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SHARP L^p HARDY TYPE AND UNCERTAINTY PRINCIPLE INEQUALITIES ON THE SPHERE

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Abstract. This paper studies L^p - version of the Hardy type inequalities on the geodesic sphere of constant sectional curvature and establishes that the corresponding constant is sharp. Furthermore, the inequalities obtained are used to derive an uncertainty principle inequality and another inequality involving the first nonzero eigenvalue of the p-Laplacian on the sphere.

1. Introduction

In this paper, we present some new version of L^p Hardy inequalities on the unit N-sphere and show that the associated constant is the best possible. Applications of this inequality yield an uncertainty principle inequality and inequality involving the bottom of the spectrum of the p-Laplacian.

1.1. Preliminaries

Let $\mathbb{R}^N$, $N \geqslant 3$ be the N-dimensional Euclidean space, the classical Hardy inequality for $f \in C_0^\infty(\mathbb{R}^N)$ and $p > 1$ states that

$$\int_{\mathbb{R}^N} |\nabla f(x)|^p dx \geqslant \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|f(x)|^p}{|x|^p} dx,$$

where $((N-p)/p)^p$ is the best constant. In recent years, several papers have been devoted to improvement and extension of the above inequality owing to its numerous applications to fields such as Analysis, Mathematical Physics and Differential Geometry, see [3, 5, 6, 7, 14] for instance. In paricular, see [8, 10, 16, 18] for the extension to complete manifolds. For more exposition see [1] and the references therein.

In the Riemannian manifold setting, Carron [4] studied weighted L^2-Hardy inequalities under some geometric assumptions on the weight and obtained the following weighted L^2-Hardy inequality on compact or noncompact manifold M

$$\int_M \rho^\alpha |\nabla f|^2 dV \geqslant \left(\frac{C+\alpha-1}{2}\right)^2 \int_M \frac{|f|^2}{\rho^{2-\alpha}} dV,$$

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for all $f \in C_0^\infty(M)$, $\alpha \in \mathbb{R}$, $C > 1$, $C + \alpha - 1 > 0$ and the positive weight function ρ satisfying $|\nabla \rho| = 1$, $\Delta \rho \geqslant \frac{C}{\rho}$. Recently, Kombe-Özaydin [8] (see also Kombe-Özaydin [9] and Kombe-Yener [10]) extended Caron's result to the general case $1 < p < \infty$. In these papers the authors proved several Hardy-type, Rellich-type and even uncertainty principle inequalities on manifolds satisfying certain geometric restrictions. Yang-Su-Kong [18] applied the above ideas to obtain the following Hardy inequality on Riemannian manifold with negative sectional curvature for $N \geqslant 3$, $1 < p < N - \alpha$, $\alpha \in \mathbb{R}$, $f \in C_0^\infty(M)$

$$\int_M \frac{|\nabla f|^p}{\rho^\alpha} dV \geqslant \left(\frac{N-p-\alpha}{p}\right)^p \int_M \frac{|f|^p}{\rho^{\alpha+p}} dV,$$

where $|\nabla \rho|^p = 1$, $\Delta \rho \geqslant \frac{N-1}{\rho}$ and the constant $\left(\frac{N-p-\alpha}{p}\right)^p$ is sharp. Hardy-Rellich type and uncertainty principle inequalities have also been established for various settings such as Poincaré model [9] and Lie groups [13].

However, a few literature has been devoted to L^p Hardy type inequalities on the sphere so far. To the best of our knowledge, the only papers found are [1] and [15], both generalising [17] for $p = 2$ (see also [19, 2] for $p = 2$ and [18] for Riemannian manifolds of negative sectional curvature). Recently, the third author in [19] studied the inequality for $p = 2$ and derived the following for any function $f \in C^\infty(\mathbb{S}^N)$, $N \geqslant 3$

$$\frac{N-2}{2} \int_{\mathbb{S}^N} f^2 dV + \int_{\mathbb{S}^N} |\nabla f|^2 dV \geqslant \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{S}^N} \frac{f^2}{\tan^2 d(q,x)} dV, \qquad (1)$$

where $q \in \mathbb{S}^N$ is a fixed point and $\dfrac{(N-2)^2}{4}$ is sharp. This is the first time Hardy type inequality is appearing with the term $\dfrac{f^2}{\tan^2 d(q,x)}$. The appearance of this term is due to the application of Laplacian of the distance function $\Delta d(q,x) = (N-1)\cot d(q,x)$, where $d(q,x)$ is smooth (in the sense of distribution).

1.2. Main theorem

The major aim of this paper therefore is to generalise (1) to the case of general p and show that the corresponding constant is also sharp. The main theorem is then stated as follows:

