Journal of
Mathematical
Inequalities
Volume 13, Number 4 (2019), 969–981 doi:10.7153/jmi-2019-13-69

# AN INEQUALITY FOR DISTANCES AMONG $n$ POINTS AND DISTANCE PRESERVING MAPPINGS

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(*Communicated by J. Pečarić*)

*Abstract.* Using familiar properties of norm and inner product, we will prove a new inequality concerning distances between each pair of $n$ points in an inner product space, where $n$ is an integer larger than 3. Moreover, we investigate the Aleksandrov-Rassias problem by proving that if the distance 1 is contractive and the golden ratio is extensive by a mapping $f$, then $f$ is a linear isometry up to translation.

## 1. Introduction

Throughout this paper, $V$ denotes a real (or complex) inner product space with inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$ defined as $\|x\| = \sqrt{\langle x,x\rangle}$ for all $x \in V$. If three points (vectors) $x_1, x_2, x_3$ are the vertices of an acute triangle or a right triangle in two dimensional Euclidean space $\mathbb{E}^2$, then the inequality

$$\|x_1 - x_3\|^2 \leqslant \|x_1 - x_2\|^2 + \|x_2 - x_3\|^2$$

is true. The equality sign holds true if and only if $x_1, x_2, x_3$ are the vertices of a right triangle and the vectors $x_1 - x_2$ and $x_2 - x_3$ are orthogonal to each other. This is called the Pythagorean theorem which is the most famous theorem in mathematics.

In connection with this subject, Jung [5] and Jung and Lee [6] proved theorems concerning the inequalities for distances between every two points among $2n$ points. We will introduce the main theorem of [6] and remark that all distances between each pair of distinct two points among $2n$ points are involved in the inequality of the following theorem, *i.e.*, the number of distances involved in the inequality of the following theorem is just the ${}_{2n}\mathrm{C}_2$.

THEOREM 1.1. (Jung and Lee [6]) *Let $V$ be a real (or complex) inner product space and let $n \geqslant 2$ be an integer. For any distinct $2n$ points $x_{11}, x_{21}, \ldots, x_{n1}, x_{12}, x_{22}, \ldots, x_{n2} \in V$, the following inequality is true:*

$$\sum_{\substack{i,j \in \{1,\ldots,n\} \\ k,\ell \in \{1,2\} \\ i<j}} \|x_{ik} - x_{j\ell}\|^2 \geqslant (n-1) \sum_{i\in\{1,\ldots,n\}} \|x_{i1} - x_{i2}\|^2.$$

*Mathematics subject classification* (2010): 51M16, 51K99, 51K05, 46C99.
*Keywords and phrases*: Inequalities, Aleksandrov-Rassias problem, distance, linear isometry, point.


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*The equality sign holds if and only if for all* $i, j \in \{1, \dots, n\}$ *with* $i < j$*, the pair of four points* $\{x_{i1}, x_{i2}, x_{j1}, x_{j2}\}$ *comprises the vertices of an appropriate (possibly degenerate) parallelogram such that* $x_{i1}$ *and* $x_{j1}$ *are the opposite vertices to* $x_{i2}$ *and* $x_{j2}$*, respectively.*

It seems harder to prove an inequality for the distances between two points of odd number of points than the even number of points. Very recently, Jung and Nam [7] have succeeded in proving an inequality for distances between every two points among five points. (We remark that the number of distances (between five distinct points) involved in the inequality of the following theorem is just the ${}_5C_2$.)

THEOREM 1.2. (Jung and Nam [7]) *Let* $V$ *be a real (or complex) inner product space and let* $\phi = \frac{1+\sqrt{5}}{2}$ *be the golden ratio. For any five points* $x_1, x_2, x_3, x_4, x_5 \in V$*, the following inequality is true:*

$$\begin{aligned} &\phi^2\{\|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_3 - x_4\|^2 + \|x_4 - x_5\|^2 + \|x_5 - x_1\|^2\} \\ &\geqslant \|x_1 - x_3\|^2 + \|x_2 - x_4\|^2 + \|x_3 - x_5\|^2 + \|x_4 - x_1\|^2 + \|x_5 - x_2\|^2. \end{aligned}$$

*The equality sign is true if and only if*

$$x_4 = x_1 - \phi x_2 + \phi x_3 \quad \textit{and} \quad x_5 = \phi x_1 - \phi x_2 + x_3,$$

*for any* $x_1, x_2, x_3 \in V$.

In Chapter 3 of this paper, we prove a new inequality for the distances between every pair of $n$ points in real (or complex) inner product space, using familiar and intrinsic properties of norm and inner product, where $n$ is an integer greater than 3.

## 2. Preliminary

From now on, let $n$ be an integer larger than 3 and let $c_n$ be defined as

$$c_n = \frac{\sin\frac{3\pi}{n}}{\sin\frac{\pi}{n}},$$

where the value of $c_n$ depends on $n$ only. Then it easily follows from (2.1) that $1 \leqslant c_n < 3$.

LEMMA 2.1. *Given an integer* $n > 3$*, let*

$$c_n = \frac{\sin\frac{3\pi}{n}}{\sin\frac{\pi}{n}} \quad \textit{and} \quad A_n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -c_n & c_n \end{pmatrix}.$$

*Then* $(A_n)^n = I$.

