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SOLUTION OF THE POMPEIU DERIVATION FUNCTIONAL INEQUALITY OF OTHER TYPE

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Abstract. In this paper, we investigate the solution of Pompeiu derivation functional inequality of other type

$$f(x+y+z+xy+xz+yz+xyz) \leqslant f(x)+f(y)+f(z)+f(x)y+xf(y)+xf(z)+yf(z)+xyf(z),$$

on R.

1. Introduction

Let R be the set of all real numbers. A function $M : R \to R$ is said to be a multiplicative function if

$$M(xy) = M(x)M(y), \tag{1}$$

for all $x,y \in R$ [1].

The functional equation

$$f(x+y+xy) = f(x)+f(y)+f(x)f(y), \tag{2}$$

is called Pompeiu functional equation [7, 8].
Therefore, the only solution f of Pompeiu functional equation (2) is given by

$$f(x) = M(x+1)-1, \tag{3}$$

where M is multiplicative [6].

A function $f : R \to R$ is called Pompeiu derivation if it satisfies the functional equation

$$f(x+y+xy) = f(x)+f(y)+f(x)y+xf(y), \tag{4}$$

for all $x,y \in R$.

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In [5] the authors studied the functional inequality

$$f(x+y+xy) \leqslant f(x)+f(y)+f(x)f(y).$$

Recently, John Rassias and others [2] investigated the solution and stability of type functional inequality

$$f(x+y+xy) \leqslant f(x)+f(y)+f(xy),$$

(see also [4]).

DEFINITION 1. Let R be the set of all real numbers. A function $f : R \to R$ is called a Pompeiu functional equation of other type if

$$\begin{aligned} & f(x+y+z+xy+xz+yz+xyz) \\ =& f(x)+f(y)+f(z)+f(x)f(y)+f(x)f(z)+f(y)f(z)+f(x)f(y)f(z), \end{aligned} \tag{5}$$

for all $x,y,z \in R$.

Recently, Wlodzimierz Fechner [3] investigated the solution of Hlawka's functional inequality

$$f(x+y)+f(y+z)+f(x+z) \leqslant f(x+y+z)+f(x)+f(y)+f(z).$$

DEFINITION 2. Let R be the set of all real numbers. A function $f : R \to R$ is called a Pompeiu derivation of other type if

$$\begin{aligned} & f(x+y+z+xy+xz+yz+xyz) \\ =& f(x)+f(y)+f(z)+f(x)y+xf(y)+f(x)z+xf(z)+f(y)z+yf(z)+f(x)yz \\ & +xf(y)z+xyf(z), \end{aligned} \tag{6}$$

for all $x,y,z \in R$.

Now, we introduce the following new functional inequality

$$\begin{aligned} & f(x+y+z+xy+xz+yz+xyz) \\ \leqslant & f(x)+f(y)+f(z)+f(x)y+xf(y)+xf(z)+yf(z)+xyf(z), \end{aligned} \tag{7}$$

on R.

Hence, it is natural that inequality (7) is called a *Pompeiu derivation functional inequality of other type* and every solution of the Eq. (7) is said to be a *Pompeiu derivation functional inequality of other type*.

In this paper, we investigate the general solution of Pompeiu derivation functional *inequality of other type* on R.

2. Pompeiu derivation inequality

In this section, we study the general solution of Pompeiu derivation *inequality*

$$f(x+y+xy) \leqslant f(x)+f(y)+xf(y), \tag{8}$$

on R.

In mathematics and specifically, real analysis, the Dini derivatives are a class of generalizations of the derivative. The upper Dini derivative for an arbitrary mapping $f: R \to R$ is defined by

$$D^{\pm} f(t) = \limsup_{h\to 0^{\pm}} \frac{f(t+h)-f(t)}{h},$$

where lim sup is the supremum limit.

