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THE FUNDAMENTAL INEQUALITY FOR ALGEBROID FUNCTIONS ON ANNULI CONCERNING SMALL ALGEBROID FUNCTIONS

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Abstract. The main aim of this article is to study the properties on value distribution of algebroid functions on annulus. We obtain the second fundamental theorems for algebroid function concerning small algebroid functions on annulus, which improves the previous result given by Tan [28].

1. Introduction

As we all know, Nevanlinna theory plays an important role in studying the properties of meromorphic functions in the fields of complex analysis (see Hayman [5], Yang [40] and Yi and Yang [41]). In fact, the value distribution theory of meromorphic functions occupies one of the central places in Complex Analysis. In the past several decades, numerous works are devoted to studying its connections with other areas of mathematics including topology, differential geometry, measure theory, potential theory and others; extending its inferences to wider classes of functions such as: meromorphic functions in arbitrary plane regions and Riemann surfaces, algebroid functions, functions of several variables, meromorphic curves.

In fact, algebroid function was firstly introduced by H. Poincaré, and after that, G. Darboux pointed out that it is a very important class of functions. Let $H_v(z),\dots,H_0(z)$ be analytic functions in a single connected domain $\mathbb{X}\subseteq\mathbb{C}$ without common zeros, then the irreducible equation

$$\Psi(z,f)=H_v(z)f^v+H_{v-1}(z)f^{v-1}+\cdots+H_0(z)=0$$

defines a v-valued algebroid function $f(z)$ in $\mathbb{X}\subseteq\mathbb{C}$ (see [7,26]). If $v=1$, then $f(z)$ is a meromorphic function in $\mathbb{X}$. Around 1930, G. Valiron, E. Ullrich, H. Selberg and K. L. Hiong [8,24,29,30] extended the second fundamental theorem to algebroid functions, M. Ru [22] in 2000 proved the second fundamental theorem concerning small meromorphic functions for algebroid functions, D. C. Sun, Z. S. Gao, H. F. Liu in

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2012 [26] further established the second fundamental theorem concerning small algebroid functions for ν-valued algebroid functions, which improved and extended those previous forms about the second fundamental theorem. About 90 years passed, many famous mathematicians (including G. Rémoundos, G. Valiron, E. Ullrich, H. Selberg, K. L. Hiong, Y. Z. He, etc.) had paid great attention to deal with the value distribution of algebroid functions in some complex domains, such as: the whole complex plane $\mathbb{C}$, the unit disc $\mathbb{D}$ and the angular domain Δ, and obtained a lot of interesting and important results (see [6, 9, 14, 16, 21, 27, 31, 32, 33, 34, 38, 39, 42, 15, 23, 35, 37, 3]). Because the whole complex plane $\mathbb{C}$, the unit disc $\mathbb{D}$ and the angular domain Δ can all be regarded as simple connected regions, thus, in a word, they had obtained many results of algebroid functions only in some simple connected regions. *Thus, there is a naturel question: what had happened for algebroid function in some multiply connected regions?* In 2016, Y. Tan [28] first studied the value distribution of algebroid functions on a special multiply connected region—double connected region—-the annulus, and established some basic theorems which is an analog of Nevanlinna theory of algebroid function in the whole complex plane. However, there were no the related results of algebroid function on annulus concerning small algebroid functions. In this way, the main purpose of this article is to further study the value distribution of algebroid functions on the annulus, and established the second fundamental theorem for algebroid functions concerning small algebroid functions on the annulus.

The structure of this paper is as follows. *In Section 2, we introduce some basic notations and fundamental theorems of algebroid functions on the annulus. Section 3 is devoted to discuss the second fundamental theorem of algebroid function concerning small algebroid functions on the annulus.*

2. Basic notations and fundamental theorems for algebroid functions on the annulus

In view of Doubly connected mapping theorem [1], each two connected domain is conformally equivalent to an annulus $\mathbb{A}_{rR} := \{z : r < |z| < R\}$, where $0 \leqslant r < R \leqslant +\infty$. Let $z \mapsto \frac{z}{\sqrt{rR}}$ and $R_0 = \sqrt{\frac{R}{r}}$, then $\mathbb{A}_{rR}$ can be reduced to the annulus $\mathbb{A}_{R_0} := \{z : \frac{1}{R_0} < |z| < R_0\}$, especially $\mathbb{A}_\infty := \{z : 0 < |z| < +\infty\}$ for $R_0 = +\infty$, i.e., $r = 0$, $R = +\infty$ simultaneously.

Similar to ref. [7, 26], the basic notions and theorems of algebroid functions on the annulus $\mathbb{A}$ will be showed as follows (see [28]) as follows. Let $A_\nu(z), \dots, A_0(z)$ be analytic functions on annulus $\mathbb{A}_{R_0}(1 < R_0 \leqslant +\infty)$ without common zeros, then the irreducible equation

$$\psi(z,Y) = A_\nu(z)Y^\nu + A_{\nu-1}(z)Y^{\nu-1} + \cdots + A_0(z) = 0 \tag{1}$$

defines a ν-valued algebroid function $Y(z)$ on the annulus $\mathbb{A}_{R_0}$ (see [28]). Then Eq. (1) defines a ν-valued algebroid function on the annulus $\mathbb{A}_{R_0}$. If $A_\nu(z), \dots, A_0(z)$ are all polynomials, then $Y(z)$ is an algebraic function, and if anyone of $A_\nu(z), \dots, A_0(z)$ is transcendental, then $Y(z)$ is called algebroid function. For an irreducible algebroid function $Y(z)$, we can divided the points in the complex plane into two classes: T_Y : a set of regular points of $Y(z)$, and $S_Y = \mathbb{C} - T_W$: a set of critical points of $Y(z)$. The set S_Y is an isolated set (see [7, 17].). For every $a \in T_Y$. there exist and only exist v number of regular function elements $\{(y_j(z), a)\}_{j=1}^{v}$. Throughout this manuscript, we usually denote $Y(z) = \{y_j(z)\}_{j=1}^{v}$ for convenience.