THEOREM 1. *Let $N \geqslant 3$, $1 < p < N$ and $q \in \mathbb{S}^N$, then there exists a positive constant $A(N,p)$ such that for all $f \in C^\infty(\mathbb{S}^N)$*

$$A(N,p) \int_{\mathbb{S}^N} \frac{|f|^p}{\sin^{p-2} d(q,x)} dV + \int_{\mathbb{S}^N} |\nabla f|^p dV \geqslant \left(\frac{N-p}{p}\right)^p \int_{\mathbb{S}^N} \frac{|f|^p}{|\tan d(x,q)|^p} dV, \quad (2)$$

for $2 \leqslant p < N$ and

$$A(N,p) \int_{\mathbb{S}^N} |f|^p dV + \int_{\mathbb{S}^N} |\nabla f|^p dV \geqslant \left(\frac{N-p}{p}\right)^p \int_{\mathbb{S}^N} \frac{|f|^p}{|\tan d(x,q)|^p} dV, \qquad (3)$$

for $1 < p < 2$, *where* $A(N,p) = \left(\frac{N-p}{p}\right)^{p-1}$ *and* $d(x,q)$ *is the geodesic distance from* x *to a fixed point* q *on* $\mathbb{S}^N$. *Moreover, the constant* $\left(\frac{N-p}{p}\right)^{p}$ *is sharp.*

REMARK 1. When $1 < p < 2$, the term $\int_{\mathbb{S}^N} \frac{|f|^p}{\sin^{p-2} d(q,x)} dV$ cannot control the right hand side of (3) since $\sin^{p-2} d(q,x)$ is large enough where x is close to q. Hence, we use $\int_{\mathbb{S}^N} |f|^p dV$ instead. Notice also that the first term in (2) (resp. (3)) cannot be removed because it will lead to contradiction if f is a nonzero constant. It is interesting also to note that if the coefficient $A(N,p)$ is arbitrary, the constant $(\frac{N-p}{p})^p$ is still sharp. When $p = 2$, the inequality (2) (resp. (1)) reduces to (3).

1.3. Notations

The unit N-sphere is denoted by $\mathbb{S}^N = \{x \in \mathbb{R}^{N+1} : |x| = 1\}$. By the geodesic polar coordinate transform one writes

$$\int_{\mathbb{S}^N} f dV = \mathrm{Vol}(\mathbb{S}^{N-1}) \int_0^{\pi} (\sin r_q)^{N-1} dr,$$

where $r_q = d(q,x)$, $\mathrm{Vol}(\mathbb{S}^{N-1})$ is the volume of the unit $(N-1)$ sphere and dV denotes the standard volume element on $\mathbb{S}^N$. The function $f = f(r)$ depending only on r is called radial and its gradient is $|\nabla f(r)| = |f'(r)|$. In this case, the Laplacian of distance function is given as $\Delta d(x,q) = (N-1)\cot d(x,q)$ and $|\nabla d| = 1$ in the distributional sense. Let $\bar{q}$ be the antipodal point of q. Then $d(q,\bar{q}) = \pi$ and for any point $x \in \mathbb{S}^N$, we have $r_q + r_{\bar{q}} = \pi$. We shall construct a function possessing a fair degree of bilateral symmetry on the sphere.

Section 2 of this paper is devoted to proving the main theorem. As in [19], we use symmetry of the sphere to modify the construction of an auxilliary function that has been used in literature and then do the calculation in two hemispheres using antipodal points. Since the auxilliary function is only continuous, we use approximation by smooth function to show sharpness of the inequalities. The appearance of general p makes our calculation more complicated, especially for the existence of the best constant. The last two sections are devoted to some applications of our inequalities. That is, Section 3 derives an uncertainty principle inequality, while Section 4 presents an inequality involving the bottom of the spectrum of the p-Laplacian on the sphere.

2. Proof of the main Theorem

Recall from [11, 16] that for any $\xi, \eta \in \mathbb{R}^N$, it holds that $|\xi + \eta|^p \geqslant |\xi|^p + p|\xi|^{p-2}\langle \xi, \eta \rangle$. Letting $\gamma = -\frac{N-p}{p}$, $f = \rho^\gamma \phi \in C^\infty(\mathbb{S}^N)$, $\rho = \sin r_q, q \in \mathbb{S}^N$, we have

$$\begin{aligned}|\nabla f|^p =& |\nabla(\rho^\gamma)\phi + \rho^\gamma \nabla \phi|^p = |\gamma \rho^{\gamma-1} \nabla \rho \phi + \rho^\gamma \nabla \phi|^p \\ \geqslant& |\gamma|^p \rho^{\gamma p - p} |\nabla \rho|^p |\phi|^p + p|\gamma|^{p-2} \gamma \rho^{\gamma p - p + 1} |\phi|^{p-2} \phi |\nabla \rho|^{p-2} \langle \nabla \rho, \nabla \phi \rangle\end{aligned}$$

$$
\begin{aligned}
&=|\gamma|^p\rho^{\gamma p-p}|\nabla\rho|^p|\phi|^p+\frac{|\gamma|^{p-2}\gamma}{\gamma p-p+2}\langle\nabla\rho^{\gamma p-p+2},\nabla\phi^p\rangle\\
&=|\gamma|^p\rho^{\gamma p-p}|\nabla\rho|^p|\phi|^p+\frac{|\gamma|^{p-2}\gamma}{\gamma p-p+2}\mathrm{div}(\phi^p\nabla\rho^{\gamma p-p+2})\\
&\quad-\frac{|\gamma|^{p-2}\gamma}{\gamma p-p+2}\langle\Delta\rho^{\gamma p-p+2},\phi^p\rangle.
\end{aligned}
$$

