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COMPARISONS BETWEEN LARGEST AND SMALLEST ORDER STATISTICS FROM PARETO DISTRIBUTIONS

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Abstract. In the paper, we discuss the problem of the stochastic comparisons of the largest and smallest order statistics from independent heterogeneous Pareto random variables with different scale and shape parameters. We study the reversed hazard rate order of smallest order statistics, usual stochastic order of the largest order statistics of type I in the sense of multivariate chain majorization. Furthermore, we investigat hazard rate order of smallest order statistics, usual stochastic order of the largest order statistics of type II in the sense of multivariate chain majorization and majorization orders respectively.

1. Introduction and preliminaries

Pareto distributions are the most popular models in finance economics and related areas. Firstly, we introduce two of the most popular Pareto distributions: the Pareto distribution of type I and the Pareto distribution of type II. A Pareto random variable of type I with shape parameter a and scale parameter b denoted by $PI(a,b)$ has the probability density and cumulative distribution by

$$f(x) = \frac{a}{b}(\frac{x}{b})^{-a-1} \quad \text{and} \quad F(x) = 1-(\frac{x}{b})^{-a},$$

respectively, for $x > b > 0$ and $a > 0$. A Pareto random variable of type II with shape parameter a and scale parameter b denoted by $PII(a,b)$ has the probability density and cumulative distribution by

$$f(x) = \frac{a}{b}(1+\frac{x}{b})^{-a-1} \quad \textit{and} \quad F(x) = 1-(1+\frac{x}{b})^{-a},$$

respectively, for $x > 0, b > 0$ and $a > 0$. Pareto distribution of type I and type II are commonly used to model random variables like income, risk and price. Some properties of Pareto distribution of type I and type II can be studied in Arnold (1985). And the maximum likelihood estimate for the Pareto distributions also can be seen in Grimshaw (1993).

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Secondly, we recall some notions of stochastic orders and majorizations. Let X (Y) be a univariate random variable with cumulative distribution function F (G), survival function $\bar{F}$ $(\bar{G})$, density function f (g), hazard rate function $h_F(=f/\bar{F})$ $(h_G(=g/\bar{G})$), reverse hazard rate function $\widetilde{r}_F(x)=(f/F)$ $(\widetilde{r}_G(x)=(g/G))$, respectively. We say that Y is smaller than X in the usual stochastic order, denoted by $X\geqslant_{st}Y$, if $\bar{F}(x)\geqslant\bar{G}(x)$ for all x; in the hazard rate order, denoted by $X\geqslant_{hr}Y$, if $h_F(x)\leqslant h_G(x)$ for all x; in the reversed hazard rate order, denoted by $X\geqslant_{rh}Y$, if $\widetilde{r}_F(x)\geqslant\widetilde{r}_G(x)$ for all x. Vector majorization is a very interesting and useful tool in statistics by sorting all components of vector in nondecreasing order. Let $\lambda_{(1)}\leqslant\ldots\leqslant\lambda_{(n)}$ and $\lambda^*_{(1)}\leqslant\ldots\leqslant\lambda^*_{(n)}$ denote ordered components corresponding to two real vectors $\lambda=(\lambda_1,\ldots,\lambda_n)$ and $\lambda^*=(\lambda^*_1,\ldots,\lambda^*_n)$, respectively. Then, λ^* is said to be majorized by λ, denoted by $\lambda\succeq_m\lambda^*$, if

$$\sum_{i=1}^{k}\lambda_{(i)}\leqslant\sum_{i=1}^{k}\lambda^*_{(i)}$$

for $k=1,2,\ldots,n-1$, and $\sum\limits_{i=1}^{n}\lambda_i=\sum\limits_{i=1}^{n}\lambda^*_i$. A real valued f defined on a set $\mathscr{A}\in\Re^n$ is said to be Schur-convex(Schur-concave) on $\mathscr{A}$ if