Let $Y(z)$ be a v-valued algebroid function on the annulus $\mathbb{A}_{R_0} := \{z : \frac{1}{R_0} < |z| < R_0\}$. For $1 < r < R_0 \leqslant +\infty$, the notations can be found in [28]

$$m(r,Y) = \frac{1}{v}\sum_{j=1}^{v} m(r,y_j) = \frac{1}{v}\sum_{j=1}^{v}\frac{1}{2\pi}\log^+|y_j(re^{i\theta})|d\theta,$$
$$N_1(r,Y) = \frac{1}{v}\int_{\frac{1}{r}}^{1}\frac{n_1(t,Y)}{t}dt, \quad N_2(r,Y) = \frac{1}{v}\int_{1}^{r}\frac{n_2(t,Y)}{t}dt,$$
$$m_0(r,Y) = m(r,Y) + m\left(\frac{1}{r},Y\right) - 2m(1,Y), \quad N_0(r,Y) = N_1(r,Y) + N_2(r,Y),$$

and

$$N_{x_1}(r,Y) = \frac{1}{v}\int_{\frac{1}{r}}^{1}\frac{n_{x_1}(t,Y)}{t}dt, \quad N_{x_2}(r,Y) = \frac{1}{v}\int_{1}^{r}\frac{n_{x_2}(t,Y)}{t}dt,$$
$$N_x(r,Y) = N_{x_1}(r,Y) + N_{x_2}(r,Y),$$

where $y_j(z)(j = 1,2,\ldots,v)$ is a one-valued branch of $Y(z)$, $n_1(t,Y)[n_2(t,Y)]$ is the counting functions of poles of the function $Y(z)$ in $\{z : t < |z| \leqslant 1\}[\{z : 1 < |z| \leqslant t\}]$ and counting multiplicity, and $n_{x_1}(t,Y)[n_{x_2}(t,Y)]$ is the counting function of branch points of the function $Y(z)$ in $\{z : t < |z| \leqslant 1\}[\{z : 1 < |z| \leqslant t\}]$. $N_x(r,Y)$ is the density index of branch point of $Y(z)$ on the annulus $\mathbb{A}$. The Nevanlinna characteristic of algebroid function Y on the annulus $\mathbb{A}$ is defined by

$$T_0(r,Y) = m_0(r,Y) + N_0(r,Y).$$

Similarly, for $\alpha \in \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, we have

$$N_0(r,\frac{1}{Y-\alpha}) = N_1(r,\frac{1}{Y-\alpha}) + N_2(r,\frac{1}{Y-\alpha}) = \frac{1}{v}\int_{\frac{1}{r}}^{1}\frac{n_1(t,\frac{1}{Y-\alpha})}{t}dt + \frac{1}{v}\int_{1}^{r}\frac{n_2(t,\frac{1}{Y-\alpha})}{t}dt,$$

where $n_1(t,\frac{1}{Y-\alpha})[n_2(t,\frac{1}{Y-\alpha})]$ is the counting functions of poles of the function $\frac{1}{Y(z)-\alpha}$ in $\{z : t < |z| \leqslant 1\}[\{z : 1 < |z| \leqslant t\}]$ and counting multiplicity. In addition, we denote by $\overline{n}_1(t,\frac{1}{Y-\alpha})$, $\overline{n}_2(t,\frac{1}{Y-\alpha})$ the counting function of distinct poles of the function $\frac{1}{Y-\alpha}$ in $\{z : t < |z| \leqslant 1\}$ and $\{z : 1 < |z| \leqslant t\}$, respectively. Similarly, the notations $\overline{N}_1(r,Y)$, $\overline{N}_2(r,Y)$, $\overline{N}_0(r,Y)$, and $\overline{N}_0(r,\frac{1}{Y-\alpha})$ can be defined.

Let $Y(z)$ be an algebroid function on the annulus $\mathbb{A}$, if there are λ branches of $Y(z)$ which take $a(\neq \infty)$ as the value in z_0 point, then the fractional power series is

$$Y(z) = \alpha + \beta_\tau(z-z_0)^{\frac{\tau}{\lambda}} + \beta_{\tau+1}(z-z_0)^{\frac{\tau+1}{\lambda}} + \cdots, \tag{2}$$

and $n_0(r,\alpha)=n_0(r,\frac{1}{Y-\alpha})=\sum_{Y=\alpha}\tau$, where $n_0(r,\alpha)$ is the counting function of zeros of $Y(z)-\alpha$ on the annulus $\mathbb{A}$ and counting multiplicity. If there are λ branches of $Y(z)$ which take ∞ as the value in z_0 point, then the fractional power series is

$$Y(z)=\beta_{-\tau}(z-z_0)^{-\frac{\tau}{\lambda}}+\beta_{-\tau+1}(z-z_0)^{\frac{-\tau+1}{\lambda}}+\cdots, \tag{3}$$

and $n_0(r,\infty)=n_0(r,Y)=\sum_{Y=\infty}\tau$, where $n_0(r,\infty)$ is the counting function of poles of $Y(z)-\alpha$ on the annulus $\mathbb{A}$ and counting multiplicity. $z=z_0$ is a branch point of $\lambda-1$ degree of $Y(z)$ on its Riemann Surface $\widetilde{\mathscr{M}}$. $n_x(r,Y)=\sum(\lambda-1)$ denotes the branch points of $Y(z)$ on its Riemann Surface on the annulus $\mathbb{A}$. Noting, assume that 0 is not a branch points of $Y(z)$ in this article. Obviously, for $\alpha\in\overline{\mathbb{C}}$, we show

$$n_0(r,\frac{1}{Y-\alpha})=n_0(r,\frac{1}{\psi(z,\alpha)}),\ \ N_0(r,\frac{1}{Y-\alpha})=N_0(r,\frac{1}{\psi(z,\alpha)}),$$

and especially, $N_0(r,\frac{1}{Y})=\frac{1}{v}N_0(r,\frac{1}{A_0})$ as $\alpha=0$, and $N_0(r,Y)=\frac{1}{k}N_0(r,\frac{1}{A_v})$ as $\alpha=\infty$.

In view of the above definitions, some relationship the classical characteristics of algebroid functions in between the whole plane $\mathbb{C}$ and annulus $\mathbb{A}$ can be listed below.

(a) $N_0(r,Y)=N(r,Y)+N(\frac{1}{r},Y)-2N(1,Y)$, for $r>1$,

(b) $T_0(r,Y)=T(r,Y)+T(\frac{1}{r},Y)-2T(1,Y)$, for $r>1$,

(c) $T(r,Y)-2T(1,Y)\leqslant T_0(r,Y)\leqslant T(r,Y)$.

