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THE LOGARITHMIC INTERSECTION BODY

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Abstract. Haberl and Ludwig extended the classical intersection body to L_p space, and they showed that the classical intersection body is the limit case of the L_p intersection body. In this paper, we introduce the logarithmic intersection body and prove that it is the limit case of the normalized L_p intersection body. The affine nature of the logarithmic intersection body operator is demonstrated. Furthermore, a positive answer to the log-Busemann-Petty problem is given.

1. Introduction

Intersection body was introduced by Lutwak [18]. For $K \in \mathscr{S}^n$, the intersection body, $\mathrm{I}K$, of K is the origin-symmetric star body, whose radial function is defined by, for all $u \in S^{n-1}$,

$$\rho(\mathrm{I}K,u) = \mathrm{vol}_{n-1}(K \cap u^{\perp}),$$

where vol_{n-1} denotes $(n-1)$-dimensional volume, and $u^{\perp}$ denotes the hyperplane orthogonal to u.

Intersection bodies have attracted increased interest during past two decades (see [5-6,8,11,14,16-17,27]). The greatest contribution of intersection bodies is to be used to solve the Busemann-Petty problem (see [7,10,28]).

Haberl and Ludwig [13] extended the classical intersection bodies to L_p space, and defined the notion of the normalized L_p intersection bodies. For $K \in \mathscr{S}^n, 0 < p < 1$, the L_p intersection body, I_pK, of K is the origin-symmetric star body, whose radial function is defined by, for all $u \in S^{n-1}$,

$$\rho(\mathrm{I}_pK,u)^p = \frac{1}{(n-p)} \int_{S^{n-1}} \rho(K,v)^n |\langle \rho(K,v)v,u\rangle|^{-p} dv.$$

Haberl and Ludwig [13] pointed out that the intersection body $\mathrm{I}K$, of K is obtained as a limit of L_p intersection body I_pK of K, that is for all $u \in S^{n-1}$,

$$\rho(\mathrm{I}K,u) = \lim_{p\to 1^-} \frac{1-p}{2} \rho(\mathrm{I}_pK,u)^p.$$

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Haberl [12] studied the symmetric (nonsymmetric) L_p-Busemann-Petty type problem (see also Yuan and Cheung [26]). More results on the L_p intersection body can be found in [1,13].

The normalized L_p intersection body was defined by Wang and Zhang [25]. For $K \in \mathscr{S}^n, p < 1, p \neq 0$, the normalized L_p intersection body, $\bar{\mathrm{I}}_p K$, of K is the origin-symmetric star body, whose radial function is defined by, for all $u \in S^{n-1}$,

$$\rho(\bar{\mathrm{I}}_p K,u)^p = \frac{1}{(n-p)V(K)} \int_{S^{n-1}} \rho(K,v)^n |\langle \rho(K,v)v,u\rangle|^{-p} dv.$$

The logarithmic Brunn-Minkowski theory was born due to the logarithmic Minkowski problem which was considered by Böröczky, Lutwak, Yang, and Zhang [3], while the planar logarithmic Minkowski problem was first studied by Stancu [21-22]. Böröczky, Lutwak, Yang, and Zhang [2] established the planar logarithmic Brunn-Minkowski inequality. Gardner, Hug, Weil, and Ye [9] established the dual logarithmic Brunn-Minkowski inequality. The (dual) logarithmic Brunn-Minkowski theory has attracted a lot of attention (see [2-3,9,19,21-24,30]).

Since the classical (dual) Brunn-Minkowski theory was extended to the (dual) L_p-Brunn-Minkowski theory, the (dual) L_p-Brunn-Minkowski theory has been developed. In particular, the (dual) logarithmic Brunn-Minkowski theory may be obtained as a limit of the (dual) L_p-Brunn-Minkowski theory when $p \to 0$.