A simple calculation yields

$$
\begin{aligned}
\Delta(\sin r_q)^{-\beta}&=\mathrm{div}\nabla(\sin r_q)^{-\beta}\\
&=\mathrm{div}(-\beta(\sin r_q)^{-\beta-1}\cos r_q\nabla r_q)\\
&=-\beta(\sin r_q)^{-\beta-1}\cos r_q\Delta r_q+\beta(\beta+1)(\sin r_q)^{-\beta-2}\cos^2 r_q+\beta(\sin r_q)^{-\beta}.
\end{aligned}
$$

Using $\Delta r_q=(N-1)\cot r_q$, we obtain

$$
\Delta(\sin r_q)^{-\beta}=\beta(N-\beta-1)(\sin r_q)^{-\beta}-\beta(N-\beta-2)(\sin r_q)^{-(\beta+2)}.
$$

Taking $\beta=-(\gamma p-p+2)$, using $\rho=\sin r_q$ and $\gamma=-\frac{N-p}{p}$, we have $|\nabla\rho|=\cos r_q$, $\Delta\rho^{\gamma p-p+2}=\Delta(\sin r_q)^{-N+2}=(N-2)(\sin r_q)^{-(N-2)}$ and $\frac{|\gamma|^{p-2}\gamma}{\gamma p-p+2}=\frac{1}{N-2}\left(\frac{N-p}{p}\right)^{p-1}$. Therefore

$$
\begin{aligned}
|\nabla f|^p\geqslant&\left(\frac{N-p}{p}\right)^p\phi^p\frac{(\sin r_q)^{\gamma p}}{|\tan r_q|^p}+\frac{|\gamma|^{p-2}\gamma}{\gamma p-p+2}\mathrm{div}(\phi^p\nabla\rho^{\gamma p-p+2})\\
&-\frac{1}{N-2}\left(\frac{N-p}{p}\right)^{p-1}\Delta(\sin r_q)^{-N+2}\phi^p.
\end{aligned}
$$

Integrating over $\mathbb{S}^N$, applying divergence theorem and using $\phi=\rho^{-\gamma}f=(\sin r_q)^{-\gamma}f$ yield

$$
\begin{aligned}
\int_{\mathbb{S}^N}|\nabla f|^p dV\geqslant&\left(\frac{N-p}{p}\right)^p\int_{\mathbb{S}^N}\frac{|f|^p}{|\tan r_q|^p}dV\\
&-\frac{1}{N-2}\left(\frac{N-p}{p}\right)^{p-1}\int_{\mathbb{S}^N}\Delta(\sin r_q)^{-N+2}\phi^p dV\\
=&\left(\frac{N-p}{p}\right)^p\int_{\mathbb{S}^N}\frac{|f|^p}{|\tan r_q|^p}dV-\left(\frac{N-p}{p}\right)^{p-1}\int_{\mathbb{S}^N}\frac{|f|^p}{\sin^{p-2}r_q}dV,
\end{aligned}
$$

which recovers inequality (2) for $p\geqslant 2$. While for $1<p<2$, we replace the term $\int_{\mathbb{S}^N}\frac{|f|^p}{\sin^{p-2}r_q}dV$ by $\int_{\mathbb{S}^N}|f|^p dV$ based on the explanation in Remark 1. Hence, inequality (3).

In what follows, we show that the constant $\left(\frac{N-p}{p}\right)^p$ is sharp. It then suffices to show that

$$
\left(\frac{N-p}{p}\right)^p\geqslant\inf_{f\in C^\infty(\mathbb{S}^N)\setminus\{0\}}\frac{\int_{\mathbb{S}^N}|\nabla f|^p dV+\left(\frac{N-p}{p}\right)^{p-1}\int_{\mathbb{S}^N}\frac{|f|^p}{\sin^{p-2}r_q}dV}{\int_{\mathbb{S}^N}\frac{|f|^p}{|\tan r_q|^p}dV}.
$$

The argument is similar to [19] and we follow it closely, see also [1, 15, 17, 18].

Let $\varphi(t):\mathbb{R}\to[0,1]$ be a smooth function such that $0\leqslant\varphi(t)\leqslant 1$ and $\varphi(t)=1$ for $|t|\leqslant 1$ and $\varphi(t)\equiv 0$ for $|t|\geqslant 2$.

Set $H(t)=1-\varphi(t)$. For sufficiently small ε, define

$$f_\varepsilon(r_q)=\begin{cases}0, & \text{for } r=0,\\ H\Big(\dfrac{r_q}{\varepsilon}\Big)(\tan r_q)^{\frac{p-N}{p}}, & \text{for } 0<r_q<\dfrac{\pi}{2},\\ H\Big(\dfrac{\pi-r_q}{\varepsilon}\Big)(\tan(\pi-r_q)^{\frac{p-N}{p}}, & \text{for } \dfrac{\pi}{2}<r_q<\pi,\\ 0, & \text{for } r=\pi.\end{cases}$$

