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BUSEMANN–PETTY PROBLEMS FOR L_p MIXED INTERSECTION BODIES

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Abstract. The notion of L_p mixed intersection bodies was introduced by Ma. In this paper, we consider the Busemann-Petty problems for the L_p mixed intersection bodies.

1. Introduction and main results

Let S^{n-1} denote the unit sphere in Euclidean space $\mathbb{R}^n$. If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, is defined by (see [9, 34])

$$\rho(K,x) = \max\{\lambda : \lambda x \in K\}.$$

If ρ_K is positive and continuous, then K will be called a star body with respect to the origin. The set of all star bodies about the origin in $\mathbb{R}^n$ is denoted by $\mathscr{S}_o^n$, and the set of all origin-symmetric star bodies in $\mathbb{R}^n$ will be denoted by $\mathscr{S}_e^n$. Two star bodies K and L are said to be dilates of one another if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean $\mathbb{R}^n$. For $u \in S^{n-1}$, $u^\perp$ denotes the $(n-1)$-dimensional subspace orthogonal to u. We use $V_k(M)$ to denote the k-dimensional volume of a k-dimensional compact convex set M. Instead of V_n we usually write V. For the standard unit ball B in $\mathbb{R}^n$, we write $\omega_n = V(B)$ for its volume. We also note that i denotes any real number in this article.

Busemann and Petty [3] posed a problem: Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. Is it true that for any $u \in S^{n-1}$,

$$V_{n-1}(K \cap u^\perp) \leqslant V_{n-1}(L \cap u^\perp) \Longrightarrow V(K) \leqslant V(L)?$$

A long list of authors contributed to the solution of this famous problem over a period of 40 years, see [1-2, 10-14, 17, 20-21, 24, 26, 33, 35, 42]. The question has

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a negative answer for $n \geqslant 5$ and an affirmative answer for $n = 3,4$. For a detailed account of the interesting history of the Busemann-Petty problems, see the books by Gardner [9, Chapter 8] and Koldobsky [22, Chapter 5].

The crucial idea solving the problem is to define a new convex body which is called the intersection body given by Lutwak [26]. For $K \in \mathscr{S}_o^n$, the intersection body, IK, of K is a star body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of K by $u^\perp$, i.e.,

$$\rho(IK,u) = V_{n-1}(K \cap u^\perp). \tag{1.1}$$

The intersection bodies have been intensively studied in recent years (see [15-16, 19, 23, 25, 31-32, 38-40] and the books [22, 36]). From (1.1) and the fact that $K \subseteq L$ for $K,L \in \mathscr{S}_o^n$ if and only if $\rho(K,\cdot) \leqslant \rho(L,\cdot)$, we see that the Busemann-Petty problems can be rephrased in the following way:

For $K,L \in \mathscr{S}_o^n$, is it true that

$$IK \subseteq IL \Longrightarrow V(K) \leqslant V(L)?$$

Lutwak [26] showed that the problem has an affirmative answer if the body K restricted to the class of intersection bodies. In addition, Lutwak proved that if L is a sufficiently smooth origin-symmetric star body with positive radial function which is not an intersection body, then there exists an origin-symmetric star body K such that $IK \subseteq IL$ but $V(K) > V(L)$. Further, the Busemann-Petty problems have been considered in the context of L_p Brunn-Minkoski theory (see [4-8, 27-30, 37, 39]). In particular, Yuan and Cheung [41] generalized the intersection body to L_p analogue, and introduced the notion of L_p intersection body: Let L be a star body and nonzero $p < 1$. The L_p intersection body I_pL, of L, is the origin-symmetric star body whose radial function is defined by

$$\rho(I_pL,u)^p = \int_L |u\cdot x|^{-p} dx. \tag{1.2}$$

In [41], they establish the affirmative version of Busemann-Petty problems for the L_p intersection body.

THEOREM 1.A. *Let K be a L_p intersection body and L be a star body in $\mathbb{R}^n$. If $I_pK \subseteq I_pL$, then for $0 < p < 1$,*

$$V(K) \leqslant V(L).$$

Equality holds if and only if $K = L$.

A further extension of L_p intersection bodies is the L_p mixed intersection bodies defined in [31]. Let K be a star body and $p \geqslant 1$. The L_p mixed intersection body, $I_{p,i}K$, of K is defined by

$$\rho(I_{p,i}K,u) = \left(\frac{\widetilde{W}_{p,i}(K, B\cap u^\perp)}{\widetilde{W}_{p,i}(B, B\cap u^\perp)}\right)^{\frac{1}{p}} \tag{1.3}$$

for $u \in S^{n-1}$, where $\widetilde{W}_{p,i}$ denotes the L_p dual mixed quermassintegrals (see (2.3)).