The lower Dini derivative of f is defined as follows:

$$D_{\pm} f(t) = \liminf_{h\to 0^{\pm}} \frac{f(t+h)-f(t)}{h},$$

where lim inf is the infimum limit.
If f is differentiable at t, then the Dini derivative at t is the usual derivative at t.

THEOREM 1. *Assume that I is a nonvoid open interval containing zero. If a function $f: I \to R$ solves (8) for all $x,y \in I$ such that $x+y+xy \in I$, satisfies the following conditions, then*

$$D^{+} f(x) \leqslant f'(0) \leqslant D_{-} f(x),$$

for all $x \in I \backslash \{-1\}$:

i) f *is differentiable at zero;*

ii) $f(0) = 0$.

Proof. Let $f: I \to R$ holds in functional inequality (8) and put $y > 0$. We have

$$f(x+(1+x)y) - f(x) \leqslant f(y) + xf(y),$$

and

$$(1+x)\frac{f(x+(1+x)y)-f(x)}{(1+x)y} \leqslant \frac{f(y)}{y} + \frac{xf(y)}{y}, \tag{9}$$

for all $x,y \in I$ such that $x+y+xy \in I$.

Assume that $x > -1$ and pick a sequence $(y_n)_{n\in N}$ of positive elements of I tending to zero in (9). Substitute $y \to y_n$ and letting $n \to +\infty$ in (9), by the definition of Dini derivatives and using the assumptions that $f(0) = 0$ and f is differentiable at zero, we have

$$(1+x) D^{+} f(x) \leqslant (1+x) f'(0),$$

that is

$$D^{+}f(x) \leqslant f'(0),$$

for all $x \in I$ such that $x > -1$.

Similarly, if $x < -1$ then

$$D_{-}f(x) \geqslant f'(0),$$

for all $x \in I$.

We put $y < 0$ for all $x, y \in I$ such that $x + y + xy \in I$. By (8), we have

$$(1+x)\frac{f(x+(1+x)y) - f(x)}{(1+x)y} \geqslant \frac{f(y)}{y} + \frac{xf(y)}{y}. \tag{10}$$

Assume that $x > -1$ and pick a sequence $(y_n)_{n\in N}$ of negative elements of I tending to zero in (10).

Substitute $y \to y_n$ and letting $n \to +\infty$ in (10), we have

$$D_{-}f(x) \geqslant f'(0),$$

for all $x \in I$.

Similarly, if $x < -1$ then

$$D^{+}f(x) \leqslant f'(0),$$

for all $x \in I$. Then

$$D^{+}f(x) \leqslant f'(0) \leqslant D_{-}f(x),$$

for all $x \in I \backslash \{-1\}$ and proof is complete. □

THEOREM 2. *Let R be the set of all real numbers. If a function $f : R \to R$ solves* (8)*, satisfies the following conditions then there exists a real constant c, that $f(x) = cx$ for all $x \in (-1, 0)$:*

i) *f is differentiable at zero;*

ii) $f(0) = 0$.

Proof. Replace x in (8) by $x + y$ and y by $-y$. We have

$$f(x+y-y-(x+y)y) \leqslant f(x+y) + f(-y) + (x+y)f(-y),$$

assume that $y > 0$, then

$$-(x+y)\frac{f(x-(x+y)y) - f(x)}{-(x+y)y} \leqslant \frac{f(x+y) - f(x)}{y} - \frac{f(-y)}{-y} - \frac{(x+y)f(-y)}{-y}, \tag{11}$$

for all $x, y \in R$.

Suppose that $x \in (-1,0)$ and pick a sequence $(y_n)_n \in N$ of positive elements of R tending to zero in (11).

Substitute $y \to y_n$ and letting $n \to +\infty$ in (11), we have

$$-xD_+f(x) \leqslant D_+f(x) - (1+x)f'(0),$$

that is $f'(0) \leqslant D_+f(x)$ for all $x \in (-1,0)$.

