x) > 0, j \in I - \{i\}\},$$

and note that $S_{I,i}$ is convex and nonempty since $C_{I,\Omega_I} \subset S_{I,i}$. Based on our assumption, we must have that for any $x \in S_{I,i}$, either $\varepsilon_i L_i(x) > 0$ or $\varepsilon_i L_i(x) < 0$. But this implies that either $\varepsilon_i L_i > 0$ throughout $S_{I,i}$ or $\varepsilon_i L_i < 0$ throughout $S_{I,i}$. To see this, note that if we had $u_1, u_2 \in S_{I,i}$ with $\varepsilon_i L_i(u_1) > 0$ and $\varepsilon_i L_i(u_2) < 0$, then $\varepsilon_i L_i(tu_1 + (1-t)u_2) = 0$ for some $t \in (0,1)$. But this is impossible by the convexity of $S_{I,i}$ and the assumption. Having shown that $\varepsilon_i L_i$ has constant sign on $S_{I,i}$, we could reach the final contradiction. Indeed, if $\varepsilon_i L_i > 0$ on $S_{I,i}$, then $\varepsilon_i L_i$ is redundant in the definition of C_{I,Ω_I}. On the other hand $\varepsilon_i L_i < 0$ on $S_{I,i}$, then we would have $C_{I,\Omega_I} = \emptyset$. As both the redundancy and the incompatibility cases have been ruled out when we defined C_{I,Ω_I}, we have $\tilde{W}_{I,i} \neq \emptyset$.

Now let $x \in C_{I,\Omega_I}$ and let $i \in I$. Next, we prove that x can be written as the sum of two vectors, one in $\tilde{W}_{I,i}$, the other in $\tilde{W}_{I,j_0}$, for some $j_0 \in I - \{i\}$. As shown above, there exists some $v \in \tilde{W}_{I,i}$. Note that $v \neq 0$. Define

$$a = \min\left\{\frac{\varepsilon_j L_j(x)}{\varepsilon_j L_j(v)}, j \in I - \{i\}\right\} = \frac{\varepsilon_{j_0} L_{j_0}(x)}{\varepsilon_{j_0} L_{j_0}(v)}$$

for some $j_0 \in I - \{i\}$. Since $\varepsilon_j L_j(x) > 0$ and $\varepsilon_j L_j(v) > 0$ for all $j \in I - \{i\}$, we have $a > 0$. For $x = av + (x - av)$, we will show that both av and $x - av$ are in $\partial \bar{C}_{I,\Omega_I}$. Clearly, $av \in \tilde{W}_{I,i} \subset \partial C_{I,\Omega_I}$ since $a > 0$. For $x - av$, note that $\varepsilon_i L_i(x - av) = \varepsilon_i L_i(x) > 0$ since $x \in C_{I,\Omega_I}$. On the other hand, for $j \in I - \{i\}$,

$$\varepsilon_j L_j(x - av) = \varepsilon_j L_j(x) - a\varepsilon_j L_j(v) \geqslant 0,$$

by the definition of a. As we also have $\varepsilon_{j_0} L_{j_0}(x - av) = 0$, we may conclude that $x - av_1 \in W_{I,j_0} \subset \partial C_{I,\Omega_I}$. $\square$

3. Extensions for four real or complex numbers

THEOREM 3.1. *Given real numbers $a_i, 1 \leqslant i \leqslant 4$, the inequality*

$$a_1 \sum_{1 \leqslant i \leqslant 4} |x_i| + a_2 \sum_{1 \leqslant i < j \leqslant 4} |x_i + x_j| + a_3 \sum_{1 \leqslant i < j < k \leqslant 4} |x_i + x_j + x_k| + a_4 |x_1 + x_2 + x_3 + x_4| \geqslant 0,$$

holds for all $x_i \in \mathbb{R}, 1 \leqslant i \leqslant 4$, if and only if the following five inequalities hold:

$$a_1 + a_2 + a_3 \geqslant 0;$$

$$a_1 + 2a_2 + a_3 \geqslant 0;$$

$$a_1 + 3a_2 + 3a_3 + a_4 \geqslant 0;$$

$$2a_1 + 3a_2 + 3a_3 + a_4 \geqslant 0;$$

$$5a_1 + 9a_2 + 3a_3 + a_4 \geqslant 0.$$

Proof. By Lemma 2.2, the inequality in the theorem above is satisfied iff it is satisfied on each one of the subspaces of $\mathbb{R}^4$ of the form $\sum_{i \in I} x_i = 0$, for all $I \subseteq \{1,2,3,4\}$. Whenever we choose one such subspace, the restriction of the function on the left side of the inequality to this subspace becomes a linear combination of absolute values of linear functions of three variables and we can use Lemma 2.2 again. If we apply the process once more, the new restrictions become constants depending on a_1, a_2, a_3, a_4 multiplied with the absolute value of one of the four variables $x_i, 1 \leqslant i \leqslant 4$. Such a function is nonnegative iff the constant is nonnegative. While the principle is simple enough, there are many cases and subcases to be investigated. However, the computations are straightforward and we summarize the results below, where $\{i,j,k,l\}$ represents any permutation of $\{1,2,3,4\}$:

(i) If $x_i + x_j = 0, x_i + x_k = 0$, and $x_j + x_l = 0$ (i.e., $x_i = -x_j = -x_k = x_l$), then

$$S = |x_i|(a_1 + a_2 + a_3).$$

(ii) If $x_i = 0$ and $x_j + x_k + x_l = 0$, then

$$S = (|x_j| + |x_k| + |x_j + x_k|)(a_1 + 2a_2 + a_3).$$

(iii) If $x_i = 0, x_j + x_k = 0$, and $x_j + x_l = 0$ (i.e., $x_i = 0, x_j = -x_k = -x_l$), then

$$S = |x_j|(3a_1 + 5a_2 + 3a_3 + a_4).$$

(iv) If $x_i + x_j + x_k = 0, x_i + x_j + x_l = 0$, and $x_i + x_k + x_l = 0$ (i.e., $x_j = x_k = x_l, x_i = -2x_j$), then

$$S = |x_i|(5a_1 + 9a_2 + 3a_3 + a_4).$$

(v) If $x_i + x_j + x_k = 0, x_i + x_l = 0$, and $x_j + x_l = 0$ (i.e., $x_i = x_j = -x_l, x_k = -2x_i$), then

$$S = |x_i|(5a_1 + 7a_2 + 5a_3 + a_4).$$

(vi) If $x_i + x_j = 0, x_i + x_k = 0$, and $x_i + x_l = 0$ (i.e., $x_j = x_k = x_l = -x_i$), then

$$S = 2|x_i|(2a_1 + 3a_2 + 3a_3 + a_4).$$

(vii) If $x_i = x_j = x_k = 0$, then

$$S = |x_l|(a_1 + 3a_2 + 3a_3 + a_4).$$

Note the seven cases above cover all possible cases of (x_1,x_2,x_3,x_4) simultaneously satisfying

$$\sum_{i\in I_1} x_i = 0, \sum_{i\in I_2} x_i = 0, \sum_{i\in I_3} x_i = 0,$$

for some $I_1,I_2,I_3 \subseteq \{1,2,3,4\}$.

Finally, we note that the number of inequalities involving the coefficients a_i can be reduced to the five in the conclusion of the theorem due to the fact that

$$3a_1+5a_2+3a_3+a_4 = \frac{2}{3}(2a_1+3a_2+3a_3+a_4)+\frac{1}{3}(5a_1+9a_2+3a_3+a_4)$$

and

$$5a_1+7a_2+5a_3+a_4=2(a_1+a_2+a_3)+\frac{1}{3}(5a_1+9a_2+3a_3+a_4)+\frac{2}{3}(2a_1+3a_2+3a_3+a_4).$$

As one can easily check, of the remaining five expressions in a_i, neither one is a linear combination with nonnegative coefficients of the other four. □

As discussed by several authors (see [4], [9]), Hlawka's inequality is not satisfied on a general normed space but it does hold on two-dimensional normed spaces. We have a similar result below.