Suppose that f is a Borel function on S^{n-1}. The spherical Radon transform Rf of f was, in [18], defined by

$$(Rf)(u) = \int_{S^{n-1}\cap u^{\perp}} f(v)dS_{n-2}(v)$$

for $u \in S^{n-1}$. The spherical Radon transform is self-adjoint, i.e., if f and g are defined on S^{n-1} bounded Borel function, then

$$\int_{S^{n-1}} f(u)Rg(u)dS(u) = \int_{S^{n-1}} Rf(u)g(u)dS(u). \tag{1.4}$$

Using the spherical Radon transform, the definition of $I_{p,i}K$ is rewritten by

$$\begin{aligned}\rho(I_{p,i}K,u) &= \left(\frac{\widetilde{W}_{p,i}(K,B\cap u^{\perp})}{\widetilde{W}_{p,i}(B,B\cap u^{\perp})}\right)^{\frac{1}{p}} = \left(\frac{1}{(n-1)\omega_{n-1}}R(\rho_K^{n-p-i})(u)\right)^{\frac{1}{p}}\\ &= \left(\frac{1}{(n-1)\omega_{n-1}}\int_{S^{n-1}\cap u^{\perp}}\rho(K,v)^{n-p-i}dS_{n-2}(v)\right)^{\frac{1}{p}} \end{aligned} \tag{1.5}$$

for $u \in S^{n-1}$.

Thus, such inequalities as the Busemann type inequality, monotonicity inequality and Brunn-Minkowski inequality were shown in [31]. The main aim of this paper is to study the Busemann-Petty problems for the L_p mixed intersection bodies. We first solve the affirmative version for the problems. For convenience, let $\mathbb{I}_{p,i}$ denote the set of L_p mixed intersection bodies.

THEOREM 1.1. *Let $p \geqslant 1$ and $K,L \in \mathscr{S}_o^n$. if $K \in \mathbb{I}_{p,i}$, then for $i < n-p$,*

$$I_{p,i}K \subseteq I_{p,i}L$$

implies

$$\widetilde{W}_i(K) \leqslant \widetilde{W}_i(L),$$

with equality if and only if $K = L$; if $L \in \mathbb{I}_{p,i}$, then for $i > n$,

$$I_{p,i}K \subseteq I_{p,i}L$$

implies

$$\widetilde{W}_i(K) \leqslant \widetilde{W}_i(L),$$

with equality if and only if $K = L$.

Here, $\widetilde{W}_i$ denotes the dual quermassintegrals (see (2.2)).

The following provides the negative version of the Busemann-Petty problems for the L_p mixed intersection bodies.

THEOREM 1.2. *Let $p \geqslant 1$. If $K \notin \mathscr{S}_e^n$, then there exists $L \in \mathscr{S}_e^n$ such that when $i < n-p$,*

$$I_{p,i}K \subset I_{p,i}L$$

has

$$\widetilde{W}_i(K) > \widetilde{W}_i(L).$$

2. Preliminaries

2.1. L_p dual mixed quermassintegrals

For $K,L \in \mathscr{S}_o^n$, $p > 0$ and $\lambda, \mu \geqslant 0$ (not both zero), the L_p radial combination, $\lambda \circ K \tilde{+}_p \mu \circ L \in \mathscr{S}_o^n$, of K and L is defined, cf. [34], by

$$\rho(\lambda \circ K \tilde{+}_p \mu \circ L, \cdot)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p. \tag{2.1}$$

The dual quermassintegrals of a body $K \in \mathscr{S}_o^n$ is

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n-i} dS(u). \tag{2.2}$$

Obviously,

$$\widetilde{W}_0(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^n dS(u) = V(K).$$

For $p \geqslant 1$, the L_p dual mixed quermassintegrals, $\widetilde{W}_{p,i}(K,L)$, of $K,L \in \mathscr{S}_o^n$ was defined, in [31], by

$$\frac{n-i}{p} \widetilde{W}_{p,i}(K,L) = \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K \tilde{+}_p \varepsilon \circ L) - \widetilde{W}_i(K)}{\varepsilon}. \tag{2.3}$$