$$x\preceq^m y\ \ on\ \ \mathscr{A}\Rightarrow f(x)\leqslant(\geqslant)f(y).$$

Last, we present the definition of multivariate chain majorization based on matrices. Let $A=\{a_{ij}\}$ and $B=\{b_{ij}\}$ be two $m\times n$ matrices. Then, A is said to chain majorize B (denoted by $A\gg B$) if there exists a finite set of $n\times n$ T-transform matrices $T_{w_1},\ldots,T_{w_k}$ such that $B=AT_{w_1}T_{w_2}\ldots T_{w_k}$. For more details on stochastic orders and majorizations, the readers may refer to Shaked and Shanthikumar(2007), Li and Li (2013) and Marshall, Olkin and Arnold (2011).

Order statistics have a remarkable contribution in statistics, applied probability theory, operations research, auction theory and so on. Let $X_{1:n}\leqslant X_{2:n}\leqslant\ldots\leqslant X_{n:n}$ denote the order statistics corresponding to the random variables $X_1,X_2,\ldots,X_n$. Some work on the stochastic comparisons for independent and non-identically distributed random variables we can learn in the following papers: Torrado (2015), Li and Li (2015), Fang and Zhang (2012, 2013) and so on. In this paper, we will utilize tool of multivariate chain majorization to investigate stochastic comparisons of smallest and largest order statistics from Pareto distributions with different scale and shape parameters.

Let

$$\mathscr{S}_n=\left\{(\mathbf{x},\mathbf{y})=\begin{pmatrix}x_1,\ldots,x_n\\ y_1,\ldots,y_n\end{pmatrix}:x_i>0,y_j>0\ and\ (x_i-x_j)(y_i-y_j)\leqslant 0,i,j=1,\ldots,n\right\}$$

$$\mathscr{T}_n=\left\{(\mathbf{x},\mathbf{y})=\begin{pmatrix}x_1,\ldots,x_n\\ y_1,\ldots,y_n\end{pmatrix}:x_i\geqslant 1,y_j>0\ and\ (x_i-x_j)(y_i-y_j)\leqslant 0,i,j=1,\ldots,n\right\}$$

THEOREM 1.1. *Let $\phi:\mathbb{R}^{+^4}\to\mathbb{R}^+$ be a differentiable function, which satisfies*

$$\phi(A)\geqslant(\leqslant)\phi(B)\ \ for\ all\ \ A,B\ \ such\ that\ \ A\in\mathscr{S}_2(\mathscr{T}_2)\ \ and\ \ A\gg B \tag{1.1}$$

if and only if
i) $\phi(A) = \phi(A\Pi)$ *for all permutation matrices* Π*;*
ii) $\sum_{i=1}^{2}(a_{ik} - a_{ij})[\phi_{ik}(A) - \phi_{ij}(A)] \geqslant (\leqslant) 0$ *for all j,k=1,2, where* $\phi_{ij}(A) = \partial\phi(A)/\partial a_{ij}$.

THEOREM 1.2. *The function* $\phi_n : \mathbb{R}^{+^{2n}} \to \mathbb{R}^+$ *be defined as*

$$\phi_n(A) = \prod_{i=1}^{n} \psi(a_{1i}, a_{2i}),$$

where $\psi : \mathbb{R}^{+^2} \to \mathbb{R}^+$ *be a differentiable function. Assume that* ϕ_2 *satisfies in (1.1). Then* $A \in \mathscr{S}_n(\mathscr{T}_n)$ *and* $B = AT_w$, *we have* $\phi_n(A) \geqslant \phi_n(B)$.

THEOREM 1.3. *The function* $\phi_n^{'} : \mathbb{R}^{+^{2n}} \to \mathbb{R}^+$ *be defined as*

$$\phi_n^{'}(A) = \sum_{i=1}^{n} \psi(a_{1i}, a_{2i}),$$

where $\psi : \mathbb{R}^{+^2} \to \mathbb{R}^+$ *be a differentiable function. Assume that* $\phi_2^{'}(A)$ *satisfies in (1.1). Then* $A \in \mathscr{S}_n(\mathscr{T}_n)$ *and* $B = AT_w$, *we have* $\phi_n^{'}(A) \geqslant \phi_n^{'}(B)$.