*Proof.* Since $\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta$, it follows that

$$c_n-1=\frac{\sin\frac{3\pi}{n}}{\sin\frac{\pi}{n}}-1=\frac{\sin\frac{3\pi}{n}-\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}}=\frac{2\cos\frac{2\pi}{n}\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}}=2\cos\frac{2\pi}{n}. \tag{2.1}$$

Using (2.1), the characteristic polynomial of $A_n$, denoted by $\phi_n(t)$, is

$$\phi_n(t)=\det(tI-A_n)=\det\begin{pmatrix} t & -1 & 0\\ 0 & t & -1\\ -1 & c_n & t-c_n\end{pmatrix}=(t-1)\big(t^2+(1-c_n)t+1\big)$$
$$=(t-1)\left(t^2-2t\cos\frac{2\pi}{n}+1\right).$$

Let $\omega=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$, and let $\overline{\omega}=\cos\frac{2\pi}{n}-i\sin\frac{2\pi}{n}$. Then the eigenvalues of $A_n$ are $1$, $\omega$, and $\overline{\omega}$. Because the eigenvalues of $A_n$ are all distinct, their corresponding eigenvectors are linearly independent. Thus $A_n$ is diagonalizable.

Define $P$ by

$$P=\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\overline{\omega}\end{pmatrix}.$$

Then there exists a matrix $U$ satisfying $U^{-1}A_nU=P$. Hence,

$$U^{-1}(A_n)^nU=P^n=\begin{pmatrix}1&0&0\\0&\omega^n&0\\0&0&\overline{\omega}^n\end{pmatrix}=I.$$

Therefore $(A_n)^n=UIU^{-1}=I$. □

## 3. An inequality for distances among $n$ points

The number of distances between $n$ distinct points that are involved in the inequality of the following theorem is $3n$, which is not equal to ${}_nC_2$ unless $n=7$. Nevertheless, this inequality is new and interesting enough.

THEOREM 3.1. *Given an integer $n>3$, let $c_n=\frac{\sin\frac{3\pi}{n}}{\sin\frac{\pi}{n}}$ and $V$ a real or complex inner product space. If arbitrary $n$ points $x_1,x_2,\dots,x_n$ are given in $V$, then the following inequality holds:*

$$\left(c_n^2+2c_n\right)\sum_{i=1}^{n}\|x_i-x_{i+1}\|^2+\sum_{i=1}^{n}\|x_i-x_{i+3}\|^2\geqslant 2c_n\sum_{i=1}^{n}\|x_i-x_{i+2}\|^2, \tag{3.1}$$

*where $x_{n+1}=x_1$, $x_{n+2}=x_2$, and $x_{n+3}=x_3$ for notational convenience. The equality sign holds if and only if for previously given $x_1,x_2,x_3\in V$, the points $x_4,x_5,\dots,x_n$ are determined by the recursion formula*

$$x_{i+3}=x_i-c_nx_{i+1}+c_nx_{i+2}, \tag{3.2}$$

*for all $i\in\{1,2,\dots,n-3\}$.*

*Proof.* Because the proof of this theorem for real inner product spaces is similar as the complex case, we will prove this theorem when $V$ is a complex inner product space.

Let $S_j = \sum_{i=1}^{n} \langle x_i, x_{i+j} \rangle$ for each $j \in \{0,1,2,3\}$. Then for each $j \in \{0,1,2,3\}$, we get

$$\begin{aligned}\sum_{i=1}^{n} \|x_i - x_{i+j}\|^2 &= \sum_{i=1}^{n} \langle x_i - x_{i+j},\, x_i - x_{i+j} \rangle \\ &= \sum_{i=1}^{n} \left\{ \langle x_i, x_i \rangle - \langle x_i, x_{i+j} \rangle - \overline{\langle x_i, x_{i+j} \rangle} + \langle x_{i+j}, x_{i+j} \rangle \right\} \\ &= \sum_{i=1}^{n} \left\{ 2\langle x_i, x_i \rangle - \langle x_i, x_{i+j} \rangle - \overline{\langle x_i, x_{i+j} \rangle} \right\} = 2S_0 - S_j - \overline{S}_j,\end{aligned} \tag{3.3}$$

where $\overline{c}$ denotes the complex conjugation of a complex number $c$.

Since

$$\begin{aligned}\sum_{i=1}^{n} \langle x_i - x_{i+3},\, x_{i+1} - x_{i+2} \rangle &= \sum_{i=1}^{n} \left\{ \langle x_i, x_{i+1} \rangle + \overline{\langle x_{i+2}, x_{i+3} \rangle} - \overline{\langle x_{i+1}, x_{i+3} \rangle} - \langle x_i, x_{i+2} \rangle \right\} \\ &= \sum_{i=1}^{n} \langle x_i, x_{i+1} \rangle + \sum_{i=1}^{n} \overline{\langle x_i, x_{i+1} \rangle} - \sum_{i=1}^{n} \langle x_i, x_{i+2} \rangle - \sum_{i=1}^{n} \overline{\langle x_i, x_{i+2} \rangle} \\ &= (S_1 + \overline{S}_1) - (S_2 + \overline{S}_2),\end{aligned} \tag{3.4}$$

it follows that

$$\sum_{i=1}^{n} \overline{\langle x_i - x_{i+3},\, x_{i+1} - x_{i+2} \rangle} = (S_1 + \overline{S}_1) - (S_2 + \overline{S}_2), \tag{3.5}$$

and by (3.3), (3.4) and (3.5), the following inequality is true:

$$\begin{aligned}0 &\leqslant \sum_{i=1}^{n} \|x_i - x_{i+3} - c_n(x_{i+1} - x_{i+2})\|^2 \qquad (3.6) \\ &= \sum_{i=1}^{n} \big\langle (x_i - x_{i+3}) - c_n(x_{i+1} - x_{i+2}),\, (x_i - x_{i+3}) - c_n(x_{i+1} - x_{i+2}) \big\rangle \\ &= \sum_{i=1}^{n} \|x_i - x_{i+3}\|^2 - c_n \sum_{i=1}^{n} \langle x_i - x_{i+3},\, x_{i+1} - x_{i+2} \rangle \\ &\quad - c_n \sum_{i=1}^{n} \overline{\langle x_i - x_{i+3},\, x_{i+1} - x_{i+2} \rangle} + c_n^2 \sum_{i=1}^{n} \|x_{i+1} - x_{i+2}\|^2 \\ &= \sum_{i=1}^{n} \|x_i - x_{i+3}\|^2 - 2c_n(S_1 + \overline{S}_1 - S_2 - \overline{S}_2) + c_n^2 \sum_{i=1}^{n} \|x_i - x_{i+1}\|^2 \\ &= \sum_{i=1}^{n} \|x_i - x_{i+3}\|^2 - 2c_n\{(2S_0 - S_2 - \overline{S}_2) - (2S_0 - S_1 - \overline{S}_1)\} + c_n^2 \sum_{i=1}^{n} \|x_i - x_{i+1}\|^2\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \|x_i - x_{i+3}\|^2 - 2c_n \sum_{i=1}^{n} \|x_i - x_{i+2}\|^2 + 2c_n \sum_{i=1}^{n} \|x_i - x_{i+1}\|^2 + c_n^2 \sum_{i=1}^{n} \|x_i - x_{i+1}\|^2 \\
&= (c_n^2 + 2c_n) \sum_{i=1}^{n} \|x_i - x_{i+1}\|^2 + \sum_{i=1}^{n} \|x_i - x_{i+3}\|^2 - 2c_n \sum_{i=1}^{n} \|x_i - x_{i+2}\|^2.
\end{aligned}
$$

Hence, inequality (3.1) is true.

*Equality condition*. If we choose arbitrary $x_1, x_2, x_3 \in V$, and define $x_4, \ldots, x_n$ recursively by substituting $i$ with $1, 2, \ldots, n-3$ in (3.2), then (3.2) is satisfied for all $i \in \{1, 2, \ldots, n-3\}$. Hence, with the same $A_n$ in Lemma 2.1, it follows that

$$
A_n \begin{pmatrix} x_i \\ x_{i+1} \\ x_{i+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -c_n & c_n \end{pmatrix} \begin{pmatrix} x_i \\ x_{i+1} \\ x_{i+2} \end{pmatrix} = \begin{pmatrix} x_{i+1} \\ x_{i+2} \\ x_i - c_n x_{i+1} + c_n x_{i+2} \end{pmatrix} = \begin{pmatrix} x_{i+1} \\ x_{i+2} \\ x_{i+3} \end{pmatrix}
$$

for all $i \in \{1, 2, \ldots, n-3\}$.

Temporarily, let us set

$$
\begin{aligned}
y_1 &= x_{n-2} - c_n x_{n-1} + c_n x_n, \\
y_2 &= x_{n-1} - c_n x_n + c_n y_1, \\
y_3 &= x_n - c_n y_1 + c_n y_2.
\end{aligned}
$$

Since $(A_n)^n = I$ by Lemma 2.1, it follows that

$$
\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= (A_n)^n \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (A_n)^3 (A_n)^{n-3} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (A_n)^3 \begin{pmatrix} x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix} \\
&= (A_n)^2 \begin{pmatrix} x_{n-1} \\ x_n \\ x_{n-2} - c_n x_{n-1} + c_n x_n \end{pmatrix} = (A_n)^2 \begin{pmatrix} x_{n-1} \\ x_n \\ y_1 \end{pmatrix} = A_n \begin{pmatrix} x_n \\ y_1 \\ x_{n-1} - c_n x_n + c_n y_1 \end{pmatrix} \\
&= A_n \begin{pmatrix} x_n \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ x_n - c_n y_1 + c_n y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.
\end{aligned}
$$

Therefore, $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3$, and

$$
\begin{aligned}
x_1 &= x_{n-2} - c_n x_{n-1} + c_n x_n, \\
x_2 &= x_{n-1} - c_n x_n + c_n x_1, \\
x_3 &= x_n - c_n x_1 + c_n x_2.
\end{aligned}
$$

Hence, condition (3.2) is satisfied for each $i \in \{1, 2, \ldots, n\}$.

Finally, it is obvious that the right-hand side of (3.6) is zero if and only if condition (3.2) is satisfied for all $i \in \{1, 2, \ldots, n\}$. $\square$

## 4. Applications to Aleksandrov-Rassias problem

In this section, suppose both $V_1$ and $V_2$ are real (or complex) normed spaces. We call a distance $\rho$ contractive (or non-expanding) by a mapping $f : V_1 \to V_2$ if and only if $\|f(x) - f(y)\| \leqslant \rho$ for all $x, y \in V_1$ with $\|x - y\| = \rho$, while we call a distance $\rho$ extensive (or non-shrinking) by $f$ if and only if $\|f(x) - f(y)\| \geqslant \rho$ for all $x, y \in V_1$ with $\|x - y\| = \rho$. In particular, $\rho$ is called preserved (or conservative) provided $\rho$ is contractive and extensive simultaneously.

Because every distance $\rho$ is preserved by an isometry, we may raise a question: *Is a mapping an isometry if the mapping preserves certain distances?* Indeed, Aleksandrov [1] raised a question whether a mapping $f : V_1 \to V_1$ is an isometry provided $f$ preserves a distance $\rho$, which is now known as the *Aleksandrov problem*. Without loss of generality, we may assume $\rho = 1$ when $V_1$ is a normed space (see [10]).