For convenience, we give a simply proof of the above conclusions as follows.

In fact, assume that $Y(0)\neq\infty$. Since $n_1(t,Y)=n(1,Y)-n(t,Y), 0<t<1$ and $n_2(t,Y)=n(t,Y)-n(1,Y), t>1$, then

$$\begin{aligned}
N_0(r,Y)=&\frac{1}{v}\int_{\frac{1}{r}}^{1}\frac{n(1,Y)-n(t,Y)}{t}dt+\frac{1}{v}\int_{1}^{r}\frac{n(t,Y)-n(1,Y)}{t}dt\\
=&\frac{1}{v}\int_{\frac{1}{r}}^{1}\frac{n(1,Y)}{t}dt-\frac{1}{v}\int_{\frac{1}{r}}^{1}\frac{n(t,Y)}{t}dt+\frac{1}{v}\int_{1}^{r}\frac{n(t,Y)}{t}dt-\frac{1}{v}\int_{1}^{r}\frac{n(1,Y)}{t}dt\\
=&\frac{1}{v}n(1,Y)\log r-\frac{1}{v}\int_{0}^{1}\frac{n(t,Y)}{t}dt+\frac{1}{v}\int_{0}^{\frac{1}{r}}\frac{n(t,Y)}{t}dt+\frac{1}{v}\int_{0}^{r}\frac{n(t,Y)}{t}dt\\
&-\frac{1}{v}\int_{0}^{1}\frac{n(t,Y)}{t}dt-\frac{1}{v}n(1,Y)\log r\\
=&N(r,Y)+N(\frac{1}{r},Y)-2N(1,Y).
\end{aligned}$$

The case $Y(0)=\infty$ can be proved similarly. Because $T(r,Y)=m(r,Y)+N(r,Y)$, from the above equality, then relation (b) follows immediately. Thus, (c) follows immediately from (b).

DEFINITION 2.1. Let $Y(z)$ be k-valued algebroid function which is determined by (1) on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\leqslant+\infty$. Then the order of $Y(z)$ is defined by:

i) if $R_0 = +\infty$,

$$\rho(Y) = \limsup_{r\to+\infty} \frac{\log^+ T_0(r,Y)}{\log r},$$

ii) if $R_0 < +\infty$,

$$\rho(Y) = \limsup_{r\to R_0} \frac{\log^+ T_0(r,Y)}{\log \frac{1}{R_0-r}}.$$

In addition, let $Y(z)$ be v-valued algebroid functions on the annulus $\mathbb{A}$, the following properties will be used in this paper (see [28]):

(d) $T_0(r,Y) = T_0\left(r,\frac{1}{Y}\right)$,

(e) $T_0(r,\frac{1}{Y-a}) = T_0(r,Y) + O(1)$, for every fixed $a \in \mathbb{C}$,

(f) $(q-2v)T_0(r,Y) < \sum_{j=1}^{q} N_0(r,\frac{1}{Y-a_j}) - N_1(r,Y) + S_0(r,Y)$, for $a_j \in \overline{\mathbb{C}}, j = 1,2,\ldots,q$,

where $N_1(r,Y)$ is the density index of all multiple values including finite or infinite, every τ multiple value counts $\tau - 1$, and

$$S_0(r,Y) = m_0(r,\frac{Y'}{Y}) + \sum_{j=1}^{q} m_0(r,\frac{Y'}{Y-a_j}) + O(1).$$

REMARK 2.1. From [28], (f) can be transformed to the following form

$$(q-2v)T_0(r,Y) < \sum_{j=1}^{q} \overline{N}_0(r,\frac{1}{Y-a_j}) + S_0(r,Y).$$

REMARK 2.2. For the remainder $S_0(r,Y)$ in (f), by combining with [11, Theorem 1] and [28], it follows:

i) in the case $R_0 = +\infty$,

$$S_0(r,Y) = O(\log(rT_0(r,Y))),$$

for $r \in (1,+\infty)$ outside a set of finite linear measure;

ii) in the case $R_0 < +\infty$,

$$S_0(r,Y) = O(\log(\frac{T_0(r,Y)}{R_0-r})),$$

for $r \in (1,R_0)$ except for the set E of r such that $\int_E \frac{dr}{(R_0-r)} < +\infty$.

REMARK 2.3. From Definition 2.1 and Remark 2.2, we obtain:

i) in the case $R_0 = +\infty$, if $\rho(Y) < +\infty$, then

$$S_0(r,Y) = O(\log r) = o(T_0(r,Y)), \text{ as } r \to +\infty;$$

if $\rho(Y) = +\infty$, then

$$S_0(r,Y) = O(\log(rT_0(r,Y))) = o(T_0(r,Y)), \quad \text{as} \quad r \to +\infty,$$

outside a set of finite linear measure;

ii) in the case $R_0 < +\infty$, if $\rho(Y) \in [0,+\infty)$, then

$$S_0(r,Y) = O(\log(\frac{T_0(r,Y)}{R_0 - r})) = O(\log(\frac{1}{R_0 - r})) = o(T_0(r,Y)),$$

except for the set E of r such that $\int_E \frac{dr}{(R_0-r)} < +\infty$, and if $\rho(Y) = +\infty$, then

$$S_0(r,Y) = O(\log(\frac{T_0(r,Y)}{R_0 - r})) = o(T_0(r,Y)), \quad \text{as} \quad r \to R_0-,$$

except for the set E of r such that $\int_E \frac{dr}{(R_0-r)} < +\infty$.

Similarly to Ref. [26], we also introduce some other definitions of algebroid function class on the annulus, which are used in our manuscript.

DEFINITION 2.2. Let $Y(z) = \{(y_j(z), a)\}_{j=1}^{v}$ be a v-valued algebroid function on the annulus $\mathbb{A}$. And let J_Y denote by the set of all algebroid mappings of $Y(z)$. Then the set

$$G_Y := \{g \circ Y(z); g \in J_Y\}$$

is said as the algebroid function class of $Y(z)$ on the annulus $\mathbb{A}$.