REMARK 1.1. For the sake of conciseness, we are not presented the proofs of Theorems 1.1-1.3, since that of are similar to the proofs of Theorems 2-3 in Balakrishnan, Haidari and Masoumifared (2014).

2. Main results

In order to prove the main results, we need the following two lemmas.

LEMMA 2.1. *Let the function* $\omega : (0,\infty) \times [1,\infty)$ *be defined as*

$$\omega(a,t) = \frac{lnt}{t^a - 1}.$$

Then,

(i) for each $t \geqslant 1, \omega(a,t)$ *is decreasing with respect to* a*;*

(ii) for each $a > 0, \omega(a,t)$ *is decreasing with respect to* t.

Proof.

(i) For each $t \geqslant 1$, we can obtain

$$(t^a - 1)^2 \frac{\partial \omega(a,t)}{\partial a} = -t^a (\ln t)^2 \leqslant 0,$$

which implies that $\omega(a,t)$ is decreasing with respect to a.

(ii) For each $a > 0$, it is easy to show that

$$(t^a-1)^2\frac{\partial\omega(a,t)}{\partial t} = t^{-1}(t^a-1) - at^{a-1}\ln t = t^{-1}(t^a-1-at^a\ln t).$$

Now, we let $u(t) = t^a - 1 - at^a\ln t, t \geqslant 1$. Then, we have

$$\frac{\partial u(t)}{\partial t} = at^{a-1} - at^{a-1} - a^2t^{a-1}\ln t = -a^2t^{a-1}\ln t \leqslant 0.$$

Therefore, $u(t) \leqslant u(1) = 0$, and we can get $\frac{\partial\omega(a,t)}{\partial t} \leqslant 0$. So, $\omega(a,t)$ is decreasing with respect to t. □

LEMMA 2.2. *Let the function* $\psi : [1,\infty)\times[1,\infty)$ *be defined as*

$$\psi(a,t) = \frac{at}{t^a-1}.$$

Then,

(i) for each $t \geqslant 1$, $\psi(a,t)$ *is decreasing with respect to* a;

(ii) for each $a \geqslant 1$, $\psi(a,t)$ *is decreasing with respect to* t.

Proof.

(i) For each $t \geqslant 1$, we have

$$(t^a-1)^2\frac{\partial\psi(a,t)}{\partial a} = t(t^a-1) - at^{a+1}\ln t = t^{a+1} - t - at^{a+1}\ln t = t(t^a-1-at^a\ln t).$$

Now, we let $u(a,t) = t^a - 1 - at^a\ln t, a \geqslant 1$. Then, we have

$$\frac{\partial u(a,t)}{\partial a} = -at^a(\ln t)^2 \leqslant 0.$$

So, $u(a,t) \leqslant u(1,t) = t - 1 - t\ln t$, and $\frac{\partial u(1,t)}{\partial t} = -\ln t \leqslant 0$, that is, $u(1,t) \leqslant u(1,1) = 0$. Therefor $u(a,t) \leqslant 0$, that $\frac{\partial\psi(a,t)}{\partial a} \leqslant 0$. Thus, $\psi(a,t)$ is decreasing with respect to a.

(ii) For each $a \geqslant 1$, we get

$$(t^a-1)^2\frac{\partial\psi(a,t)}{\partial t} = a(t^a-1) - a^2t^a = at^a - a - a^2t^a = a(t^a(1-a)-1) \leqslant 0.$$

Thus, $\psi(a,t)$ is decreasing with respect to t. □

THEOREM 2.1. *Let X_1, X_2 be independent random variables with $X_i \sim PI(a_i, b_i)$, $i = 1,2$. Further, let Y_1, Y_2 be another set of independent random variables with $Y_i \sim PI(c_i, d_i), i = 1,2$. Then, for $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in \mathscr{S}_2$, we have*