COROLLARY 3.2. *Theorem 3.1 remains valid on any two-dimensional normed space E if $|\cdot|$ is replaced by the norm on E.*

Proof. If we assume that the five inequalities in $a_i, 1 \leqslant i \leqslant 4$ are satisfied, then the conclusion follows based on the same argument used in extending an inequality from $\mathbb{R}$ to $\mathbb{C}$ in Lemma 2.1. The converse also follows based on the proof of same lemma by choosing four complex numbers $v_i, 1 \leqslant i \leqslant 4$ that are x_i- multiples of a some complex number u. □

4. An affine version of Hlawka's inequality

In this section we prove an affine version of Hlawka's inequality on inner product spaces. The result can be seen as an extension of Hlawka's inequality. While interesting in its own right, we will only make use of it in the last section in order to derive the equality case for a certain Hlawka-type inequality involving four vectors.

THEOREM 4.1. *Let E be any inner product space. The inequality*

$$\sum_{i=1}^{3}\|x_i-p\| - \sum_{1\leqslant i<j\leqslant 3}\left\|x_i+x_j-p\right\| + \|x_1+x_2+x_3-p\| + \|p\| \geqslant 0 \tag{4.1}$$

is satisfied for all $x_1,x_2,x_3,p \in X$.

Proof. By Lemma 2.1, we may assume that x_1, x_2, x_3 are real numbers and $\|\cdot\|$ is the usual absolute value function. By Lemma 2.2, it is enough to show that the inequality holds in each of the four cases below:

(i) If $x_1 = p$, we need to prove that

$$|x_2 - x_1| + |x_3 - x_1| + |x_2 + x_3| + |x_1| \geqslant |x_2| + |x_3| + |x_2 + x_3 - x_1|.$$

By Hlawka's inequality applied to $x_2 - x_1, x_3 - x_1, x_1$, we have

$$|x_2 - x_1| + |x_3 - x_1| + |x_1| \geqslant |x_2 + x_3 - 2x_1| + |x_2| + |x_3| - |x_2 + x_3 - x_1|.$$

Since

$$|x_2 + x_3| + |x_2 + x_3 - 2x_1| \geqslant 2|x_2 + x_3 - x_1|$$

by the triangle inequality, we may conclude the validity of inequality (4.1) in this case.

(ii) If $x_1 + x_2 = p$, then the inequality to be proven becomes

$$|x_2| + |x_1| + |x_3 - x_1 - x_2| + |x_3| \geqslant |x_3 - x_2| + |x_3 - x_1| - |x_1 + x_2|.$$

Applying Hlawka's inequality for $-x_1, -x_2, x_3$ yields

$$|x_1| + |x_2| + |x_3| + |x_3 - x_2 - x_1| \geqslant |x_3 - x_1| + |x_3 - x_2| + |x_1 + x_2|.$$

Clearly, the right side of the inequality above is greater than or equal to $|x_3 - x_2| + |x_3 - x_1| - |x_1 + x_2|$.

(iii) If $x_1 + x_2 + x_3 = p$, then the inequality we need to prove becomes

$$|x_2 + x_3| + |x_1 + x_3| + |x_1 + x_2| - (|x_1| + |x_2| + |x_3|) + |x_1 + x_2 + x_3| \geqslant 0.$$

By considering the subcases $x_1 + x_2 = 0, x_1 = 0$, and $x_1 + x_2 + x_3 = 0$, the expression on the left side of the inequality becomes $(|x_3 - x_1| + |x_3 + x_1| - 2|x_1|), 2|x_2 + x_3|$, and 0, respectively. In either case, the left side is nonnegative.

(iv) If $p = 0$, then inequality 4.1 is simply Hlawka's inequality. $\square$

OBSERVATION 4.2. If we let $f : E \to \mathbb{R}$ be defined as $f(x) = \|x - p\| + \|p\|$. Inequality (4.1) can be written as

$$\sum_{i=1}^{3} f(x_i) - \sum_{1 \leqslant i < j \leqslant 3} f(x_i + x_j) + f(x_1 + x_2 + x_3) \geqslant 0.$$

This is the reason why we refer to inequality (4.1) as the affine version of Hlawka's inequality.