Note that

$$\begin{aligned}
&\lim_{p\to 0} \log\Big[\frac{1}{(n-p)V(K)} \int_{S^{n-1}} \rho(K,v)^n |\langle v,u\rangle \rho(K,v)|^{-p} dv\Big]^{\frac{1}{p}} \\
&= \lim_{p\to 0} \frac{\log\Big[\frac{1}{(n-p)V(K)} \int_{S^{n-1}} \rho(K,v)^n |\langle v,u\rangle \rho(K,v)|^{-p} dv\Big]}{p} \\
&= \lim_{p\to 0} \frac{\log \frac{n}{n-p}}{p} + \lim_{p\to 0} \frac{\log\Big[\frac{1}{nV(K)} \int_{S^{n-1}} \rho(K,v)^n |\langle v,u\rangle \rho(K,v)|^{-p} dv\Big]}{p} \qquad (1.1)\\
&= \frac{1}{n} + \lim_{p\to 0} \frac{1}{nV(K)} \int_{S^{n-1}} \rho(K,v)^n \langle v,u\rangle \rho(K,v)|^{-p} (-\log|\langle v,u\rangle \rho(K,v)| dv) \\
&= \frac{1}{n} - \frac{1}{nV(K)} \int_{S^{n-1}} \rho(K,v)^n \log|\langle v,u\rangle \rho(K,v)| dv.
\end{aligned}$$

For $K \in \mathscr{S}^n$, the logarithmic intersection body, $\mathrm{I}_0 K$, of K is the origin-symmetric star body, whose radial function is defined by, for all $u \in S^{n-1}$,

$$\log \rho(\mathrm{I}_0 K,u) = \frac{1}{n} - \frac{1}{nV(K)} \int_{S^{n-1}} \rho(K,v)^n \log|\langle v,u\rangle \rho(K,v)| dv. \qquad (1.2)$$

Applying (1.1) and (1.2), the logarithmic intersection body $\mathrm{I}_0 K$, of K can be obtained as a limit of the normalized L_p intersection body $\bar{\mathrm{I}}_pK$ of K, that is for all $u\in S^{n-1}$,

$$\rho(\mathrm{I}_0K,u)=\lim_{p\to 0}\rho(\bar{\mathrm{I}}_pK,u). \tag{1.3}$$

Recently, Gardner, Hug, Weil and Ye [9] defined the Orlicz intersection body, $\mathrm{I}_\phi K$, of a star body K as whose radial function is given by (also see [29])

$$\rho(\mathrm{I}_\phi K,u)=\inf\{\lambda>0:\int_K\phi(\frac{1}{\lambda|u\cdot y|})dy\leqslant 1\}, \tag{1.4}$$

where $\phi:(0,\infty)\to(0,\infty)$ is a strictly decreasing function and $\phi(1)=1$. If $\phi:(0,\infty)\to(0,\infty)$ is a strictly increasing function, the inequality of the integral (1.4) is reserve.

The main purpose of this paper is to study the log-Busemann-Petty problem. Our main result can be stated as follows.

THEOREM 1.1. *Let K be a logarithmic intersection body and L be an origin-symmetric body in $\mathbb{R}^n$. If*

$$I_0K\subset I_0L,$$

then

$$V(K)\geqslant V(L),$$

with equality if and only if $K=L$.

2. Notation and background material

For general reference for the theory of convex (star) bodies the reader may wish to consult the books of Gardner [8] and Schneider [20].

The unit ball and its surface in $\mathbb{R}^n$ are denoted by B and S^{n-1}, respectively. We write $V(K)$ for the volume of the compact set K in $\mathbb{R}^n$. The radial function $\rho_K:S^{n-1}\to[0,\infty)$ of a compact star-shaped about the origin, $K\in\mathbb{R}^n$, is defined, for $u\in S^{n-1}$, by

$$\rho_K(u)=\rho(K,u)=\max\{\lambda\geqslant 0:\ \ \lambda u\in K\}. \tag{2.1}$$

If $\rho_K(\cdot)$ is positive and continuous, then K is called a star body about the origin. The set of star bodies about the origin in $\mathbb{R}^n$ is denoted by $\mathscr{S}^n$. Obviously, for $K,L\in\mathscr{S}^n$,

$$K\subseteq L\Leftrightarrow\rho_K(u)\leqslant\rho_L(u),\ \ \forall\, u\in S^{n-1}. \tag{2.2}$$

If $\dfrac{\rho_K(u)}{\rho_L(u)}$ is independent of $u\in S^{n-1}$, then we say star bodies K and L are dilates.