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \gg \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} \Rightarrow Y_{1:2} \geqslant_{rh} X_{1:2}.$$

Proof. The reversed hazard rate function of $X_{1:2}$ is, for $x \geqslant \max(b_1, b_2)$,

$$\widetilde{r}_{X_{1:2}}(x) = \frac{(\frac{a_1}{x} + \frac{a_2}{x})(\frac{x}{b_1})^{-a_1}(\frac{x}{b_2})^{-a_2}}{1 - (\frac{x}{b_1})^{-a_1}(\frac{x}{b_2})^{-a_2}}.$$

So we can know that the function $\widetilde{r}_{X_{1:2}}(x)$ is permutation invariant in $(a_i, b_i)' st$, and so condition (i) of Theorem 1.1 is satisfied. Next, we have to show that condition (ii) of Theorem 1.1 also holds.

Obviously, the partial derivatives of $\widetilde{r}_{X_{1:2}}(x)$ with respect to a_i and b_i are,

$$\begin{aligned}
&\left(1 - (\frac{x}{b_1})^{-a_1}(\frac{x}{b_2})^{-a_2}\right)^2 \frac{\partial \widetilde{r}_{X_{1:2}}(x)}{\partial a_i} \\
=&(\frac{x}{b_1})^{-a_1}(\frac{x}{b_2})^{-a_2}\left(\frac{1}{x} - (\frac{a_1}{x} + \frac{a_2}{x})\ln\frac{x}{b_i} - \frac{1}{x}(\frac{x}{b_1})^{-a_1}(\frac{x}{b_2})^{-a_2})\right)
\end{aligned}$$

and

$$\left(1 - (\frac{x}{b_1})^{-a_1}(\frac{x}{b_2})^{-a_2}\right)^2 \frac{\partial \widetilde{r}_{X_{1:2}}(x)}{\partial b_i} = (\frac{a_1}{x} + \frac{a_2}{x})(\frac{x}{b_1})^{-a_1}(\frac{x}{b_2})^{-a_2}\frac{a_i}{b_i},$$

respectively.

For fixed $x \geqslant \max(b_1, b_2)$, let us define the function $\varphi(a,b)$ as follows,

$$\begin{aligned}
\varphi(a,b) =&(a_1 - a_2)(\frac{\partial \widetilde{r}_{X_{1:2}}(x)}{\partial a_1} - \frac{\partial \widetilde{r}_{X_{1:2}}(x)}{\partial a_2}) \\
&+ (b_1 - b_2)(\frac{\partial \widetilde{r}_{X_{1:2}}(x)}{\partial b_1} - \frac{\partial \widetilde{r}_{X_{1:2}}(x)}{\partial b_2}) \\
\overset{sign}{=}&(a_1 - a_2)\ln\frac{b_1}{b_2} + (b_1 - b_2)(\frac{a_1}{b_1} - \frac{a_2}{b_2}).
\end{aligned}$$

The assumption that $(a,b) \in \mathscr{S}_2$ implies that $(a_1 - a_2)(b_1 - b_2) \leqslant 0$. This means that $a_1 \geqslant a_2 > 0$ and $0 < b_1 \leqslant b_2$, or $0 < a_1 \leqslant a_2$ and $b_1 \geqslant b_2 > 0$. We present the proof only for the case when $a_1 \geqslant a_2 > 0$ and $0 < b_1 \leqslant b_2$, since the proof for the other case is quite similar.