Without loss of generality, we assume $0<\varepsilon<1/2$ and $f_\varepsilon(r_q)$ can be approximated by smooth function on $\mathbb{S}^N$. Let $\bar{q}$ be the antipodal point of q on $\mathbb{S}^N$ and $r_q(x,\bar{q})=\pi-r_q(x)$ be the distance from point $\bar{q}$. Then we have

$$\int_{\mathbb{S}^N}\frac{|f_\varepsilon|^p}{(\sin r_q)^{p-2}}dV=\int_{B_q(\frac{\pi}{2})}\frac{|f_\varepsilon|^p}{(\sin r_q)^{p-2}}dV+\int_{B_{\bar{q}}(\frac{\pi}{2})}\frac{|f_\varepsilon|^p}{(\sin r_{\bar{q}})^{p-2}}dV,$$

where

$$\begin{aligned}\int_{B_q(\frac{\pi}{2})}\frac{|f_\varepsilon|^p}{(\sin r_q)^{p-2}}dV&=\mathrm{Vol}(\mathbb{S}^{N-1})\int_\varepsilon^{\frac{\pi}{2}}H^p\Big(\frac{r_q}{\varepsilon}\Big)(\tan r_q)^{p-N}(\sin r_q)^{2-p}(\sin r_q)^{N-1}dr\\ &\leqslant \mathrm{Vol}(\mathbb{S}^{N-1})\int_\varepsilon^{\frac{\pi}{2}}r_q^{p-N}r_q^{N-p+1}dr=\frac{\mathrm{Vol}(\mathbb{S}^{N-1})}{2}\Big(\frac{\pi^2}{4}-\varepsilon^2\Big)\end{aligned}$$

and

$$\begin{aligned}\int_{B_{\bar{q}}(\frac{\pi}{2})}\frac{|f_\varepsilon|^p}{(\sin r_{\bar{q}})^{p-2}}dV&=\mathrm{Vol}(\mathbb{S}^{N-1})\int_{\frac{\pi}{2}}^{\pi-\varepsilon}H^p\Big(\frac{\pi-r_{\bar{q}}}{\varepsilon}\Big)(\tan(\pi-r_{\bar{q}})^{p-N}\\ &\qquad\times(\sin(\pi-r_{\bar{q}})^{2-p}(\sin(\pi-r_{\bar{q}})^{N-1}dr\\ &=\mathrm{Vol}(\mathbb{S}^{N-1})\int_\varepsilon^{\frac{\pi}{2}}H^p\Big(\frac{r_{\bar{q}}}{\varepsilon}\Big)(\tan r_{\bar{q}})^{p-N}(\sin r_{\bar{q}})^{N-p+1}dr\\ &\leqslant \mathrm{Vol}(\mathbb{S}^{N-1})\int_\varepsilon^{\frac{\pi}{2}}r_{\bar{q}}^{p-N}r_{\bar{q}}^{N-p+1}dr=\frac{\mathrm{Vol}(\mathbb{S}^{N-1})}{2}\Big(\frac{\pi^2}{4}-\varepsilon^2\Big).\end{aligned}$$

Therefore

$$\int_{\mathbb{S}^N}\frac{|f_\varepsilon|^p}{(\sin r_q)^{p-2}}dV\leqslant\mathrm{Vol}(\mathbb{S}^{N-1})\Big(\frac{\pi^2}{4}-\varepsilon^2\Big).$$

Similarly,

$$\int_{\mathbb{S}^N}\frac{|f_\varepsilon|^p}{\tan^p r_q}dV=\int_{B_q(\frac{\pi}{2})}\frac{|f_\varepsilon|^p}{\tan^p r_q}dV+\int_{B_{\bar{q}}(\frac{\pi}{2})}\frac{|f_\varepsilon|^p}{\tan^p r_{\bar{q}}}dV,$$

where

$$\begin{aligned}\int_{B_q(\frac{\pi}{2})} \frac{|f_\varepsilon|^p}{\tan^p r_q} dV &= \mathrm{Vol}(\mathbb{S}^{N-1}) \int_\varepsilon^{\frac{\pi}{2}} H^p\Big(\frac{r_q}{\varepsilon}\Big)(\tan r_q)^{-N}(\sin r_q)^{N-1} dr \\ &\geqslant \mathrm{Vol}(\mathbb{S}^{N-1}) \int_{2\varepsilon}^{\frac{\pi}{2}} H^p\Big(\frac{r_q}{\varepsilon}\Big)(\tan r_q)^{-N}(\sin r_q)^{N-1} dr \\ &= \mathrm{Vol}(\mathbb{S}^{N-1}) \int_{2\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-N}(\sin r_q)^{N-1} dr\end{aligned}$$

and

$$\begin{aligned}\int_{B_{\bar{q}}(\frac{\pi}{2})} \frac{|f_\varepsilon|^p}{\tan^p r_{\bar{q}}} dV &= \mathrm{Vol}(\mathbb{S}^{N-1}) \int_{\frac{\pi}{2}}^{\pi-\varepsilon} H^p\Big(\frac{\pi - r_{\bar{q}}}{\varepsilon}\Big)(\tan(\pi - r_{\bar{q}})^{-N}(\sin(\pi - r_{\bar{q}})^{N-1} dr \\ &= \mathrm{Vol}(\mathbb{S}^{N-1}) \int_\varepsilon^{\frac{\pi}{2}} H^p\Big(\frac{r_{\bar{q}}}{2\varepsilon}\Big)(\tan r_{\bar{q}})^{-N}(\sin r_{\bar{q}})^{N-1} dr \\ &\geqslant \mathrm{Vol}(\mathbb{S}^{N-1}) \int_{2\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-N}(\sin r_q)^{N-1} dr.\end{aligned}$$