From definition (2.3), the following integral representation of L_p dual mixed quermassintegrals was given in [31]: If $K,L \in \mathscr{S}_o^n$ and $p \geqslant 1$, then

$$\widetilde{W}_{p,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n-p-i} \rho(L,u)^p dS(u). \tag{2.4}$$

Apparently,

$$\widetilde{W}_{p,i}(K,K) = \widetilde{W}_i(K).$$

The Minkowski inequalities for the L_p dual mixed quermassintegrals were established in [31]: If $K,L \in \mathscr{S}_o^n$ and $p \geqslant 1$, then for $i < n-p$,

$$\widetilde{W}_{p,i}(K,L) \leqslant \widetilde{W}_i(K)^{\frac{n-p-i}{n-i}} \widetilde{W}_i(L)^{\frac{p}{n-i}}; \tag{2.5}$$

for $n-p < i < n$ or $i > n$,

$$\widetilde{W}_{p,i}(K,L) \geqslant \widetilde{W}_i(K)^{\frac{n-p-i}{n-i}} \widetilde{W}_i(L)^{\frac{p}{n-i}}. \tag{2.6}$$

In every case, equality holds if and only if K is a dilate of L.

2.2. L_p dual mixed Blaschke body

For $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\lambda, \mu \geqslant 0$ (not both zero), the L_p dual mixed Blaschke combination, $\lambda \star K \check{+}_{p,i} \mu \star L \in \mathscr{S}_o^n$, of K and L is defined by

$$\rho(\lambda \star K \check{+}_{p,i} \mu \star L, \cdot)^{n-p-i} = \lambda \rho(K, \cdot)^{n-p-i} + \mu \rho(L, \cdot)^{n-p-i}. \tag{2.7}$$

Taking $\lambda = \mu = \frac{1}{2}$, $L = -K$ in (2.7), the L_p dual mixed Blaschke body, $\breve{\nabla}_{p,i}K$, of K is defined by

$$\breve{\nabla}_{p,i}K = \frac{1}{2} \star K \breve{+}_{p,i} \frac{1}{2} \star (-K). \tag{2.8}$$

Obviously, the L_p dual mixed Blaschke body is origin-symmetric.

3. Proofs of Theorems 1.1-1.2

The proof of Theorem 1.1 needs the following lemma.

LEMMA 3.1. *If $K, L \in \mathscr{S}_o^n$, then for $p \geqslant 1$,*

$$\widetilde{W}_{p,i}(K, I_{p,i}L) = \widetilde{W}_{p,i}(L, I_{p,i}K).$$

Proof. From (1.4), (1.5) and (2.4), it follows that

$$\begin{aligned}
\widetilde{W}_{p,i}(K, I_{p,i}L) &= \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-p-i}\rho(I_{p,i}L,u)^p dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \frac{1}{(n-1)\omega_{n-1}}\rho(K,u)^{n-p-i}R(\rho_L^{n-p-i})(u)dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \frac{1}{(n-1)\omega_{n-1}}R(\rho_K^{n-p-i})(u)\rho(L,u)^{n-p-i}dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{n-p-i}\rho(I_{p,i}K,u)^p dS(u) = \widetilde{W}_{p,i}(L, I_{p,i}K). \quad \square
\end{aligned}$$

Proof of Theorem 1.1. For a star body $\overline{K}$ with $I_{p,i}\overline{K} = K$, it follows from Lemma 3.1 that

$$\begin{aligned}
\widetilde{W}_i(K) = \widetilde{W}_{p,i}(K,K) = \widetilde{W}_p(K, I_{p,i}\overline{K}) = \widetilde{W}_{p,i}(\overline{K}, I_{p,i}K); \\
\widetilde{W}_{p,i}(L,K) = \widetilde{W}_{p,i}(L, I_{p,i}\overline{K}) = \widetilde{W}_{p,i}(\overline{K}, I_{p,i}L).
\end{aligned}$$

Since

$$\begin{aligned}
\widetilde{W}_{p,i}(\overline{K}, I_{p,i}K) &= \frac{1}{n}\int_{S^{n-1}} \rho(\overline{K},u)^{n-p-i}\left(\frac{\rho(I_{p,i}K,u)}{\rho(I_{p,i}L,u)}\right)^p \rho(I_{p,i}L,u)^p dS(u) \\
&\leqslant \max_{u\in S^{n-1}}\left(\frac{\rho(I_{p,i}K,u)}{\rho(I_{p,i}L,u)}\right)^p \widetilde{W}_{p,i}(\overline{K}, I_{p,i}L),
\end{aligned}$$

we have

$$\frac{\widetilde{W}_i(K)}{\widetilde{W}_{p,i}(\overline{K}, I_{p,i}L)} \leqslant \max_{u\in S^{n-1}}\left(\frac{\rho(I_{p,i}K,u)}{\rho(I_{p,i}L,u)}\right)^p.$$