If $s>0$, we have

$$\rho_{sK}(u)=s\rho_K(u),\ \ \text{for all } u\in S^{n-1}. \tag{2.3}$$

If $A\in GL(n)$, we have

$$\rho_{AK}(u)=\rho_K(A^{-1}u),\ \ \text{for all } u\in S^{n-1}. \tag{2.4}$$

Let K and L be two star bodies in $\mathbb{R}^n$ and $0 \leqslant \lambda \leqslant 1$, then the log radial sum, $(1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L$, is defined by[9,24]

$$\rho_{(1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L}(u) = \rho_K(u)^{1-\lambda}\rho_L(u)^{\lambda},\ \forall u\in S^{n-1}. \tag{2.5}$$

In particular, if $\lambda = 0$, then $(1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L = K$, If $\lambda = 1$, then $(1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L = L$.

The dual log mixed volume was defined by Gardner, Hug, Weil and Ye [9] (see also [24]). Let $K,L\in\mathscr{S}^n$, the dual log mixed volume $\widetilde{V}_0(K,L)$ is defined by

$$\widetilde{V}_0(K,L) = \frac{1}{n}\int_{S^{n-1}}\log(\frac{\rho_L(u)}{\rho_K(u)})\rho_K(u)^n dS(u). \tag{2.6}$$

In particular, $\widetilde{V}_0(K,K) = 0$.

Moreover, they proved the following dual log-Minkowski inequality.

LEMMA 2.1. [9] *If K and L are two star bodies in $\mathbb{R}^n$, then*

$$\frac{\widetilde{V}_0(K,L)}{V(K)} \leqslant \frac{1}{n}\log\frac{V(L)}{V(K)}, \tag{2.7}$$

with equality if and only if K and L are dilates.

3. Main results

THEOREM 3.1. *Let $K\in\mathscr{S}^n$ and $c>0$. Then*

$$I_0(cK) = \frac{1}{c}I_0K.$$

Proof. By (1.2) and (2.3), we obtain that, for $\forall u\in S^{n-1}$,

$$\begin{aligned}\log\rho(\mathrm{I}_0 cK,u) &= \frac{1}{n} - \frac{1}{nV(cK)}\int_{S^{n-1}}\rho(cK,v)^n\log|\langle v,u\rangle\rho(cK,v)|dv\\ &= \frac{1}{n} - \frac{1}{nV(K)}\int_{S^{n-1}}\rho(K,v)^n\log|\langle v,u\rangle\rho(K,v)|dv - \log c\\ &= \log\rho(\mathrm{I}_0(K,u)) - \log c = \log\frac{\rho(\mathrm{I}_0K,u)}{c}.\end{aligned}$$

Thus, we have that $\mathrm{I}_0(cK) = \frac{1}{c}\mathrm{I}_0K$. □

It is well known that the I_p is GL(n) contravariant of weight $\frac{1}{p}$, i.e., for every $A\in\mathrm{GL}(n)$ and every star body K,

$$\mathrm{I}_p(AK) = |detA|^{\frac{1}{p}}A^{-t}\mathrm{I}_pK.$$

However, the logarithmic intersection operator I_0 is GL(n) contravariant of weight 0.

THEOREM 3.2. *Let $K \in \mathscr{S}^n$, and $A \in GL(n)$. Then*

$$I_0(AK) = A^{-t} I_0 K.$$

Proof. From (1.2) and (2.4), it follows that, for $\forall u \in S^{n-1}$,

$$\begin{aligned}\log\rho(\mathrm{I}_0 AK, u) &= \frac{1}{n} - \frac{1}{nV(AK)} \int_{S^{n-1}} \rho(AK, v)^n \log|\langle v, u\rangle \rho(AK, v)| dv \\ &= \frac{1}{n} - \frac{1}{n|A|V(K)} \int_{S^{n-1}} \rho(K, A^{-1}v)^n \log|\langle v, u\rangle \rho(K, A^{-1}v)| dv \\ &= \frac{1}{n} - \frac{1}{nV(K)} \int_{S^{n-1}} \rho(K, v)^n \log|\langle v, A^t u\rangle \rho(K, v)| dv = \log\rho(\mathrm{I}_0 K, A^t u) \\ &= \log\rho(A^{-t}\mathrm{I}_0 K, u). \quad \square\end{aligned}$$

In order to prove Theorem 1.1, the following lemmas are required.