Therefore

$$\int_{\mathbb{S}^N} \frac{|f_\varepsilon|^p}{\tan^p r_q} dV \geqslant 2\mathrm{Vol}(\mathbb{S}^{N-1}) \int_{2\varepsilon}^{\frac{\pi}{2}} \tan r_q)^{-N}(\sin r_q)^{N-1} dr.$$

Next we want to evaluate

$$\int_{\mathbb{S}^N} |\nabla f_\varepsilon|^p dV = \int_{B_q(\frac{\pi}{2})} |\nabla f_\varepsilon|^p dV + \int_{B_{\bar{q}}(\frac{\pi}{2})} |\nabla f_\varepsilon|^p dV.$$

A straightforward computation yields

$$\begin{aligned}&\int_{B_q(\frac{\pi}{2})} |\nabla f_\varepsilon|^p dV \\ =&\mathrm{Vol}(\mathbb{S}^{N-1}) \int_\varepsilon^{\frac{\pi}{2}} \Big|\frac{1}{\varepsilon} H'\Big(\frac{r_q}{\varepsilon}\Big)(\tan r_q)^{\frac{p-N}{p}} + \frac{p-N}{p} H\Big(\frac{r_q}{\varepsilon}\Big)(\tan r_q)^{-\frac{N}{p}} \sec^2 r_q\Big|^p (\sin r_q)^{N-1} dr \\ =&\mathrm{Vol}(\mathbb{S}^{N-1}) \int_\varepsilon^{2\varepsilon} \Big|\frac{1}{\varepsilon} H'\Big(\frac{r_q}{\varepsilon}\Big)(\tan r_q)^{\frac{p-N}{p}} + \frac{p-N}{p} H\Big(\frac{r_q}{\varepsilon}\Big)(\tan r_q)^{-\frac{N}{p}} (1+\tan^2 r_q)\Big|^p \\ &\qquad \times (\sin r_q)^{N-1} dr \\ &+ \mathrm{Vol}(\mathbb{S}^{N-1}) \Big(\frac{p-N}{p}\Big)^p \int_{2\varepsilon}^{\frac{\pi}{2}} H\Big(\frac{r_q}{\varepsilon}\Big)(\tan r_q)^{-\frac{N}{p}} (1+\tan^2 r_q)\Big|^p (\sin r_q)^{N-1} dr \\ \leqslant&\frac{\mathrm{Vol}(\mathbb{S}^{N-1})}{\varepsilon^p} \int_\varepsilon^{2\varepsilon} \Big|H'\Big(\frac{r_q}{\varepsilon}\Big)\Big|^p (\tan r_q)^{p-N}(\sin r_q)^{N-1} dr \\ &+ \mathrm{Vol}(\mathbb{S}^{N-1}) \Big|\frac{p-N}{p}\Big|^p \int_\varepsilon^{2\varepsilon} H\Big(\frac{r_q}{\varepsilon}\Big)(\tan r_q)^{-N}(\sin r_q)^{N-1} dr \\ &+ \mathrm{Vol}(\mathbb{S}^{N-1}) \Big|\frac{p-N}{p}\Big|^p \int_\varepsilon^{2\varepsilon} H\Big(\frac{r_q}{\varepsilon}\Big)(\tan r_q)^{-N+2p}(\sin r_q)^{N-1} dr\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{Vol}(\mathbb{S}^{N-1})\left|\frac{p-N}{p}\right|^p\int_{2\varepsilon}^{\frac{\pi}{2}}H\Big(\frac{r_q}{\varepsilon}\Big)(\tan r_q)^{-N}(\sin r_q)^{N-1}dr\\
&+\mathrm{Vol}(\mathbb{S}^{N-1})\left|\frac{p-N}{p}\right|^p\int_{2\varepsilon}^{\frac{\pi}{2}}H\Big(\frac{r_q}{\varepsilon}\Big)(\tan r_q)^{-N+2p}(\sin r_q)^{N-1}dr\\
\leqslant&\frac{\mathrm{Vol}(\mathbb{S}^{N-1})}{\varepsilon^p}\Big(\max_{t\in[0,2]}H'(t)\Big)^p\int_{\varepsilon}^{2\varepsilon}r_q^{p-1}dr\\
&+\mathrm{Vol}(\mathbb{S}^{N-1})\left|\frac{p-N}{p}\right|^p\int_{\varepsilon}^{2\varepsilon}(\tan r_q)^{-N}(\sin r_q)^{N-1}dr\\
&+\mathrm{Vol}(\mathbb{S}^{N-1})\left|\frac{p-N}{p}\right|^p\int_{\varepsilon}^{2\varepsilon}(\tan r_q)^{-N+2p}(\sin r_q)^{N-1}dr+\mathrm{Vol}(\mathbb{S}^{N-1})\Big(\frac{p-N}{p}\Big)^p\\
&\times\int_{2\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-N}(\sin r_q)^{N-1}dr+\mathrm{Vol}(\mathbb{S}^{N-1})\Big(\frac{p-N}{p}\Big)^p\int_{2\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-N+2p}(\sin r_q)^{N-1}dr\\
=&\frac{2^p-1}{p}\mathrm{Vol}(\mathbb{S}^{N-1})\Big(\max_{t\in[0,2]}H'(t)\Big)^p+\mathrm{Vol}(\mathbb{S}^{N-1})\left|\frac{p-N}{p}\right|^p\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-N}(\sin r_q)^{N-1}dr\\
&+\mathrm{Vol}(\mathbb{S}^{N-1})\Big(\frac{p-N}{p}\Big)^p\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-N+2p}(\sin r_q)^{N-1}dr.
\end{aligned}
$$