OBSERVATION 4.3. The affine version of the triangle inequality, i.e.,

$$\|x_1 - p\| + \|x_2 - p\| - \|x_1 + x_2 - p\| + \|p\| \geqslant 0 \tag{4.2}$$

is satisfied as well by the triangle inequality. Equality is achieved iff $x_1 = \lambda_1 p, x_2 = \lambda_2 p$, with $\lambda_1, \lambda_2 \geqslant 1$.

5. Extensions to inner product spaces

Let $(E,\langle\cdot,\cdot\rangle)$ be an inner product space with $\|\cdot\|$ as the induced norm and let $x_i\in E, 1\leqslant i\leqslant 4$. By Theorem 3.1 and Lemma 2.1, we obtain the following:

THEOREM 5.1. *With notations as above, given $a_i\in\mathbb{R}, 1\leqslant i\leqslant 4$, the inequality*

$$a_1\sum_{1\leqslant i\leqslant 4}\|x_i\|+a_2\sum_{1\leqslant i<j\leqslant 4}\left\|x_i+x_j\right\|+a_3\sum_{1\leqslant i<j<k\leqslant 4}\left\|x_i+x_j+x_k\right\|+a_4\left\|\sum_{i=1}^4 x_i\right\|\geqslant 0$$

is valid iff the following inequalities hold:

$$a_1+a_2+a_3\geqslant 0;$$

$$a_1+2a_2+a_3\geqslant 0;$$

$$a_1+3a_2+3a_3+a_4\geqslant 0;$$

$$2a_1+3a_2+3a_3+a_4\geqslant 0;$$

$$5a_1+9a_2+3a_3+a_4\geqslant 0.$$

OBSERVATION 5.2. Looking at the expression (1.1) considered when attempting to naively extend Hlawka's inequality, we have $a_1=1, a_2=-1, a_3=1$, and $a_4=-1$ in the formula for S above. For this choice of a_i, all inequalities in the conclusion of the theorem above are satisfied except for $5a_1+9a_2+3a_3+a_4\geqslant 0$. To find "best"valid extensions, we fix three of the coefficients a_i as in the naive extension and find the correct remaining coefficient based on the five inequalities in the theorem.

COROLLARY 5.3. *With notations as in Theorem 5.1 we have following:*

(i) If $a_1=1, a_2=-1, a_3=1$, then $a_4\geqslant 1$ and

$$\sum_{1\leqslant i\leqslant 4}\|x_i\|-\sum_{1\leqslant i<j\leqslant 4}\left\|x_i+x_j\right\|+\sum_{1\leqslant i<j<k\leqslant 4}\left\|x_i+x_j+x_k\right\|+\left\|\sum_{i=1}^4 x_i\right\|\geqslant 0.$$

(ii) If $a_1=1, a_2=-1, a_4=-1$, then $a_3\geqslant \frac{5}{3}$ and

$$\sum_{1\leqslant i\leqslant 4}\|x_i\|-\sum_{1\leqslant i<j\leqslant 4}\left\|x_i+x_j\right\|+\frac{5}{3}\sum_{1\leqslant i<j<k\leqslant 4}\left\|x_i+x_j+x_k\right\|-\left\|\sum_{i=1}^4 x_i\right\|\geqslant 0.$$

(iii) If $a_1=1, a_3=1, a_4=-1$, then $a_2\geqslant -\frac{7}{9}$ and

$$\sum_{1\leqslant i\leqslant 4}\|x_i\|-\frac{7}{9}\sum_{1\leqslant i<j\leqslant 4}\left\|x_i+x_j\right\|+\sum_{1\leqslant i<j<k\leqslant 4}\left\|x_i+x_j+x_k\right\|-\left\|\sum_{i=1}^4 x_i\right\|\geqslant 0.$$

(iv) If $a_2=-1, a_3=1, a_4=-1,$ then $a_1 \geqslant \frac{7}{5}$ and

$$\frac{7}{5}\sum_{1\leqslant i\leqslant 4}\|x_i\| - \sum_{1\leqslant i<j\leqslant 4}\left\|x_i+x_j\right\| + \sum_{1\leqslant i<j<k\leqslant 4}\left\|x_i+x_j+x_k\right\| - \left\|\sum_{i=1}^{4}x_i\right\| \geqslant 0.$$