DEFINITION 2.3. The small algebroid function set X_Y of $Y(z)$ is defined as

$$X_Y := \left\{ f \in G_Y : T_0(r,f) = o[T_0(r,Y)](r \to R_0, r \notin E_f) \right\},$$

where E_f is a real number set of finite linear measure depending on f if $R_0 = +\infty$, and E_f is a set of r such that $\int_E \frac{dr}{(R_0-r)} < +\infty$ if $R_0 < +\infty$. Thus, the element in X_Y is called the small algebroid function of $Y(z)$.

REMARK 2.4. Note that all the finite or infinite complex constants belong to the set X_Y, and all the small meromorphic functions and all the small algebroid functions also belong this set.

DEFINITION 2.4. (see [26]). Let the set of all algebroid mappings of $Y(z)$ be J_Y and $G_Y := \{g \circ Y(z); g \in J_Y\}$. For any $g_1, g_2 \in J_Y$, we define:

1) Addition: $(g_1+g_2)\circ Y(z)=g_1\circ Y(z)+g_2\circ Y(z)$.

2) Subtraction: $(g_1-g_2)\circ Y(z)=g_1\circ Y(z)-g_2\circ Y(z)$.

3) Multiplication: $(g_1\cdot g_2)\circ Y(z)=(g_1\circ Y(z))\cdot(g_2\circ Y(z))$.

4) Division: $(\frac{g_1}{g_2})\circ Y(z)=g_1\circ Y(z)\cdot\frac{1}{g_2}\circ Y(z)$.

Next, we list the following theorem which is used in this paper.

THEOREM 2.1. *Let $Y(z)=\{(y_j(z),a)\}_{j=1}^{\nu}$ and $G(z)=\{(g_j(z),a)\}_{j=1}^{\nu}\in G_Y$ be two ν-valued algebroid functions. Then:*

i) $T_0(r,Y+G)\leqslant T_0(r,Y)+T_0(r,G)+O(1)$;

ii) $T_0(r,Y\cdot G)\leqslant T_0(r,Y)+T_0(r,G)+O(1)$.

Proof. By cutting the annulus $\mathbb{A}$, let $\{y_j(z)\}_{j=1}^{\nu}$ and $\{g_j(z)\}_{j=1}^{\nu}$ be ν simple-valued branches of $Y(z)$ and $G(z)$, respectively. Then

$$\begin{aligned} m_0(r,Y+G)=&\frac{1}{2\pi}\sum_{j=1}^{\nu}\int_0^{2\pi}\log^+\left|y_j(re^{i\theta})+g_j(re^{i\theta})\right|d\theta\\ &+\frac{1}{2\pi}\sum_{j=1}^{\nu}\int_0^{2\pi}\log^+\left|y_j(\frac{1}{r}e^{i\theta})+g_j(\frac{1}{r}e^{i\theta})\right|d\theta\\ &-\frac{1}{2\pi}\sum_{j=1}^{\nu}\int_0^{2\pi}\log^+\left|y_j(e^{i\theta})+g_j(e^{i\theta})\right|d\theta. \end{aligned}$$

With a view of

$$\int_0^{2\pi}\log^+\left|y_j(e^{i\theta})\right|d\theta=O(1),\quad \int_0^{2\pi}\log^+\left|g_j(e^{i\theta})\right|d\theta=O(1),\quad for\ \ j=1,2,\ldots,\nu,$$

thus, it yields

$$\begin{aligned} m_0(r,Y+G)=&\frac{1}{2\pi}\sum_{j=1}^{\nu}\left\{\int_0^{2\pi}\log^+\left|y_j(re^{i\theta})\right|d\theta+\int_0^{2\pi}\log^+\left|g_j(re^{i\theta})\right|d\theta\right\}\\ &+\frac{1}{2\pi}\sum_{j=1}^{\nu}\left\{\int_0^{2\pi}\log^+\left|y_j(\frac{1}{r}e^{i\theta})\right|d\theta+\int_0^{2\pi}\log^+\left|g_j(\frac{1}{r}e^{i\theta})\right|d\theta\right\}\\ &-\frac{1}{2\pi}\sum_{j=1}^{\nu}\left\{\int_0^{2\pi}\log^+\left|y_j(e^{i\theta})\right|d\theta+\int_0^{2\pi}\log^+\left|g_j(e^{i\theta})\right|d\theta\right\}+O(1)\\ \leqslant& m_0(r,Y)+m_0(r,G)+O(1), \end{aligned}\tag{4}$$

and by using the argument as in (4), we have

$$N_0(r,Y+G)=\frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{n_1(t,Y+G)}{t}dt+\frac{1}{\nu}\int_1^r\frac{n_2(t,Y+G)}{t}dt$$

$$\leqslant N_0(r,Y)+N_0(r,G). \tag{5}$$

Thus, from (4) and (5), i) follows. Similar to the same argument as in (4) and (5), we can prove ii) easily. □

REMARK 2.5. From the process of the proof of Theorem 2.1, it is easily to get

$$m_0(r,Y+G)\leqslant m_0(r,Y)+m_0(r,G)+O(1),\ \ N_0(r,Y+G)\leqslant N_0(r,Y)+N_0(r,G),$$
$$m_0(r,Y\cdot G)\leqslant m_0(r,Y)+m_0(r,G)+O(1),\ \ N_0(r,Y\cdot G)\leqslant N_0(r,Y)+N_0(r,G).$$

3. The fundamental theorem for algebroid function on the annulus concerning small algebroid functions

In this article, our main aim is to investigate the question: *Could the conclusions of Lemma 3.5 in [28] still hold when p distinct complex numbers $a_j(j=1,2,\dots,p)$ are replaced by p small algebroid functions $a_j(z)(j=1,2,\dots,p)$?* We obtain the following theorem, which is a positive answer to this question.