Similarly, we compute

$$
\begin{aligned}
&\int_{B_{\bar{q}}(\frac{\pi}{2})}|\nabla f_\varepsilon|^p dV\\
\leqslant&\frac{2^p-1}{p}\mathrm{Vol}(\mathbb{S}^{N-1})\Big(\max_{t\in[0,2]}H'(t)\Big)^p+\mathrm{Vol}(\mathbb{S}^{N-1})\left|\frac{p-N}{p}\right|^p\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_{\bar{q}})^{-N}(\sin r_{\bar{q}})^{N-1}dr\\
&+\mathrm{Vol}(\mathbb{S}^{N-1})\Big(\frac{p-N}{p}\Big)^p\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_{\bar{q}})^{-N+2p}(\sin r_{\bar{q}})^{N-1}dr
\end{aligned}
$$

and then

$$
\begin{aligned}
\int_{\mathbb{S}^N}|\nabla f_\varepsilon|^p dV\leqslant&\frac{2(2^p-1)}{p}\mathrm{Vol}(\mathbb{S}^{N-1})\Big(\max_{t\in[0,2]}H'(t)\Big)^p\\
&+2\mathrm{Vol}(\mathbb{S}^{N-1})\left|\frac{p-N}{p}\right|^p\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-N}(\sin r_q)^{N-1}dr\\
&+2\mathrm{Vol}(\mathbb{S}^{N-1})\Big(\frac{p-N}{p}\Big)^p\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-N+2p}(\sin r_q)^{N-1}dr.
\end{aligned}
$$

Since $f_\varepsilon(r)$ can be approximated by smooth functions on the sphere $\mathbb{S}^N$, it follows that

$$
\inf_{f\in C^\infty(\mathbb{S}^N)\setminus\{0\}}\frac{\int_{\mathbb{S}^N}|\nabla f|^p dV+\left(\frac{N-p}{p}\right)^{p-1}\int_{\mathbb{S}^N}\frac{|f|^p}{\sin^{p-2}r_q}dV}{\int_{\mathbb{S}^N}\frac{|f|^p}{|\tan r_q|^p}dV}
$$

$$\leqslant \frac{\int_{\mathbb{S}^N} |\nabla f_\varepsilon|^p dV + \left(\frac{N-p}{p}\right)^{p-1} \int_{\mathbb{S}^N} \frac{|f_\varepsilon|^p}{\sin^{p-2} r_q} dV}{\int_{\mathbb{S}^N} \frac{|f_\varepsilon|^p}{|\tan r_q|^p} dV}$$

$$\leqslant \frac{\frac{2(2^p-1)}{p}\mathrm{Vol}(\mathbb{S}^{N-1})\left(\max_{t\in[0,2]} H'(t)\right)^p}{2\int_{2\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-N}(\sin r_q)^{N-1} dr} + \frac{2\left|\frac{p-N}{p}\right|^p \int_{\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-N}(\sin r_q)^{N-1} dr}{2\int_{2\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-N}(\sin r_q)^{N-1} dr}$$

$$+ \frac{2\left(\frac{p-N}{p}\right)^p \int_{\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-N+2p}(\sin r_q)^{N-1} dr}{2\int_{2\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-N}(\sin r_q)^{N-1} dr} + \frac{\left(\frac{\pi^2}{4} - \varepsilon^2\right)}{2\int_{2\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-N}(\sin r_q)^{N-1} dr}$$

$$=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.$$

Passing to the limit as $\varepsilon \to 0^+$ gives

$$\lim_{\varepsilon\to 0^+} \int_{2\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-N}(\sin r_q)^{N-1} dr = +\infty.$$

Applying L'Hopital rule we have

$$\lim_{\varepsilon\to 0^+} \frac{\int_{\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-N}(\sin r_q)^{N-1} dr}{\int_{2\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-N}(\sin r_q)^{N-1} dr} = 1 \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \frac{\int_{\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-N+2p}(\sin r_q)^{N-1} dr}{\int_{2\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-N}(\sin r_q)^{N-1} dr} = 0.$$

Thus, $\mathrm{I} \equiv 0, \mathrm{II} \equiv 0$ and $\mathrm{IV} \equiv 0$, and then

$$\inf_{f\in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} |\nabla f|^p dV + \left(\frac{N-p}{p}\right)^{p-1} \int_{\mathbb{S}^N} \frac{|f|^p}{\sin^{p-2} r_q} dV}{\int_{\mathbb{S}^N} \frac{|f|^p}{|\tan r_q|^p} dV} \leqslant \left|\frac{p-N}{p}\right|^p.$$

This implies that the constant $\left(\frac{N-p}{p}\right)^p$ of the inequality (2) (resp. (3)) is sharp and the proof is therefore complete. □

3. Uncertainty principle inequality

The classical uncertainty principle as introduced in quantum mechanics says the position and momentum of a particle cannot be determined exactly at the same time but only with an 'uncertainty'. The uncertainty inequality on $\mathbb{R}^N$ can be stated in the following way

$$\left(\int_{\mathbb{R}^N} |x|^2 |f(x)|^2 dx\right)\left(\int_{\mathbb{R}^N} |\nabla f(x)|^2 dx\right) \geqslant \left(\frac{N-2}{2}\right)^2 \left(\int_{\mathbb{R}^N} |f(x)|^2 dx\right)^2$$

for all $f \in L^2(\mathbb{R}^N)$ with inequality being attained when f is a Gaussian like function $f(x) = A\exp(-\lambda|x|^2)$ for some $A \in \mathbb{R}$ and $\lambda > 0$.