OBSERVATION 5.4. Djoković's inequality for $n=4$ can be obtained from Theorem 5.1. If $k=2,$ the left side of the inequality reads

$$2\sum_{1\leqslant i\leqslant 4}\|x_i\| - \sum_{1\leqslant i<j\leqslant 4}\left\|x_i+x_j\right\| + \|x_1+x_2+x_3+x_4\| \geqslant 0,$$

with $a_1=2, a_2=-1, a_3=0, a_4=1$. As one can easily check, these coefficients verify the five conditions in Theorem 5.1. Similarly, if $k=3,$ we have

$$\sum_{1\leqslant i\leqslant 4}\|x_i\| - \sum_{1\leqslant i<j<k\leqslant 4}\left\|x_i+x_j+x_k\right\| + 2\|x_1+x_2+x_3+x_4\| \geqslant 0$$

and the coefficients $a_1=1, a_2=0, a_3=-1, a_4=2$ check the conditions as well.

Lastly, we address the equality case for the first inequality in consequence 5.3 above.

PROPOSITION 5.5. *Given an inner product space E and $x_i \in E$, we have*

$$\sum_{1\leqslant i\leqslant 4}\|x_i\| - \sum_{1\leqslant i<j\leqslant 4}\left\|x_i+x_j\right\| + \sum_{1\leqslant i<j<k\leqslant 4}\left\|x_i+x_j+x_k\right\| + \|x_1+x_2+x_3+x_4\| = 0$$

if and only if there exists a permutation σ of $\{1,2,3,4\}$, a vector $u\in E$, and real numbers $\lambda_i, 1\leqslant i\leqslant 4,$ such that $x_{\sigma(i)}=\lambda_{\sigma(i)}u, \lambda_{\sigma(i)}\geqslant 0, 1\leqslant i\leqslant 3,$ and $0\leqslant \lambda_{\sigma(1)}+\lambda_{\sigma(2)}+\lambda_{\sigma(3)}+\lambda_{\sigma(4)} \leqslant min\{\lambda_{\sigma(1)},\lambda_{\sigma(2)},\lambda_{\sigma(3)}\}$.

Proof. Note that the equality in the proposition can be reorganized as follows:

$$\begin{aligned}&\Big(\sum_{1\leqslant i\leqslant 3}\|x_i\| - \sum_{1\leqslant i<j\leqslant 3}\left\|x_i+x_j\right\| + \|x_1+x_2+x_3\|\Big)\\ &+\Big(\sum_{1\leqslant i\leqslant 3}\|x_i-p\| - \sum_{1\leqslant i<j\leqslant 3}\left\|x_i+x_j-p\right\| + \|x_1+x_2+x_3-p\| + \|p\|\Big)=0,\end{aligned}$$

where $p=x_1+x_2+x_3+x_4$. By Theorem 4.1 and Hlawka's inequality, both of the terms on the left side of the equality above are nonnegative. Thus, in order to have equality, both terms need to be zero. In particular, we must have equality in Hlawka's inequality for x_1,x_2,x_3. Noting that the equality in the proposition can be reorganized by using not just x_1,x_2,x_3 in the relation above but any three of the vectors x_1,x_2,x_3,x_4, we get equality in Hlawka's inequality for any three of the vectors x_1,x_2,x_3,x_4. After analyzing the possibilities, the only viable option is

$$x_{i_1}=\lambda_{i_1}u, x_{i_2}=\lambda_{i_2}u, x_{i_3}=\lambda_{i_3}u, x_{i_4}=\lambda_{i_4}u$$

for some $u\in E$, $\lambda_{i_j}\geqslant 0, 1\leqslant j\leqslant 3, 0\leqslant \lambda_{i_1}+\lambda_{i_2}+\lambda_{i_3}+\lambda_{i_4}\leqslant \min\{\lambda_{i_1},\lambda_{i_2},\lambda_{i_3}\}$, and $\{1,2,3,4\}=\{i_1,i_2,i_3,i_4\}$. □

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Marius Munteanu
Department of Mathematics, Computer Science, and Statistics
SUNY Oneonta
Oneonta, NY 13820
e-mail: Marius.Munteanu@oneonta.edu