THEOREM 3.1. *Suppose that $Y(z)=\{(y_j(z),a)\}_{j=1}^{v}$ is a v-valued nonconstant algebroid function on the annulus $\mathbb{A}:=\{z:\frac{1}{R_0}<|z|<R_0\}(1<R_0\leqslant+\infty)$, $\{a_t(z)\}_{t=1}^{p}\subset X_Y$ are $p>2$ distinct small algebroid functions of $Y(z)$. Then for any $\varepsilon\in(0,1)$ and $r>r_0$, we have*

$$m_0(r,Y)+\sum_{t=1}^{p}m_0(r,\frac{1}{Y(z)-a_t(z)})\leqslant(2+\varepsilon)T_0(r,Y)+2N_x(r,Y)+S_0(r,Y), \tag{6}$$

or

$$(p-2-\varepsilon)T_0(r,Y)\leqslant\sum_{t=1}^{p}N_0(r,\frac{1}{Y-a_j})+2N_x(r,Y)+S_0(r,Y), \tag{7}$$

or

$$(p-4v+2-\varepsilon)T_0(r,Y)\leqslant\sum_{t=1}^{p}N_0(r,\frac{1}{Y-a_j})+S_0(r,Y), \tag{8}$$

where $S_0(r,Y)$ is stated as in Remark 2.3.

To prove this theorem, we require some lemmas as follows.

LEMMA 3.1. *(see [26]). Suppose that $Y(z)$ is a v-valued nonconstant algebroid function and n is a positive integer. Then $\frac{Y^{(n)}}{Y}$ is the differential polynomial of $\frac{Y'}{Y}$.*

LEMMA 3.2. *(see [26]). Let $f_1,f_2,\dots,f_q$, $g\in G_Y$. Then*

$$W(f_1,f_2,\dots,f_q):=\begin{vmatrix} f_1 & f_2 & \cdots & f_q\\ f_1' & f_2' & \cdots & f_q'\\ \cdots & \cdots & & \\ f_1^{(q-1)} & f_2^{(q-1)} & \cdots & f_q^{(q-1)}\end{vmatrix}=g^qW\left(\frac{f_1}{g},\frac{f_2}{g},\cdots,\frac{f_q}{g}\right).$$

LEMMA 3.3. *(see [26]). Suppose that* $A_u = \{a_t := a_t(z)\}_{t=1}^{u} \subset X_Y$ *are* $u \geqslant 1$ *distinct small algebroid funtions. Let* $L(\chi, A_u)$ *denote the vector space spanned by finitely many* $a_1^{p_1}, a_2^{p_2}, \ldots, a_u^{p_u}$*, where integer* $p_t \geqslant 0 (t = 1,2,\ldots,u)$ *and* $\sum_{t=1}^{u} p_t = \chi (\geqslant 1)$*. Let* $dimL(\chi, A_u)$ *denote the dimension of the vector space* $L(\chi, A_u)$*. Then for any* $\varepsilon > 0$*, there exists* $\chi \geqslant 1$ *such that*

$$\frac{dimL(\chi+1, A_u)}{dimL(\chi, A_u)} < 1 + \varepsilon.$$

LEMMA 3.4. *Let* $Y(z) = \{(y_j(z), a)\}$ *be a* v*-valued nonconstant algebroid function on the annulus* $\mathbb{A}_{R_0} := \{z : \frac{1}{R_0} < |z| < R_0\} (1 < R_0 \leqslant +\infty)$*, and* $\{a_t(z)\}_{t=0}^{p} \subset X_Y$ *be* p *distinct small algebroid function with respect to* $Y(z)$*. Then for any* $\frac{1}{R_0} < r < R_0$*, we have*

$$\left| m_0\left(r, \sum_{t=1}^{p} \frac{1}{Y(z) - a_t(z)}\right) - \sum_{t=1}^{p} m_0\left(r, \frac{1}{Y(z) - a_t(z)}\right) \right| = S_0(r, Y),$$

where $S_0(r,Y)$ *is state as in Remark 2.3.*

Proof. By using the curve V through all branch points of $Y(z)$, $Y(z)$ can be cut into v single-valued branch $\{y_j(z)\}_{j=1}^{v}$ on the annulus $\mathbb{A}$. Similarly, every $a_t(z)$ can be cut into v single-valued branch $\{a_{t,j}(z)\}_{j=1}^{v}$ on the annulus $\mathbb{A}$. For any $j = 1,2,\ldots,v$, set

$$F_j(z) := \sum_{t=1}^{p} \frac{1}{y_j(z) - a_{t,j}(z)}. \tag{9}$$

In view of $a_t(z) \in X_Y$ and $\frac{1}{R_0} < r < R_0$, then it yields $m_0(r, a_t) \leqslant T_0(r, a_t) = o(T_0(r,Y))$ for $t = 1,2,\ldots,p$. Hence, we can conclude that

$$\begin{aligned}
m_0\left(r, \sum_{t=1}^{p} \frac{1}{Y(z) - a_t(z)}\right) &= \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \sum_{t=1}^{p} \frac{1}{y_j(re^{i\theta}) - a_t(re^{i\theta})} \right| d\theta \\
&\quad + \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \sum_{t=1}^{p} \frac{1}{y_j(\frac{1}{r}e^{i\theta})) - a_t(\frac{1}{r}e^{i\theta})} \right| d\theta \\
&\quad - \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \sum_{t=1}^{p} \frac{1}{y_j(e^{i\theta}) - a_t(e^{i\theta})} \right| d\theta \\
&\leqslant \sum_{j=1}^{v} \sum_{t=1}^{p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{y_j(re^{i\theta}) - a_t(re^{i\theta})} \right| d\theta \\
&\quad + \sum_{j=1}^{v} \sum_{t=1}^{p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{y_j(\frac{1}{r}e^{i\theta})) - a_t(\frac{1}{r}e^{i\theta})} \right| d\theta \\
&\quad - \sum_{j=1}^{v} \sum_{t=1}^{p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{y_j(e^{i\theta}) - a_t(e^{i\theta})} \right| d\theta
\end{aligned}$$

$$
\begin{aligned}
&+o(T_0(r,Y))+K\log p\\
&\leqslant \sum_{t=1}^{p} m_0\left(r,\frac{1}{Y(z)-a_t(z)}\right)+S_0(r,Y). \qquad (10)
\end{aligned}
$$

Now, we will show the estimation of the lower bound of $\sum_{j=1}^{v} m_0(r,F_j)$ for any $z\in\mathbb{A}$. Denote

$$\delta_j(z):=\min_{1\leqslant t<u\leqslant p}\{|a_{t,j}(z)-a_{u,j}(z)|\}\geqslant 0.$$

Since $\delta_j(z)$ is the function of $z\in\mathbb{A}$, in virtue of the uniqueness theorem, its zeros must be isolated. Take arbitrary $z\in\{z:\delta_j(z)\neq 0\}$.