In [9], the authors proved Heisenberg uncertainty principle inequalities on complete noncampact Riemannian maifolds and determined an explicit constant in the case of hyperbolic space. In [1] we derived an analogue of these inequality on the sphere. At present we shall apply our hardy type inequalities to derive a new form in the following proposition.

PROPOSITION 1. *Let the assumptions of Theorem 1 hold. Then there holds the following inequality for all functions* $f \in C^\infty(\mathbb{S}^N)$

$$\begin{aligned}&\Big(\int_{\mathbb{S}^N} |f|^p |\tan d(x,\xi)^q| dV\Big)^{p/q}\Big(\int_{\mathbb{S}^N} |\nabla f|^p dV + A(N,p)\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin^{p-2} d(x,\xi)} dV\Big)\\ &\geqslant \Big(\frac{N-p}{p}\Big)^p \Big(\int_{\mathbb{S}^N} |f|^p dV\Big)^p,\end{aligned} \tag{4}$$

where p *and* q *are Hölders conjugate, that is,* $1/p+1/q=1$.

Proof. By Hölder's inequality, we have

$$\int_{\mathbb{S}^N} \frac{|f|^p}{\tan^p d(x,\xi)} dV \geqslant \Big(\int_{\mathbb{S}^N} |f|^p dV\Big)^p \Big(\int_{\mathbb{S}^N} |f|^p |\tan d(x,\xi)^q| dV\Big)^{-p/q}, \tag{5}$$

where $1/p+1/q=1$. Combining (5) and the Hardy type inequality (2) (or (3)) we obtain

$$\begin{aligned}&C\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin^{p-2} d(x,\xi)} dV + \int_{\mathbb{S}^N} |\nabla f|^p dV\\ &\geqslant \Big(\frac{N-p}{p}\Big)^p \Big(\int_{\mathbb{S}^N} |f|^p dV\Big)^p \Big(\int_{\mathbb{S}^N} |f|^p |\tan d(x,\xi)^q| dV\Big)^{-p/q}\end{aligned}$$

from where the result follows at once. □

4. Eigenvalue of the p-Laplacian

Define a p-energy on $\mathbb{S}^N$

$$\mathscr{E}_p(f) := \frac{\int_{\mathbb{S}^N} |\nabla f|^p dV}{\int_{\mathbb{S}^N} |f|^p dV}$$

whose infimum is the first nonzero eigenvalue, λ_p^*, of the p-Laplacian $\Delta_p := \operatorname{div}(|\nabla f|^{p-2}\nabla f)$. Thus $\lambda_p^* = \inf \mathscr{E}_p(f)$ subject to the constraint $\int_{\mathbb{S}^N} |f|^{p-2} f dV = 0$ with infimum taken over all $f \in W^{1,p}(\mathbb{S}^N)$. It is well known that λ_p^* satisfies the Euler-Langrage equation

$$\Delta_p f = -\lambda_p^* |f|^{p-2} f \quad \text{on } \mathbb{S}^N$$

with f being the associated eigenfunction. λ_p^* has been computed explicitly to be $\lambda_p^* = N$ when $p = 2$ and $\lambda_p^* \geqslant \left(\frac{N-1}{p-1}\right)^{\frac{p}{2}}$ for the case $p \geqslant 2$ [12].

We derive the following corollary from Theorem 1.

COROLLARY 1. *Let the assumptions of Theorem 1 hold. Then, we have*

$$\frac{p[(N-p)^{p-1} + \lambda_p^* p^{p-1}]}{(N-p)^p} \geqslant \inf_{f \in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} |f|^p \cos^p d(x,q) dV}{\int_{\mathbb{S}^N} |f|^p \sin^2 d(x,q) dV}, \tag{6}$$

for $2 \leqslant p < N$ *and*

$$\frac{p[(N-p)^{p-1} + \lambda_p^* p^{p-1}]}{(N-p)^p} \geqslant \inf_{f \in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} |f|^p \cos^p d(x,q) dV}{\int_{\mathbb{S}^N} |f|^p \sin^p d(x,q) dV}, \tag{7}$$

for $1 < p < 2$.

Proof. For the case $1 < p < 2$. By (3) of Theorem 1, we have

$$\left(\frac{N-p}{p}\right)^{p-1} + \inf_{f \in C^\infty(\mathbb{S}^N)\setminus\{0\}} \mathscr{E}_p(f) \geqslant \left(\frac{N-p}{p}\right)^p \inf_{f \in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} \frac{|f|^p}{\tan^p d(x,q)} dV}{\int_{\mathbb{S}^N} |f|^p dV}.$$

Replacing f by $f \sin d(x,q)$ in the last equation we have

$$\left(\frac{N-p}{p}\right)^{p-1} + \lambda_p^* \geqslant \left(\frac{N-p}{p}\right)^p \inf_{f \in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} |f|^p \cos^p d(x,q) dV}{\int_{\mathbb{S}^N} |f|^p \sin^p d(x,q) dV},$$

from where (7) which is the desired result follows.