About 20 years earlier than Aleksandrov, the Aleksandrov problem was investigated by Beckman and Quarles [2] for the $n$-dimensional real Euclidean space $\mathbb{E}^n$.

THEOREM 4.1. (Beckman and Quarles [2]) *Assume that $n$ is an integer larger than 1 and $\rho$ is an arbitrarily given positive number. Every mapping $f : \mathbb{E}^n \to \mathbb{E}^n$ preserving the distance $\rho$ is a linear isometry up to translation.*

They could construct non-isometric mappings preserving unit distance for one-dimensional or for infinite-dimensional real Euclidean spaces (cf. [8]). Thereafter, Rassias [9] raised the question: *Is a mapping between normed spaces an isometry if it preserves two (or more) distances?* Such a problem is called the *Aleksandrov-Rassias problem*. For a strictly convex vector space, Benz gave an affirmative answer to this problem (see [3] and also [4]):

THEOREM 4.2. (Benz [3]) *Assume that $V_1$ is a real normed space with $\dim V_1 \geqslant 2$ and $V_2$ is a real normed space which is strictly convex. Suppose $N$ is an integer larger than 1. If a distance $\rho$ is contractive and $N\rho$ is extensive by a mapping $f : V_1 \to V_2$, then $f$ is a linear isometry up to translation.*

Now, assume that $V_1$ is a real (or complex) inner product space and $c_{12}$, $c_{23}$, $c_{34}$, $c_{45}$, $c_{56}$, $c_{67}$, $c_{71}$, $e_{13}$, $e_{24}$, $e_{35}$, $e_{46}$, $e_{57}$, $e_{61}$, $e_{72}$, $c_{14}$, $c_{25}$, $c_{36}$, $c_{47}$, $c_{51}$, $c_{62}$, $c_{73}$ are positive numbers such that there exist points (vectors) $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, $x_6$, $x_7$ of $V_1$ such that they satisfy the condition (3.2) as well as

$$\begin{aligned}
&\|x_1 - x_2\| = c_{12}, \quad \|x_1 - x_3\| = e_{13}, \quad \|x_1 - x_4\| = c_{14},\\
&\|x_2 - x_3\| = c_{23}, \quad \|x_2 - x_4\| = e_{24}, \quad \|x_2 - x_5\| = c_{25},\\
&\|x_3 - x_4\| = c_{34}, \quad \|x_3 - x_5\| = e_{35}, \quad \|x_3 - x_6\| = c_{36},\\
&\|x_4 - x_5\| = c_{45}, \quad \|x_4 - x_6\| = e_{46}, \quad \|x_4 - x_7\| = c_{47},\\
&\|x_5 - x_6\| = c_{56}, \quad \|x_5 - x_7\| = e_{57}, \quad \|x_5 - x_1\| = c_{51},\\
&\|x_6 - x_7\| = c_{67}, \quad \|x_6 - x_1\| = e_{61}, \quad \|x_6 - x_2\| = c_{62},\\
&\|x_7 - x_1\| = c_{71}, \quad \|x_7 - x_2\| = e_{72}, \quad \|x_7 - x_3\| = c_{73},
\end{aligned} \tag{4.1}$$

as we see in the following figure. (Obviously, due to (3.2), the seven points $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, $x_6$, $x_7$ lie on a two dimensional subspace of $V_1$.)

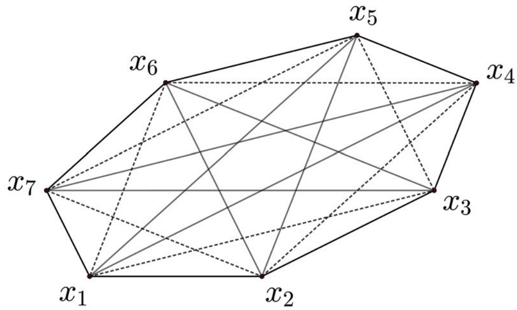


Figure 4.1. $x_{i+3} = x_i - c_7 x_{i+1} + c_7 x_{i+2}$

THEOREM 4.3. *Let $V_1$ and $V_2$ be either real inner product spaces or complex inner product spaces. Assume that the distances $c_{12}$, $c_{23}$, $c_{34}$, $c_{45}$, $c_{56}$, $c_{67}$, $c_{71}$, $c_{14}$, $c_{25}$, $c_{36}$, $c_{47}$, $c_{51}$, $c_{62}$, $c_{73}$ are contractive and the distances $e_{13}$, $e_{24}$, $e_{35}$, $e_{46}$, $e_{57}$, $e_{61}$, $e_{72}$ are extensive by a mapping $f : V_1 \to V_2$, where $c_{ij}$'s and $e_{ij}$'s are given by* (4.1) *and the corresponding $x_i$'s satisfy the condition* (3.2) *with $n = 7$ (see Figure* 4.1)*. Then $f$ preserves all the distances $c_{ij}$'s and $e_{ij}$'s.*