Case 1. If for any $t\in\{1,2,\ldots,p\}$,

$$|y_j(z)-a_{t,j}(z)|\geqslant\frac{\delta_j(z)}{2p},$$

then it yields

$$\sum_{t=1}^{p}\log^+\frac{1}{|y_j(z)-a_{t,j}(z)|}\leqslant p\log^+\frac{2q}{\delta_j(z)}. \qquad (11)$$

Case 2. If there exists some $u\in\{1,2,\ldots,p\}$ such that

$$|y_j(z)-a_{u,j}(z)|\leqslant\frac{\delta_j(z)}{2p}. \qquad (12)$$

Thus, for $t\neq u$, it follows

$$|y_j(z)-a_{t,j}(z)|\geqslant|a_{u,j}(z)-a_{t,j}(z)|-|y_j(z)-a_{u,j}(z)|\geqslant\delta_j(z)-\frac{\delta_j(z)}{2p}=\frac{2p-1}{2p}\delta_j(z).$$

Hence, by combining with (12), it yields

$$
\begin{aligned}
\frac{1}{|y_j(z)-a_{t,j}(z)|}&\leqslant\frac{1}{2p-1}\frac{2p}{\delta_j(z)} \qquad (13)\\
&<\frac{1}{2p-1}\frac{1}{|y_j(z)-a_{u,j}(z)|}. \qquad (14)
\end{aligned}
$$

Thus, with a view of (9) and (14), we can deduce that

$$
\begin{aligned}
|F_j(z)|&\geqslant\frac{1}{|y_j(z)-a_{u,j}(z)|}-\sum_{t\neq u}\frac{1}{|y_j(z)-a_{t,j}(z)|}\\
&\geqslant\frac{1}{|y_j(z)-a_{u,j}(z)|}-\frac{p-1}{2p-1}\frac{1}{|y_j(z)-a_{t,j}(z)|}>\frac{1}{2|y_j(z)-a_{u,j}(z)|},
\end{aligned}
$$

this leads to

$$\log^+|F_j(z)|>\log^+\frac{1}{|y_j(z)-a_{u,j}(z)|}-\log 2$$

$$
\begin{aligned}
&= \sum_{t=1}^{p} \log^+ \frac{1}{|y_j(z) - a_{t,j}(z)|} - \sum_{t \neq u} \log^+ \frac{1}{|y_j(z) - a_{t,j}(z)|} - \log 2 \\
&\geqslant \sum_{t=1}^{p} \log^+ \frac{1}{|y_j(z) - a_{t,j}(z)|} - \sum_{t \neq u} \frac{2p}{(2p-1)\delta_j(z)} - \log 2 \\
&> \sum_{t=1}^{p} \log^+ \frac{1}{|y_j(z) - a_{t,j}(z)|} - p \log^+ \frac{2p}{\delta_j(z)} - \log 2. \qquad (15)
\end{aligned}
$$

Hence, combining with Case 1 and Case 2, we can conclude from (11) and (15) that

$$
\log^+ |F_j(z)| > \sum_{t=1}^{p} \log^+ \frac{1}{|y_j(z) - a_{t,j}(z)|} - p \log^+ \frac{2p}{\delta_j(z)} - \log 2. \qquad (16)
$$

With a view of definition $\delta_j(z)$ and the choice of z, then there exists $t(z) \neq u(z)$ such that $\delta_j(z) = a_{t(z),j}(z) - a_{u(z),j}(z)$. Thus, it follows

$$
\frac{1}{\delta_j(z)} = \frac{1}{|a_{t(z),j}(z) - a_{u(z),j}(z)|} \leqslant \sum_{1 \leqslant t < u \leqslant p} \frac{1}{|a_{t,j}(z) - a_{u,j}(z)|}.
$$

This leads to

$$
\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\delta_j(re^{i\theta})} d\theta &\leqslant \sum_{1 \leqslant t < u \leqslant p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|a_{t,j}(re^{i\theta}) - a_{u,j}(re^{i\theta})|} d\theta + O(1) \\
&= \sum m(r, a_{t,j}(z) - a_{u,j}(z)) + O(1) \\
&\leqslant \sum m(r, a_{t,j}) + m(r, a_{u,j}) + O(1), \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\delta_j(\frac{1}{r}e^{i\theta})} d\theta &\leqslant \sum_{1 \leqslant t < u \leqslant p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|a_{t,j}(\frac{1}{r}e^{i\theta}) - a_{u,j}(\frac{1}{r}e^{i\theta})|} d\theta + O(1) \\
&= \sum m\left(\frac{1}{r}, a_{t,j} - a_{u,j}\right) + O(1) \\
&\leqslant \sum m\left(\frac{1}{r}, a_{t,j}\right) + m\left(\frac{1}{r}, a_{u,j}\right) + O(1), \qquad (18)
\end{aligned}
$$

and

$$
\frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\delta_j(e^{i\theta})} d\theta = O(1). \qquad (19)
$$

Thus, it yields from (17)-(19) that

$$
\begin{aligned}
&\sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{d\theta}{\delta_j(re^{i\theta})} + \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{d\theta}{\delta_j(\frac{1}{r}e^{i\theta})} - \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{d\theta}{\delta_j(e^{i\theta})} \\
&\leqslant \sum T_0(r, a_{t,j}) + T_0(r, a_{u,j}) + O(1) = S_0(r, Y). \qquad (20)
\end{aligned}
$$