LEMMA 3.3. *Let $K, L_1, L_2 \in \mathscr{S}^n$. If $L_1 \subseteq L_2$, then*

$$\widetilde{V}_0(K, L_1) \leqslant \widetilde{V}_0(K, L_2).$$

Proof. By (2.6), and the fact that the exponential function $\log(\cdot)$ is increasing on $(0, \infty)$, we have

$$\begin{aligned}\widetilde{V}_0(K, L_1) &= \frac{1}{n} \int_{S^{n-1}} \log(\frac{\rho_{L_1}(u)}{\rho_K(u)}) \rho_K(u)^n dS(u) \leqslant \frac{1}{n} \int_{S^{n-1}} \log(\frac{\rho_{L_2}(u)}{\rho_K(u)}) \rho_K(u)^n dS(u) \\ &= \widetilde{V}_0(K, L_2). \quad \square\end{aligned}$$

LEMMA 3.4. *Let $K, L \in \mathscr{S}^n$. Then*

$$\frac{\widetilde{V}_0(K, I_0 L)}{V(K)} = \frac{\widetilde{V}_0(L, I_0 K)}{V(L)}.$$

Proof. From (1.2), (2.6) and Fubini's theorem, it follows that

$$\begin{aligned}&\frac{\widetilde{V}_0(K, \mathrm{I}_0 L)}{V(K)} \\ &= \frac{1}{nV(K)} \int_{S^{n-1}} \rho_K(u)^n \log\frac{\rho_{\mathrm{I}_0 L}(u)}{\rho_K(u)} du = \frac{1}{nV(K)} \int_{S^{n-1}} \rho_K(u)^n (\log\rho_{\mathrm{I}_0 L}(u) - \log\rho_K(u)) du \\ &= \frac{1}{nV(K)} \int_{S^{n-1}} \rho_K(u)^n [\frac{1}{n} - \frac{1}{nV(L)} \int_{S^{n-1}} \rho_L(v)^n (\log|\langle u, v\rangle \rho_L(v)| + \log\rho_K(u)) dv] du\end{aligned}$$

$$=\frac{1}{n}-\frac{1}{n^2V(K)V(L)}\int_{S^{n-1}}\rho_K(u)^n\int_{S^{n-1}}\rho_L(v)^n\log|\langle u,v\rangle\rho_L(v)\rho_K(u)|dvdu$$

$$=\frac{1}{n}-\frac{1}{n^2V(K)V(L)}\int_{S^{n-1}}\rho_L(v)^n\int_{S^{n-1}}\rho_K(u)^n\log|\langle u,v\rangle\rho_L(v)\rho_K(u)|dudv$$

$$=\frac{1}{nV(L)}\int_{S^{n-1}}\rho_L(v)^n[\frac{1}{n}-\frac{1}{nV(K)}\int_{S^{n-1}}\rho_K(u)^n(\log|\langle u,v\rangle\rho_K(u)|+\log\rho_L(v))du]dv$$

$$=\frac{1}{nV(L)}\int_{S^{n-1}}\rho_L(v)^n(\log\rho_{\mathrm{I}_0K}(v)-\log\rho_L(v))dv=\frac{\widetilde{V}_0(L,\mathrm{I}_0K)}{V(L)}. \qquad \square$$

Now, we consider the following the log-Busemann-Petty problem. Let $K,L\in\mathscr{S}^n$. If

$$\mathrm{I}_0K\subseteq\mathrm{I}_0L,$$

does it follow that

$$V(K)\geqslant V(L)?$$

Just as the classical Busemann-Petty problem, we will show that the log-Busemann-Petty problem has an affirmative answer if K is a logarithmic intersection body.

Proof of Theorem 1.1. Since K is a logarithmic intersection body, there exists a star body M such that $K=\mathrm{I}_0M$. Using Lemma 3.4 and Lemma 3.3, we can conclude that

$$\frac{\widetilde{V}_0(L,K)}{V(L)}=\frac{\widetilde{V}_0(L,\mathrm{I}_0M)}{V(L)}=\frac{\widetilde{V}_0(M,\mathrm{I}_0L)}{V(M)}\geqslant\frac{\widetilde{V}_0(M,\mathrm{I}_0K)}{V(M)}=\frac{\widetilde{V}_0(K,\mathrm{I}_0M)}{V(K)}=0.$$

Applying the dual log-Minkowski inequality (2.7), we obtain that

$$V(K)\geqslant V(L),$$

with equality if and only if K and L are dilates. Obviously, if $V(K)=V(L)$, we must have $K=L$. $\square$

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