*Proof.* First, we set $x_i' = f(x_i)$ for all $i \in \{1,2,\ldots,7\}$. Because the distances $c_{12}$, $c_{23}$, $c_{34}$, $c_{45}$, $c_{56}$, $c_{67}$, $c_{71}$, $c_{14}$, $c_{25}$, $c_{36}$, $c_{47}$, $c_{51}$, $c_{62}$, $c_{73}$ are contractive and the distances $e_{13}$, $e_{24}$, $e_{35}$, $e_{46}$, $e_{57}$, $e_{61}$, $e_{72}$ are extensive by $f$, we can use Theorem 3.1 to get

$$\begin{aligned}
&\left(c_7^2+2c_7\right)\left(\|x_1-x_2\|^2+\|x_2-x_3\|^2+\|x_3-x_4\|^2+\|x_4-x_5\|^2+\|x_5-x_6\|^2\right.\\
&\quad\left.+\|x_6-x_7\|^2+\|x_7-x_1\|^2\right)+\|x_1-x_4\|^2+\|x_2-x_5\|^2+\|x_3-x_6\|^2+\|x_4-x_7\|^2\\
&\quad+\|x_5-x_1\|^2+\|x_6-x_2\|^2+\|x_7-x_3\|^2\\
&\geqslant \left(c_7^2+2c_7\right)\left(\|x_1'-x_2'\|^2+\|x_2'-x_3'\|^2+\|x_3'-x_4'\|^2+\|x_4'-x_5'\|^2+\|x_5'-x_6'\|^2\right.\\
&\quad\left.+\|x_6'-x_7'\|^2+\|x_7'-x_1'\|^2\right)+\|x_1'-x_4'\|^2+\|x_2'-x_5'\|^2+\|x_3'-x_6'\|^2+\|x_4'-x_7'\|^2\\
&\quad+\|x_5'-x_1'\|^2+\|x_6'-x_2'\|^2+\|x_7'-x_3'\|^2\\
&\geqslant 2c_7\left(\|x_1'-x_3'\|^2+\|x_2'-x_4'\|^2+\|x_3'-x_5'\|^2+\|x_4'-x_6'\|^2+\|x_5'-x_7'\|^2\right. \qquad (4.2)\\
&\quad\left.+\|x_6'-x_1'\|^2+\|x_7'-x_2'\|^2\right)\\
&\geqslant 2c_7\left(\|x_1-x_3\|^2+\|x_2-x_4\|^2+\|x_3-x_5\|^2+\|x_4-x_6\|^2+\|x_5-x_7\|^2+\|x_6-x_1\|^2\right.\\
&\quad\left.+\|x_7-x_2\|^2\right)\\
&= \left(c_7^2+2c_7\right)\left(\|x_1-x_2\|^2+\|x_2-x_3\|^2+\|x_3-x_4\|^2+\|x_4-x_5\|^2+\|x_5-x_6\|^2\right.
\end{aligned}$$

$$+\|x_6-x_7\|^2+\|x_7-x_1\|^2\big)+\|x_1-x_4\|^2+\|x_2-x_5\|^2+\|x_3-x_6\|^2+\|x_4-x_7\|^2$$
$$+\|x_5-x_1\|^2+\|x_6-x_2\|^2+\|x_7-x_3\|^2,$$

where the last equality follows from the condition (3.2).

On the other hand, our hypotheses imply that

$$\begin{aligned}
&c_{12}=\|x_1-x_2\|\geqslant\|x_1'-x_2'\|, && c_{14}=\|x_1-x_4\|\geqslant\|x_1'-x_4'\|,\\
&c_{23}=\|x_2-x_3\|\geqslant\|x_2'-x_3'\|, && c_{25}=\|x_2-x_5\|\geqslant\|x_2'-x_5'\|,\\
&c_{34}=\|x_3-x_4\|\geqslant\|x_3'-x_4'\|, && c_{36}=\|x_3-x_6\|\geqslant\|x_3'-x_6'\|,\\
&c_{45}=\|x_4-x_5\|\geqslant\|x_4'-x_5'\|, && c_{47}=\|x_4-x_7\|\geqslant\|x_4'-x_7'\|,\\
&c_{56}=\|x_5-x_6\|\geqslant\|x_5'-x_6'\|, && c_{51}=\|x_5-x_1\|\geqslant\|x_5'-x_1'\|,\\
&c_{67}=\|x_6-x_7\|\geqslant\|x_6'-x_7'\|, && c_{62}=\|x_6-x_2\|\geqslant\|x_6'-x_2'\|,\\
&c_{71}=\|x_7-x_1\|\geqslant\|x_7'-x_1'\|, && c_{73}=\|x_7-x_3\|\geqslant\|x_7'-x_3'\|,\\
&e_{13}=\|x_1-x_3\|\leqslant\|x_1'-x_3'\|, && e_{24}=\|x_2-x_4\|\leqslant\|x_2'-x_4'\|,\\
&e_{35}=\|x_3-x_5\|\leqslant\|x_3'-x_5'\|, && e_{46}=\|x_4-x_6\|\leqslant\|x_4'-x_6'\|,\\
&e_{57}=\|x_5-x_7\|\leqslant\|x_5'-x_7'\|, && e_{61}=\|x_6-x_1\|\leqslant\|x_6'-x_1'\|,\\
&e_{72}=\|x_7-x_2\|\leqslant\|x_7'-x_2'\|.
\end{aligned}\tag{4.3}$$