Substituting $z = re^{i\theta}$, $z = \frac{1}{r}e^{i\theta}$ and $z = e^{i\theta}$ into (16), respectively, and integrating on θ from 0 to 2π, then we conclude from (20) that

$$\begin{aligned}
& m_0\left(r, \sum_{t=1}^{p} \frac{1}{Y(z)-a_t(z)}\right) \\
= & \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F_j(re^{i\theta})| d\theta + \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F_j(\frac{1}{r}e^{i\theta})| d\theta \\
& - \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F_j(e^{i\theta})| d\theta \\
\geqslant & \sum_{j=1}^{v} \sum_{t=1}^{p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|y_j(re^{i\theta}) - a_{t,j}(re^{i\theta})|} d\theta \\
& + \sum_{j=1}^{v} \sum_{t=1}^{p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|y_j(\frac{1}{r}e^{i\theta}) - a_{t,j}(\frac{1}{r}e^{i\theta})|} d\theta \\
& - \sum_{j=1}^{v} \sum_{t=1}^{p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|y_j(e^{i\theta}) - a_{t,j}(e^{i\theta})|} d\theta - \sum_{j=1}^{v} \frac{p}{2\pi} \int_0^{2\pi} \log^+ \frac{2p}{\delta_j(re^{i\theta})} d\theta \\
& - \sum_{j=1}^{v} \frac{p}{2\pi} \int_0^{2\pi} \log^+ \frac{2p}{\delta_j(\frac{1}{r}e^{i\theta})} d\theta + \sum_{j=1}^{v} \frac{p}{2\pi} \int_0^{2\pi} \log^+ \frac{2p d\theta}{\delta_j(e^{i\theta})} + O(1) \\
\geqslant & \sum_{t=1}^{p} m_0\left(r, \frac{1}{Y(z)-a_j(z)}\right) + S_0(r,Y).
\end{aligned} \tag{21}$$

Therefore, this completes the proof of this lemma from (10) and (21). □

LEMMA 3.5. *(see [28, Lemma 3.3]). Let $Y(z)$ be a v-valued algebroid function which is determined by* (1) *on the annulus $\mathbb{A}$, then*

$$N_x(r,Y) \leqslant 2(v-1)T_0(r,Y) + O(1).$$

The proof of Theorem 3.1. : We will adapt the method of [26]. Let $\Delta_p = \{a_1, a_2, \ldots, a_p\}$ and $L(\chi, \Delta_p)$ be the vector space spanned by finitely many $a_1^{n_1} a_2^{n_2} \ldots a_p^{n_p}$, where $n_t \geqslant 0 (t = 1, 2, \ldots, p)$ and $\sum_{t=1}^{p} = \chi$, and $dimL(\chi, \Delta_p) = n$ for given χ. Further, let $b_1, b_2, \ldots, b_n$ be a basis of $L(\chi, \Delta_p)$. In addition, let $dimL(\chi+1, \Delta_p) = l$, thus we further denote by $\lambda_1, \lambda_2, \ldots, \lambda_l$ a basis of $L(\chi+1, \Delta_p)$. By Lemma 3.3, for any $\varepsilon > 0$, there exists some χ such that

$$1 \leqslant \frac{l}{n} < 1 + \varepsilon. \tag{22}$$

Let

$$\gamma(Y) := W(\lambda_1, \lambda_2, \ldots, \lambda_l, Yb_1, Yb_2, \ldots, Yb_n).$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_l, Yb_1, Yb_2, \ldots, Yb_n$ are linearly independent and $P(Y) \not\equiv 0$, then in view of the definition of the Wronskian determinant, it follows

$$\gamma(Y) = \sum C_q(z) \prod_{t=0}^{n+l-1} (Y^{(t)})^{q_t} = Y^n \sum C_q(z) \prod_{t=0}^{n+l-1} \left(\frac{Y^{(t)}}{Y}\right)^{q_t}. \tag{23}$$

From [11, Theorem 1] and [28], we have $m_0\left(r, \frac{Y'}{Y}\right) = S_0(r,Y)$. Thus, by combining this with (23), it follows

$$m_0(r,\gamma(Y)) \leqslant nm_0(r,Y) + S_0(r,Y). \tag{24}$$

Meanwhile, in view of Lemma 3.2, we have

$$W(\lambda_1,\lambda_2,\ldots,\lambda_l,Yb_1,\ldots,Yb_n) = \gamma(Y) = Y^{n+l}\cdot W\left(\frac{\lambda_1}{Y},\ldots,\frac{\lambda_l}{Y},b_1,\ldots,b_n\right). \tag{25}$$

(i) Let $(q(z),z_0)$ be a meromorphic fuction element or multivalent algebraic function element of $Y(z)$. If z_0 is a τ-fold pole of $q(z)$, then in view of the right side of Eq. (25), it follows that $(q(z),z_0)$ is the pole of $\gamma(Y)$ with order $(n+l)\tau$, outside the poles of the small algebroid functions $\{\lambda_i\},\{b_t\}$; If z_0 is a zero of $q(z)$, then in view of the left side of Eq. (25), it follows that $(q(z),z_0)$ is not the pole of $\gamma(Y)$, outside the poles of the small algebroid functions $\{\lambda_i\},\{b_t\}$.

(ii) Denote

$$W_j(\lambda_1,\ldots,\lambda_l,Yb_1,\ldots,Yb_n) := W(\lambda_1,\ldots,\lambda_{l-1},\lambda_{l+1},\ldots,\lambda_k,Yb_1,\ldots,Yb_n),$$

for any $1\leqslant j\leqslant l$, and

$$W_j(\lambda_1,\ldots,\lambda_l,Yb_1,\ldots,Yb_n) := W(\lambda_1,\ldots,\lambda_l,\ldots,\lambda_k,Yb_1,\ldots,Yb_{l-1},Yb_{l+1},\ldots,Yb_n),$$

if $l<j\leqslant n+l$. Assume that z_0 is not the pole of $q(z)$ and $(q(z),z_0)$ is any λ-sheeted algebraic function element of $Y(z)$. Then z_0 is at most the pole of $q(z)$ with the order $\lambda-1$. By Lemma 3.2, it follows

$$\begin{aligned}
\gamma(Y) &= \sum_{j=1}^{l}[(-1)^{j+1}\lambda_j W_j(\lambda_1',\ldots,\lambda_l',(Yb_1)',\ldots,(Yb_n)')] \\
&\quad + \sum_{j=l+1}^{n+l}[(-1)^{j+1}Yb_j\cdot W_j(\lambda_1',\ldots,\lambda_l',(Yb_1)',\ldots,(Yb_n)')] \\
&= \sum_{j=1}^{l}(-1)^{j+1}\lambda_j(Y'b_j+Yb_j')^{n+l-1} \\
&\qquad \times W_j\left(\frac{\lambda_1'}{(Yb_j)'},\ldots,\frac{\lambda_l'}{(Yb_j)'},\frac{(Yb_1)'}{(Yb_j)'},\ldots,\frac{(Yb_l)'}{(Yb_j)'}\right) \\
&\quad + \sum_{j=l+1}^{n+l}(-1)^{j+1}Yb_j(Y'b_j+Yb_j')^{n+l-1} \\
&\qquad \times W_j\left(\frac{\lambda_1'}{(Yb_j)'},\ldots,\frac{\lambda_l'}{(Yb_j)'},\frac{(Yb_1)'}{(Yb_j)'},\ldots,\frac{(Yb_l)'}{(Yb_j)'}\right).
\end{aligned}$$