From $I_{p,i}K \subseteq I_{p,i}L$, we obtain that for $i < n-p$,

$$\widetilde{W}_i(K) \leqslant \widetilde{W}_{p,i}(\overline{K}, I_{p,i}L) = \widetilde{W}_{p,i}(L,K) \leqslant \widetilde{W}_i(L)^{\frac{n-p-i}{n-i}}\widetilde{W}_i(K)^{\frac{p}{n-i}},$$

i.e.,

$$\widetilde{W}_i(K) \leqslant \widetilde{W}_i(L).$$

From the equality condition of (2.5) and the condition $I_{p,i}K \subseteq I_{p,i}L$, we know that equality holds if and only if $K = L$.

Let $I_{p,i}\overline{L} = L$ for a star body $\overline{L}$. By Lemma 3.1, we have

$$\widetilde{W}_i(L) = \widetilde{W}_{p,i}(L,L) = \widetilde{W}_{p,i}(L, I_{p,i}\overline{L}) = \widetilde{W}_{p,i}(\overline{L}, I_{p,i}L);$$
$$\widetilde{W}_{p,i}(K,L) = \widetilde{W}_{p,i}(K, I_{p,i}\overline{L}) = \widetilde{W}_{p,i}(\overline{L}, I_{p,i}K).$$

Thus

$$\begin{aligned}\widetilde{W}_{p,i}(\overline{L}, I_{p,i}K) &= \frac{1}{n}\int_{S^{n-1}} \rho(\overline{L},u)^{n-p-i}\left(\frac{\rho(I_{p,i}K,u)}{\rho(I_{p,i}L,u)}\right)^p \rho(I_{p,i}L,u)^p dS(u)\\ &\leqslant \max_{u\in S^{n-1}}\left(\frac{\rho(I_{p,i}K,u)}{\rho(I_{p,i}L,u)}\right)^p \widetilde{W}_{p,i}(\overline{L}, I_{p,i}L),\end{aligned}$$

i.e.

$$\frac{\widetilde{W}_{p,i}(K,L)}{\widetilde{W}_i(L)} \leqslant \max_{u\in S^{n-1}}\left(\frac{\rho(I_{p,i}K,u)}{\rho(I_{p,i}L,u)}\right)^p.$$

From $I_{p,i}K \subseteq I_{p,i}L$, it follows that

$$\widetilde{W}_i(L) \geqslant \widetilde{W}_{p,i}(K,L).$$

The above inequality implies that for $i > n$,

$$\widetilde{W}_i(K) \leqslant \widetilde{W}_i(L). \quad \square$$

LEMMA 3.2. *If $K, L \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\lambda, \mu \geqslant 0$ (not both zero), then for $i < n-p$,*

$$\widetilde{W}_i(\lambda \star K \check{+}_{p,i} \mu \star L)^{\frac{n-p-i}{n-i}} \leqslant \lambda \widetilde{W}_i(K)^{\frac{n-p-i}{n-i}} + \mu \widetilde{W}_i(L)^{\frac{n-p-i}{n-i}}; \tag{3.1}$$

for $n-p < i < n$ or $i > n$,

$$\widetilde{W}_i(\lambda \star K \check{+}_{p,i} \mu \star L)^{\frac{n-p-i}{n-i}} \geqslant \lambda \widetilde{W}_i(K)^{\frac{n-p-i}{n-i}} + \mu \widetilde{W}_i(L)^{\frac{n-p-i}{n-i}}. \tag{3.2}$$

In every inequality, with equality if and only if K and L are dilates.