Similarly, assume that $y < 0$ and pick a sequence $(y_n)_{n\in N}$ of negative elements of R tending to zero in (11) for all $x \in (-1,0)$. Substitute $y \to y_n$ and letting $n \to +\infty$ in (11). By using Theorem 1, proof is complete. □

3. Solution

In this section, we study the general solution of Pompeiu derivation of other type. In [6], Kannappan and Sahoo defined a genealization of the Pompeiu functional equation (2), such that

$$f(ax+by+cxy) = f(x)+f(y)+f(x)f(y), \tag{12}$$

for all $x, y \in R$ and a, b, c are arbitrary real constants.

DEFINITION 3. Let R be the set of all real numbers. A function $f : R \to R$ is called a multiplicative derivation if

$$f(xy) = f(x)y+xf(y),$$

for all $x, y \in R$.

Hence, the function $f : R \to R$ is called a 3-multiplicative derivation if

$$f(xyz) = f(x)yz+xf(y)z+xyf(z), \tag{13}$$

for all $x, y, z \in R$.

THEOREM 3. *Let R be the set of all real numbers. A function $f : R \to R$ satisfies the functional equation (6) if only if there exists a 3-multiplicative derivation $D : R \to R$ such that*

$$f(x) = D(x+1),$$

for all $x, y, z \in R$.

Proof. Putting $f(x) = D(x+1)$, we have

$$\begin{aligned}
& f(x+y+z+xy+xz+yz+xyz) = D(x+y+z+xy+xz+yz+xyz+1) \\
=& D((x+1)(y+1)(z+1)) \\
=& (D(x+1))(y+1)(z+1) + (x+1)D(y+1)(z+1) + (x+1)(y+1)(D(z+1)) \\
=& f(x)(y+1)(z+1) + (x+1)f(y)(z+1) + (x+1)(y+1)f(z) \\
=& f(x)(y+z+yz+1) + (x+1)f(y)(z+1) + (x+y+xy+1)f(z) \\
=& f(x)+f(y)+f(z)+f(x)y+xf(y)+f(x)z+xf(z)+f(y)z+yf(z)+f(x)yz \\
& +xf(y)z+xyf(z),
\end{aligned}$$

for all $x,y,z \in R$.

Similarly, we put $D(x) = f(x-1)$ and proof is complete. □

We now, investigate the solution of Pompeiu derivation functional equation of other type for $z = 1$ such that introduced the following inequality

$$f(2x+2y+2xy+1) \leqslant 2f(x)+2f(y)+2xf(y), \tag{14}$$

for all $x,y \in R$.

THEOREM 4. *Let R be the set of all real numbers. If a function $f : R \to R$ solves (14), satisfies the following conditions then there exists a real constant c, that $f(x) = cx$ for all $x \in (-1,0)$:*

i) *f is differentiable at zero;*

ii) $f(0) = 0$.

Proof. We put $z = 1$ in (6) by inequality (8), we have

$$f(2x+2y+2xy+1) = 2f(x+y+xy) \leqslant 2f(x)+2f(y)+2xf(y), \tag{15}$$

similar of Theorem 2, proof is complete. □

4. Pompeiu derivation inequality of other type

In this section, we investigate the general solution of Pompeiu derivation functional inequality of other type on R.

THEOREM 5. *Assume that I is a nonvoid open interval containing zero. If a function $f : I \to R$ solves (7) for all $x,y,z \in I$ such that $x+y+z+xy+xz+yz+xyz \in I$, satisfies the following conditions, then*

$$D^{+}f(x+y+xy) \leqslant f'(0) \leqslant D_{-}f(x+y+xy),$$

for all $x,y \in I \backslash \{-1\}$:

i) f is differentiable function;

ii) $f(0)=0$.

Proof. Let $f: I \to R$ holds in functional inequality (7) and put $z>0$. We have

$$f(x+y+xy+(1+x+y+xy)z)-f(x+y+xy) \leqslant (1+x+y+xy)f(z),$$

and

$$(1+x)(1+y)\frac{f(x+y+xy+(1+x)(1+y)z)-f(x+y+xy)}{(1+x)(1+y)z} \leqslant (1+x)(1+y)\frac{f(z)}{z}, \tag{16}$$

for all $x,y,z \in I$ such that $x+y+z+xy+xz+yz+xyz \in I$.