By combining (4.2) and (4.3), we conclude that

$$\begin{aligned}
&\|x_1-x_2\|=c_{12}=\|x_1'-x_2'\|, && \|x_1-x_4\|=c_{14}=\|x_1'-x_4'\|,\\
&\|x_2-x_3\|=c_{23}=\|x_2'-x_3'\|, && \|x_2-x_5\|=c_{25}=\|x_2'-x_5'\|,\\
&\|x_3-x_4\|=c_{34}=\|x_3'-x_4'\|, && \|x_3-x_6\|=c_{36}=\|x_3'-x_6'\|,\\
&\|x_4-x_5\|=c_{45}=\|x_4'-x_5'\|, && \|x_4-x_7\|=c_{47}=\|x_4'-x_7'\|,\\
&\|x_5-x_6\|=c_{56}=\|x_5'-x_6'\|, && \|x_5-x_1\|=c_{51}=\|x_5'-x_1'\|,\\
&\|x_6-x_7\|=c_{67}=\|x_6'-x_7'\|, && \|x_6-x_2\|=c_{62}=\|x_6'-x_2'\|,\\
&\|x_7-x_1\|=c_{71}=\|x_7'-x_1'\|, && \|x_7-x_3\|=c_{73}=\|x_7'-x_3'\|,\\
&\|x_1-x_3\|=e_{13}=\|x_1'-x_3'\|, && \|x_2-x_4\|=e_{24}=\|x_2'-x_4'\|,\\
&\|x_3-x_5\|=e_{35}=\|x_3'-x_5'\|, && \|x_4-x_6\|=e_{46}=\|x_4'-x_6'\|,\\
&\|x_5-x_7\|=e_{57}=\|x_5'-x_7'\|, && \|x_6-x_1\|=e_{61}=\|x_6'-x_1'\|,\\
&\|x_7-x_2\|=e_{72}=\|x_7'-x_2'\|.
\end{aligned}$$

For arbitrarily given $x_1,x_2\in V_1$ with $\|x_1-x_2\|=c_{12}$, we can choose five points (vectors) $x_3,x_4,x_5,x_6,x_7$ in $V_1$ such that $x_1,\ldots,x_7$ determine a geometrical figure congruent to the one in Figure 4.1. In view of the above argument, we may conclude that $\|x_1'-x_2'\|=c_{12}$. For other distances such as $c_{23}$, $c_{34}$, $c_{45}$, $c_{56}$, $c_{67}$, $c_{71}$, $c_{14}$, $c_{25}$, $c_{36}$, $c_{47}$, $c_{51}$, $c_{62}$, $c_{73}$, $e_{13}$, $e_{24}$, $e_{35}$, $e_{46}$, $e_{57}$, $e_{61}$, and $e_{72}$, we can apply a similar argument. Therefore, $f$ preserves the distances $c_{12}$, $c_{23}$, $c_{34}$, $c_{45}$, $c_{56}$, $c_{67}$, $c_{71}$, $c_{14}$, $c_{25}$, $c_{36}$, $c_{47}$, $c_{51}$, $c_{62}$, $c_{73}$, $e_{13}$, $e_{24}$, $e_{35}$, $e_{46}$, $e_{57}$, $e_{61}$, and $e_{72}$. $\square$

REMARK 4.4. Assume that $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ are the vertices of a regular heptagon $S$ with a unit side length (see Figure 4.2 below). Let $\alpha = \frac{\sin\frac{2\pi}{7}}{\sin\frac{\pi}{7}} \approx 1.8019\ldots$ be the shorter diagonal and $\beta = c_7 = \frac{\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}} \approx 2.2469\ldots$ be the longer diagonal of $S$. If we set $c_{12} = c_{23} = c_{34} = c_{45} = c_{56} = c_{67} = c_{71} = 1$, $e_{13} = e_{24} = e_{35} = e_{46} = e_{57} = e_{61} = e_{72} = \alpha$ and $c_{14} = c_{25} = c_{36} = c_{47} = c_{51} = c_{62} = c_{73} = \beta$ in Theorem 4.3, then the function $f$ given in Theorem 4.3 preserves the distances $1$, $\alpha$, and $\beta$.

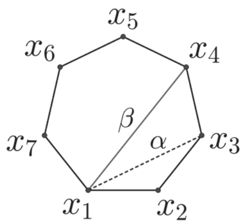


Figure 4.2. Regular heptagon

THEOREM 4.5. *Assume that $V_1$ and $V_2$ are real inner product spaces, and let $f$ be a function from $V_1$ into $V_2$. Let $v_1$, $v_2$, $v_3$, $v_4$ be arbitrary four points in $V_1$ and let $w_i = f(v_i)$ be four points in $V_2$. If $v_1$, $v_2$, $v_3$, $v_4$ lie on one plane, and if $\|v_i - v_j\| = \|w_i - w_j\|$ for all $1 \leqslant i < j \leqslant 4$, then $w_1$, $w_2$, $w_3$, $w_4$ also lie on one plane.*

*Proof.* Translation preserves distances between points, and does not affect coplanarity of four points. Hence, we can assume that $v_4 = w_4 = 0$. Then the condition of this theorem becomes simple:

$$\begin{aligned}
\|v_1\| &= \|w_1\|, \quad \|v_1 - v_2\| = \|w_1 - w_2\|, \\
\|v_2\| &= \|w_2\|, \quad \|v_2 - v_3\| = \|w_2 - w_3\|, \\
\|v_3\| &= \|w_3\|, \quad \|v_3 - v_1\| = \|w_3 - w_1\|.
\end{aligned}$$

From the above condition we obtain

$$\langle v_1, v_2\rangle = \frac{1}{2}(\|v_1\|^2 + \|v_2\|^2 - \|v_1 - v_2\|^2) = \frac{1}{2}(\|w_1\|^2 + \|w_2\|^2 - \|w_1 - w_2\|^2) = \langle w_1, w_2\rangle,$$

and in similar way we get $\langle v_2, v_3\rangle = \langle w_2, w_3\rangle$ and $\langle v_3, v_1\rangle = \langle w_3, w_1\rangle$.