Proof. From (2.4), (2.5) and (2.7), we have for any $Q \in \mathscr{S}_o^n$ and $i < n-p$

$$\begin{aligned}\widetilde{W}_{p,i}(\lambda \star K \check{+}_{p,i} \mu \star L, Q) &= \lambda \widetilde{W}_{p,i}(K,Q) + \mu \widetilde{W}_{p,i}(L,Q)\\ &\leqslant \left[\lambda \widetilde{W}_i(K)^{\frac{n-p-i}{n-i}} + \mu \widetilde{W}_i(L)^{\frac{n-p-i}{n-i}}\right] \widetilde{W}_i(Q)^{\frac{p}{n-i}}.\end{aligned} \tag{3.3}$$

Let $Q = \lambda \star K \check{+}_{p,i} \mu \star L$ in (3.3). Thus we have

$$\widetilde{W}_i(\lambda \star K \check{+}_{p,i} \mu \star L)^{\frac{n-p-i}{n-i}} \leqslant \lambda \widetilde{W}_i(K)^{\frac{n-p-i}{n-i}} + \mu \widetilde{W}_i(L)^{\frac{n-p-i}{n-i}}.$$

For $n-p<i<n$ or $i>n$, similar to the above method, we have

$$\widetilde{W}_i(\lambda\star K\check{+}_{p,i}\mu\star L)^{\frac{n-p-i}{n-i}}\geqslant\lambda\widetilde{W}_i(K)^{\frac{n-p-i}{n-i}}+\mu\widetilde{W}_i(L)^{\frac{n-p-i}{n-i}}.$$

Together with the equality conditions of (2.5) and (2.6), we see that equality in every inequality holds if and only if K and L are dilates. □

Let $\lambda=\mu=\frac{1}{2}$, $L=-K$ in (3.1) and (3.2). Then the following is an immediate result of Lemma 3.2.

COROLLARY 3.1. *If $K\in\mathscr{S}_o^n$ and $p\geqslant 1$, then for $i<n-p$ or $n-p<i<n$,*

$$\widetilde{W}_i(\check{\nabla}_{p,i}K)\leqslant\widetilde{W}_i(K); \tag{3.4}$$

for $i>n$,

$$\widetilde{W}_i(\check{\nabla}_{p,i}K)\geqslant\widetilde{W}_i(K). \tag{3.5}$$

In every inequality, with equality if and only if K is origin-symmetric.

LEMMA 3.3. *If $K\in\mathscr{S}_o^n$, then for $p\geqslant 1$,*

$$I_{p,i}(\check{\nabla}_{p,i}K)=I_{p,i}K.$$

Proof. From (1.5), (2.7) and (2.8), we have

$$\begin{aligned}\rho(I_{p,i}(\check{\nabla}_{p,i}K),u)^p&=\frac{1}{(n-1)\omega_{n-1}}\int_{S^{n-1}\cap u^\perp}\rho\left(\frac{1}{2}\star K\check{+}_{p,i}\frac{1}{2}\star(-K),v\right)^{n-p-i}dS_{n-2}(v)\\&=\frac{1}{(n-1)\omega_{n-1}}\int_{S^{n-1}\cap u^\perp}\left[\frac{1}{2}\rho(K,v)^{n-p-i}+\frac{1}{2}\rho(-K,v)^{n-p-i}\right]dS_{n-2}(v)\\&=\frac{1}{2}\rho(I_{p,i}K,u)^p+\frac{1}{2}\rho(I_{p,i}(-K),u)^p.\end{aligned}$$

From formula (1.5) we easily see $I_{p,i}(-K)=I_{p,i}(K)$. Thus, we have

$$\rho(I_{p,i}(\check{\nabla}_{p,i}K),u)^p=\rho(I_{p,i}K,u)^p,$$

i.e.,

$$I_{p,i}(\check{\nabla}_{p,i}K)=I_{p,i}K. \quad \square$$

Proof of Theorem 1.2. Since $K\notin\mathscr{S}_e^n$, (3.4) implies that for $i<n-p$,

$$\widetilde{W}_i(\check{\nabla}_{p,i}K)<\widetilde{W}_i(K).$$

Let $\varepsilon>0$ such that $\widetilde{W}_i((1+\varepsilon)\check{\nabla}_{p,i}K)<\widetilde{W}_i(K)$. Taking $L=(1+\varepsilon)\check{\nabla}_{p,i}K$ we have

$$\widetilde{W}_i(K)>\widetilde{W}_i(L).$$

However, from formula (1.5) and Lemma 3.3 we get

$$I_{p,i}L=I_{p,i}((1+\varepsilon)\check{\nabla}_{p,i}K)=(1+\varepsilon)^{\frac{n-p-i}{p}}I_{p,i}(\check{\nabla}_{p,i}K)=(1+\varepsilon)^{\frac{n-p-i}{p}}I_{p,i}K\supset I_{p,i}K. \quad \square$$

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