Assume that $x>-1$, $y>-1$ and pick a sequence $(z_n)_{n\in N}$ of positive elements of I tending to zero in (16).

Substitute $z \to z_n$ and letting $n \to +\infty$ in (16), we have

$$(1+x)(1+y)D^{+}f(x+y+xy) \leqslant (1+x)(1+y)f'(0),$$

so

$$D^{+}f(x+y+xy) \leqslant f'(0),$$

for all $x,y \in I$ such that $x>-1$ and $y>-1$.

Similarly, if $x<-1$ and $y<-1$ then

$$D_{-}f(x+y+xy) \geqslant f'(0).$$

We put $z<0$ in (7), we have

$$f(x+y+xy+(1+x+y+xy)z)-f(x+y+xy) \geqslant (1+x+y+xy)f(z),$$

and

$$(1+x)(1+y)\frac{f(x+y+xy+(1+x)(1+y)z)-f(x+y+xy)}{(1+x)(1+y)z} \geqslant (1+x)(1+y)\frac{f(z)}{z}, \tag{17}$$

for all $x,y,z \in I$ such that $x+y+z+xy+xz+yz+xyz \in I$.

Assume $x>-1$, $y>-1$ and pick a sequence $(z_n)_{n\in N}$ of negative elements of I tending to zero in (17).

Substitute $z \to z_n$ and letting $n \to +\infty$ in (17), we have

$$D_{-}f(x+y+xy) \geqslant f'(0),$$

for all $x,y \in I$ such that $x>-1$ and $y>-1$.

Similarly, if $x<-1$ and $y<-1$ then

$$D^{+}f(x+y+xy) \leqslant f'(0).$$

So

$$D^{+}f(x+y+xy) \leqslant f'(0) \leqslant D_{-}f(x+y+xy),$$

for all $x,y \in I\backslash\{-1\}$ and proof is complete. □

THEOREM 6. *Let R be the set of all real numbers. If a function $f : R \to R$ solves (7), satisfies the following conditions then there exists a real constant c, that $f(x+y+xy) = c(x+y+xy)$ for all $x,y \in (-1,0)$:*

i) *f is differentiable function;*

ii) $f(0) = 0$.

Proof. Similar of Theorem 5, proof is complete. □

REFERENCES

[1] J. ACZEL AND J. DHOMBRES, *Functional equations in several variables*, Cambridge University Press, Cambridge (1989).
[2] B. BOUIKHALENE, J. M. RASSIAS, A. CHARIFI AND S. KABBAJ, *On the approximate solution of Hosszu functional equation*, Int. J. Nonlinear Anal. Appl. 3. No. 1 (2012) 40–44.
[3] W. FECHNER, *Hlawka's functional inequality*, Aequationes Math (2014) 71–87.
[4] W. FECHNER, *On a question of J. M. Rassias*, Bulletin of the Australian Mathematical Society, Volume 89, Issue 03 (2014) 494–499.
[5] S. M. JUNG AND P. K. SAHOO, *On the stability of the Pompeiu functional equation*, Math. Inequal. Appl. 5. No. 2 (2002) 257–261.
[6] P. L. KANNAPPAN, P. K. SAHOO, *On the generalization of the Pompeiu functional equation*, Int. J. Math. Math. Sci. 21 (1998) 117–124.
[7] E. L. KOH, *The Cauchy functional equations in distributions*, Proc. Amer. Math. Soc. 106 (1989) 641–646.
[8] M. NEAGU, *About the Pompeiu equation in distributions*, Inst. Politehn. "Traian Vuia" Timisoara. Lucrar. Sem. Mat. Fiz (1984) 62–66.

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