Because $v_1$, $v_2$, $v_3$, $v_4 (= 0)$ lie on one plane, *i.e.*, they are coplanar, there exists $r_1, r_2 \in \mathbb{R}$ satisfying $v_3 = r_1 v_1 + r_2 v_2$. Hence, $0 = v_3 - r_1 v_1 - r_2 v_2$, and so

$$\begin{aligned}
0 &= \|v_3 - r_1 v_1 - r_2 v_2\|^2 \\
&= \|v_3\|^2 + r_1^2\|v_1\|^2 + r_2^2\|v_2\|^2 - 2r_1\langle v_3, v_1\rangle - 2r_2\langle v_3, v_2\rangle + 2r_1 r_2\langle v_1, v_2\rangle \\
&= \|w_3\|^2 + r_1^2\|w_1\|^2 + r_2^2\|w_2\|^2 - 2r_1\langle w_3, w_1\rangle - 2r_2\langle w_3, w_2\rangle + 2r_1 r_2\langle w_1, w_2\rangle \\
&= \|w_3 - r_1 w_1 - r_2 w_2\|^2
\end{aligned}$$

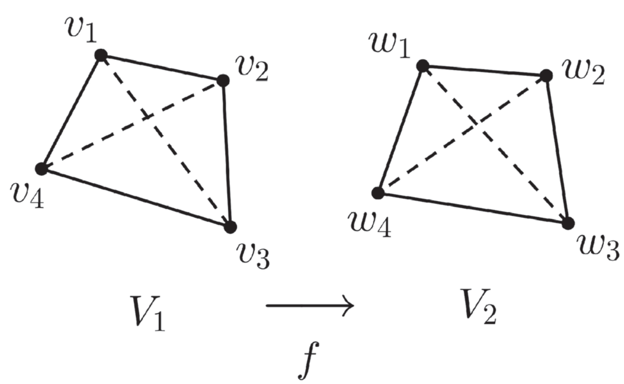


Figure 4.3. $f$ preserves coplanarity

is satisfied, *i.e.*, $0 = w_3 - r_1w_1 - r_2w_2$. Therefore, $w_3 = r_1w_1 + r_2w_2$, and $w_1$, $w_2$, $w_3$, $w_4(=0)$ also lie on one plane. □

THEOREM 4.6. *Assume that $V_1$ and $V_2$ are real Hilbert spaces with* $\dim V_1 \geqslant 3$. *If the distances* 1 *and* $\alpha$ *are contractive and the distance* $\beta$ *is extensive by a mapping* $f : V_1 \to V_2$*, then* $f$ *is a linear isometry up to translation.*

*Proof.* By regarding Theorem 4.3 and Remark 4.4, $f$ preserves the distances 1, $\alpha$, and $\beta$. We will show that $f$ preserves the distance $\sqrt{2}$. Then because $f$ preserves the distances 1 and $\sqrt{2}$, we can conclude that $f$ is an isometry up to translation by [11, Theorem 2.8].

Assume that the distance between $v_1$ and $v_3$ of $V_1$ is $\sqrt{2}$, *i.e.*, $\|v_1 - v_3\| = \sqrt{2}$. Because $\dim V_1 \geqslant 3$, there exists a subspace $U$ of $V_1$ containing $v_1$ and $v_3$ such that $\dim U = 3$. It is well-known that if two finite dimensional inner product spaces have same dimension, then there exists an inner product space isomorphism between them. Hence because $\dim U = 3 = \dim \mathbb{E}^3$, there exists an inner product space isomorphism $L_1 : \mathbb{E}^3 \to U$. From a regular heptagon having unit side length, we can get an isosceles trapezoid whose shape is illustrated in Figure 4.4.

Let "shape 1" denote the isosceles trapezoid illustrated in Figure 4.4, "shape 2" denote a triangle whose side lengths are 1, $\alpha$, $\alpha$, and "shape 3" denote a square whose side length is 1. By attaching two "shape 1", two "shape 2", and one "shape 3" in $\mathbb{E}^3$, we can get the geometrical figure illustrated in Figure 4.5.

Let "shape 4" denote this figure. There exists $u_1, u_3 \in \mathbb{E}^3$ satisfying $L_1(u_1) = v_1$ and $L_1(u_3) = v_3$. Because inner product space isomorphism preserves distance, we

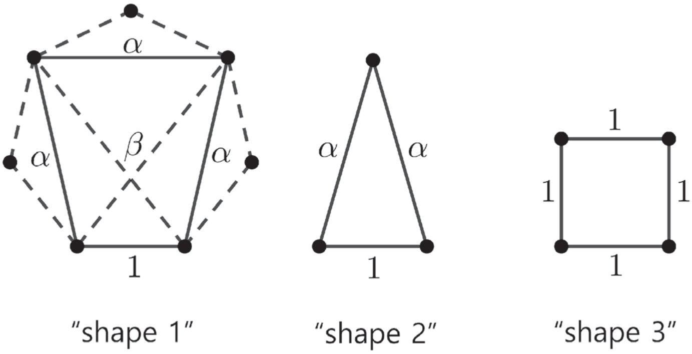


Figure 4.4. Shape 1, shape 2, and shape 3

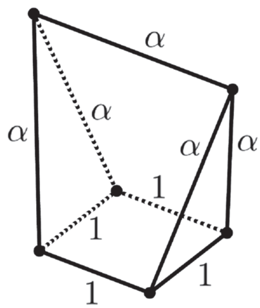


Figure 4.5. Shape 4

get $\|u_1 - u_3\| = \|L_1(u_1) - L_1(u_3)\| = \|v_1 - v_3\| = \sqrt{2}$. Because $\|u_1 - u_3\| = \sqrt{2}$, we can choose $u_2$, $u_4$, $u_5$, and $u_6$ in $\mathbb{E}^3$ such that $\{u_1, \ldots, u_6\}$ composes "shape 4". Let $v_i = L_1(u_i)$ and $w_i = f(v_i)$ for $i \in \{1, \ldots, 6\}$.