Thus the order of pole of $\gamma(Y)$ at $(q(z),z_0)$ is less than $(\lambda-1)(n+l-1)$, outside the poles of the small algebroid functions $\{\lambda_i\},\{b_t\}$. Hence it follows from (i) and (ii) that

$$N_0(r,\gamma(Y)) \leqslant (n+l)N_0(r,Y) + (n+l-1)N_x(r,Y) + S_0(r,Y).$$

By applying Theorem 2.1 and in view of (25), we can deduce

$$T_0(r,\gamma(Y)) \leqslant nT_0(r,Y)+lN_0(r,Y)+(n+l-1)N_x(r,Y)+S_0(r,Y). \tag{26}$$

Let a be a linear combination of $\{a_t\}$, then

$$\begin{aligned}\gamma(Y-a) &= W(\lambda_1,\lambda_2,\dots,\lambda_l,Yb_1-ab_1,Yb_2-ab_2,\dots,Yb_n-ab_n)\\ &= W(\lambda_1,\lambda_2,\dots,\lambda_l,Yb_1,Yb_2,\dots,Yb_n) \pm \sum W(\lambda_1,\lambda_2,\dots,\lambda_l,\dots),\end{aligned}$$

where the element "..." behind λ_l in $\sum W(\lambda_1,\lambda_2,\dots,\lambda_l,\dots)$ consists of ab_t. But ab_t and $\lambda_1,\lambda_2,\dots,\lambda_l$ are linearly dependent, thus it follows $\sum W(\lambda_1,\lambda_2,\dots,\lambda_l,\dots)=0$. So, we obtain

$$\gamma(Y-a)=\gamma(Y). \tag{27}$$

Thus, it follows from Lemma 3.1 and (24) that

$$\gamma(Y)=Y^n\cdot\varphi\left(\frac{Y'}{Y}\right), \tag{28}$$

where $\varphi\left(\frac{Y'}{Y}\right)$ is the differential polynomial of $\frac{Y'}{Y}$. Let

$$V_t := Y-a_t,\ \ \varphi_t := \varphi\left(\frac{V_t'}{V_t}\right),\quad t=1,2,\dots,p.$$

Hence, by combining with (27) and (28), we obtain $\gamma(Y)=\gamma(V_t)=V_t^n\varphi_t$, that is,

$$\frac{1}{(Y-a_t)^n}=\frac{\varphi_t}{\gamma(Y)}.$$

Thus, this leads to

$$\frac{1}{|Y-a_t|}=\frac{|\varphi_t|^{n^{-1}}}{|\gamma(Y)|^{n^{-1}}}. \tag{29}$$

Set

$$F(z):=\sum_{t=1}^{p}\frac{1}{Y(z)-a_j(z)},$$

then in view of Lemma 3.4 and (29), it yields

$$m_0(r,F)=m_0\left(r,\sum_{t=1}^{p}\frac{1}{Y(z)-a_t(z)}\right)=\sum_{t=1}^{p}m_0\left(r,\frac{1}{Y(z)-a_t(z)}\right)+S_0(r,Y), \tag{30}$$

and

$$|F(z)|\leqslant\sum_{t=1}^{p}\frac{1}{|Y(z)-a_t(z)|}\leqslant\frac{1}{|\gamma(Y)|^{n^{-1}}}\sum_{t=1}^{p}|\varphi_t|^{n^{-1}}, \tag{31}$$

which leads to

$$
\begin{aligned}
m_0(r,F) &\leqslant \frac{1}{n}m_0\left(r,\frac{1}{\gamma(Y)}\right)+\frac{1}{n}\sum_{t=1}^{p}m_0(r,\varphi_t)+O(1)\\
&\leqslant \frac{1}{n}T_0(r,\gamma(Y))-\frac{1}{n}N_0\left(r,\frac{1}{\gamma(Y)}\right)+S_0(r,Y)\\
&\leqslant T_0(r,Y)+\frac{l}{n}N_0(r,Y)+\frac{n+l-1}{n}N_x(r,Y)-\frac{1}{n}N_0\left(r,\frac{1}{\gamma(Y)}\right)+S_0(r,Y)\\
&\leqslant T_0(r,Y)+\frac{l}{n}N_0(r,Y)+2N_x(r,Y)-\frac{1}{n}N_0\left(r,\frac{1}{\gamma(Y)}\right)+S_0(r,Y). \quad (32)
\end{aligned}
$$

Hence from (22), (30) and (32), it yields

$$
\begin{aligned}
m_0(r,Y)+\sum_{t=1}^{p}m_0\left(r,\frac{1}{Y(z)-a_t(z)}\right) &\leqslant \frac{l}{n}m_0(r,Y)+m_0(r,F)\\
&\leqslant (1+\frac{l}{n})T_0(r,Y)+2N_x(r,Y)+S_0(r,Y)\\
&< (2+\varepsilon)T_0(r,Y)+2N_x(r,Y)+S_0(r,Y), \quad (33)
\end{aligned}
$$

which leads to (6).

Since

$$
\begin{aligned}
m_0\left(r,\frac{1}{Y(z)-a_t(z)}\right) &\leqslant T_0(r,Y-a_t)-N_0\left(r,\frac{1}{Y-a_t}\right)+O(1)\\
&\leqslant T_0(r,Y)-N_0\left(r,\frac{1}{Y-a_t}\right)+S_0(r,Y), \quad (34)
\end{aligned}
$$

and substituting (34) into (6), and with a view of $N_0(r,Y)\leqslant T_0(r,Y)$, we obtain (7). By Lemma 3.5 and (7), then (8) holds immediately.

Therefore, this completes the proof of Theorem 3.1. □

Competing interests

The authors declare that none of the authors have any competing interests in the manuscript.

Author's contributions

HYX completed the main part of this article, HYX and ZJW corrected the main theorems. All authors read and approved the final manuscript.

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