For the case $P \geqslant 2$. By (2) of Theorem 1, we have

$$\left(\frac{N-p}{p}\right)^{p-1} + \inf_{f \in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} |\nabla f|^p dV}{\int_{\mathbb{S}^N} \frac{|f|^p}{\sin^{p-2} d(x,q)} dV}$$
$$\geqslant \left(\frac{N-p}{p}\right)^p \inf_{f \in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} \frac{|f|^p}{\tan^p d(x,q)} dV}{\int_{\mathbb{S}^N} \frac{|f|^p}{\sin^{p-2} d(x,q)} dV}.$$

One can show that

$$\int_{\mathbb{S}^N} \frac{|f|^p}{\sin^{p-2} d(x,q)} dV \geqslant \int_{\mathbb{S}^N} |f|^p dV, \quad p \geqslant 2$$

and then write

$$\left(\frac{N-p}{p}\right)^{p-1} + \inf_{f \in C^\infty(\mathbb{S}^N)\setminus\{0\}} \mathscr{E}_p(f) \geqslant \left(\frac{N-p}{p}\right)^p \inf_{f \in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} \frac{|f|^p}{\tan^p d(x,q)} dV}{\int_{\mathbb{S}^N} \frac{|f|^p}{\sin^{p-2} d(x,q)} dV}.$$

Replacing f by $f \sin d(x,q)$ and using $\lambda_p^* = \inf \mathscr{E}_p(f)$ in the last equation, we have

$$\left(\frac{N-p}{p}\right)^{p-1} + \lambda_p^* \geqslant \left(\frac{N-p}{p}\right)^p \inf_{f \in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} |f|^p \cos^p d(x,q) dV}{\int_{\mathbb{S}^N} |f|^p \sin^2 d(x,q) dV},$$

from where (6) follows. □

REMARK 2. When $p = 2$ the inequality (6) (resp. (7)) reduces to [19, Corollary].

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REFERENCES

[1] A. ABOLARINWA, T. APATA, *L^p-Hardy-Rellich and uncertainty principle inequalities on the sphere*, Adv. Oper. Theory, 3(4), (2018), 745–762.

[2] A. ABOLARINWA, K. RAUF, *Optimal L p -Hardy-Rellich type inequalities on the sphere*, Submitted.

[3] H. BREZIS, J. L. VÁZQUEZ, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Comp. Madrid 10 (1997), 443–469.

[4] G. CARRON, *Inégalités de Hardy sur les vari ét és riemanniennes non-compactes*, J. Math. Pures Appl., 76(10) (1997), 883–891.

[5] D. G. COSTA, *On Hardy-Rellich type inequalities in $\mathbb{R}^N$*, Appl. Math. Lett., 22(6), (2009), 902–905.

[6] E. B. DAVIES, A. M. HINZ, *Explicit constants for Rellich inequalities in $L^p(\Omega)$*, Math. Z. 227 (1998), 511–532.

[7] F. GAZZOLA, H.-CH. GRUNAU AND E. MITIDIERI, *Hardy inequalities with optimal constants and remainder terms*, Trans. Amer. Math. Soc., 356(6), (2004), 2149–2168.

[8] I. KOMBE, M. ÖZAYDIN, *Improved Hardy and Rellich inequalities on Riemannian manifolds*, Trans. Amer. Math. Soc. 361 (2009), 6191–6203.

[9] I. KOMBE, M. ÖZAYDIN, *Hardy-Poincaré, Rellich and uncertainty principle inequalities on Riemannian manifolds*, Trans. Amer. Math. Soc. 365 (2013), 5035–5050.

[10] I. KOMBE, A. YENER, *Weighted Hardy and Rellich type inequalities on Riemannian manifolds*, Math. Nachr. 289, (8-9) (2016), 994–1004.

[11] P. LINDQVIST, *On the equation $div(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0$*, Proc. Amer. Math. Soc. 109, 157–164 (1990).

[12] A. M. MATEI, *First eigenvalue for the p -Laplace operator*, Nonl. Anal. 39(2000) 1051–1068.

[13] A. SITARAM, M. SUNDARI, S. THANGAVELU, *Uncertainty principles on certain Lie groups*, Proc. Ind. Acad. Sci. (Math. Sci.) 105 (1995), 133–151.

[14] A. TERTIKAS, N. ZOGRAPHOPOULOS, *Best constants in the Hardy-Rellich Inequalities and Related Improvements*, Adv. Math., 209, (2007), 407–459.

[15] X. SUN, F. PAN, *Hardy type inequalities on the sphere*, J. Ineq. Appl., 148, (2017), 1–8.

[16] C. XIA, *Hardy and Rellich type inequalities on complete manifolds*, J. Math. Anal Appl., 409 (2014), 84–90.

[17] Y. XIAO, *Some Hardy inequalities on the sphere*, J. Math. Inequal. 10, (2016), 793–805.

[18] Q. YANG, D. SU, Y. KONG, *Hardy inequalities on Riemannian manifolds with negative curvature*, Commun. Contemp. Math. 16, (2) (2014), 1350043.

[19] S. YIN, *Hardy type inequalities on the sphere*, New York. J. Math. 24(2018), 1101–1110.

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