We know each of $\{u_1, u_2, u_5, u_6\}$ and $\{u_3, u_4, u_5, u_6\}$ is coplanar. Because both $L_1$ and $f$ preserve the distances $1$, $\alpha$, and $\beta$, we conclude that each of $\{v_1, v_2, v_5, v_6\}$ and $\{v_3, v_4, v_5, v_6\}$ is coplanar, and each of $\{w_1, w_2, w_5, w_6\}$ and $\{w_3, w_4, w_5, w_6\}$ is coplanar by Theorem 4.5. All of them compose "shape 1". Note that we want to show $\|w_1 - w_3\| = \sqrt{2}$.

Define $T : V_2 \to V_2$ by $T(x) = x - w_6$. Let $w'_i = T(w_i) = w_i - w_6$ for $i \in \{1, \ldots, 5\}$. Since $T$ is a translation, $T$ preserves distance. By Theorem 4.5, each of $\{0, w'_1, w'_2, w'_5\}$ and $\{0, w'_3, w'_4, w'_5\}$ is coplanar, and all of them compose "shape 1". Thus $w'_1$ is a linear combination of $\{w'_2, w'_5\}$, and $w'_4$ is a linear combination of $\{w'_3, w'_5\}$. Let $W$ be a subspace of $V_2$ spanned by $\{w'_2, w'_3, w'_5\}$. Then $\{0, w'_1, \ldots, w'_5\} \subset W$ and $\dim W \leqslant 3$.

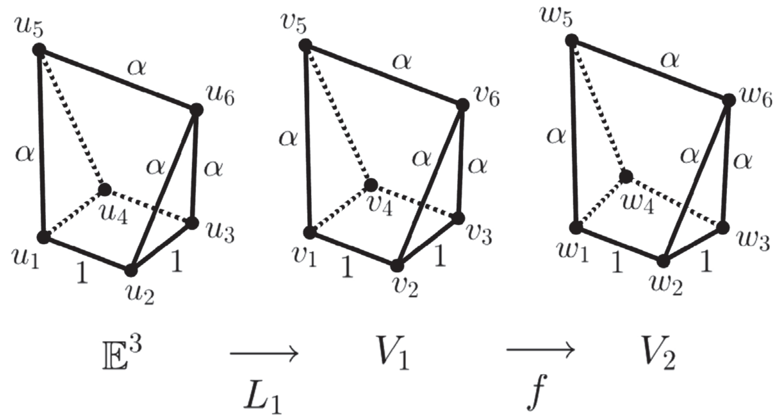


Figure 4.6. Diagram for $L_1$ and $f$

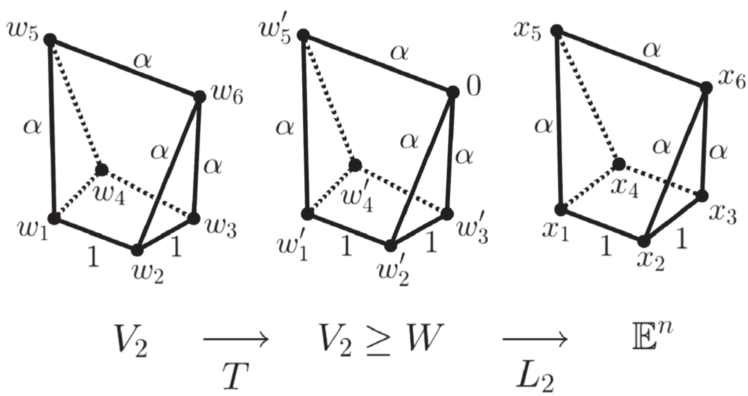


Figure 4.7. Diagram for $T$ and $L_2$

Thus there exists an inner product space isomorphism $L_2 : W \longrightarrow \mathbb{E}^n$ with $n \leqslant 3$. $L_2$ preserves distance.

Let $L_2(w'_i) = x_i$ for $i \in \{1,\ldots,5\}$ and $L_2(0) = x_6$. Then each of $\{x_1,x_2,x_5,x_6\}$ and $\{x_3,x_4,x_5,x_6\}$ is coplanar by Theorem 4.5. Hence, each of $\{x_1,x_2,x_5,x_6\}$ and $\{x_3,x_4,x_5,x_6\}$ composes "shape 1". And each of $\{x_1,x_4,x_5\}$ and $\{x_2,x_3,x_6\}$ composes "shape 2", and we know $\|x_2 - x_3\| = \|x_1 - x_4\| = 1$. To make such a structure, $n$ should be 3.

Because $\{x_1,\ldots,x_6\}$ is located in $\mathbb{E}^3$ with the condition that each of $\{x_1,x_2,x_5,x_6\}$ and $\{x_3,x_4,x_5,x_6\}$ composes "shape 1", and each of $\{x_1,x_4,x_5\}$ and $\{x_2,x_3,x_6\}$ composes "shape 2", we conclude that $\{x_1,x_2,x_3,x_4\}$ should composes "shape 3". Thus

$\|x_3 - x_1\| = \sqrt{2}$. Because $T$ and $L_2$ preserves distance, we get $\|w_3 - w_1\| = \sqrt{2}$. Hence $f$ preserves distance $\sqrt{2}$. Since $f$ preserves distances $1$ and $\sqrt{2}$, we conclude that $f$ is an isometry up to translation by using [11, Theorem 2.8]. □

*Acknowledgement.* This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B03931061).

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(Received March 28, 2019)

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