According to the assumption $a_1 \geqslant a_2 > 0$ and $0 < b_1 \leqslant b_2$, for $x \geqslant \max(b_1, b_2) = b_2$, we have

$$\frac{a_1}{b_1} \geqslant \frac{a_2}{b_1} \geqslant \frac{a_2}{b_2}.$$

So, we obtain $\varphi(a,b) \leqslant 0$. By Theorem 2.1, $\widetilde{r}_{X_{1:2}}(x) \leqslant \widetilde{r}_{Y_{1:2}}(x)$ is hold, that is, $Y_{1:2}(x) \geqslant_{rh} X_{1:2}(x)$. $\square$

THEOREM 2.2. *Let X_1, X_2 be independent random variables with $X_i \sim PI(a_i, b_i)$, $i = 1,2$. Further, let Y_1, Y_2 be another set of independent random variables with $Y_i \sim PI(c_i, d_i), i = 1,2$. Then, for $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in \mathscr{T}_2$, we have*

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \gg \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} \Rightarrow X_{2:2} \geqslant_{st} Y_{2:2}.$$

Proof. The distribution function of $X_{2:2}$ is given by, for $x \geqslant \max(b_1, b_2)$,

$$F_{X_{2:2}}(x) = \prod_{i=1}^{2} [1 - (\frac{x}{b_i})^{-a_i}].$$

We can know that the function $F_{X_{2:2}}(x)$ is permutation invariant in $(a_i, b_i)' st$, and so condition (i) of Theorem 1.1 is satisfied. Next, we have to show that condition (ii) of Theorem 1.1 also holds. For fixed $x \geqslant \max(b_1, b_2)$, let us define the function $\varphi(a,b)$ as follows:

$$\varphi(a,b) = \varphi_1(a,b) + \varphi_2(a,b),$$

where

$$\varphi_1(a,b) = (a_1 - a_2)(\frac{\partial F_{X_{2:2}}(x)}{\partial a_1} - \frac{\partial F_{X_{2:2}}(x)}{\partial a_2}), \tag{2.1}$$

and

$$\varphi_2(a,b) = (b_1 - b_2)(\frac{\partial F_{X_{2:2}}(x)}{\partial b_1} - \frac{\partial F_{X_{2:2}}(x)}{\partial b_2}). \tag{2.2}$$

The assumption that $(a,b) \in \mathscr{T}_2$ implies that $(a_1 - a_2)(b_1 - b_2) \leqslant 0$. This means that $a_1 \geqslant a_2 \geqslant 1$ and $0 < b_1 \leqslant b_2$, or $1 \leqslant a_1 \leqslant a_2$ and $b_1 \geqslant b_2 > 0$. We present the proof only for the case when $a_1 \geqslant a_2 \geqslant 1$ and $0 < b_1 \leqslant b_2$, since the proof for the other case is quite similar.

The partial derivatives of $F_{X_{2:2}}(x)$ with respect to a_i and b_i are

$$\frac{\partial F_{X_{2:2}}(x)}{\partial a_i} = F_{X_{2:2}}(x) \cdot \frac{(\frac{x}{b_i})^{-a_i} \ln \frac{x}{b_i}}{1 - (\frac{x}{b_i})^{-a_i}}, \tag{2.3}$$

and

$$\frac{\partial F_{X_{2:2}}(x)}{\partial b_i} = -\frac{1}{x} F_{X_{2:2}}(x) \cdot \frac{a_i (\frac{x}{b_i})^{1-a_i}}{1 - (\frac{x}{b_i})^{-a_i}}, \tag{2.4}$$

respectively.

Upon using (2.3) in (2.1), we get

$$\varphi_1(a,b) = F_{X_{2:2}}(x)(a_1 - a_2)\left(\omega(a_1, \frac{x}{b_1}) - \omega(a_2, \frac{x}{b_2})\right).$$

From Lemma 2.1, it follows that $\omega(a, \frac{x}{b})$ is decreasing with respect to a for fixed $\frac{x}{b}$, and is decreasing with respect to $\frac{x}{b}$ for fixed a.

Therefore, we can conclude that

$$\omega(a_1, \frac{x}{b_1}) \leqslant \omega(a_2, \frac{x}{b_1}) \leqslant \omega(a_2, \frac{x}{b_2}),$$

which in turn implies

$$\varphi_1(a,b) \leqslant 0.$$

On the other hand, upon using (2.4) in (2.2), we get

$$\varphi_2(a,b) = -\frac{1}{x}F_{X_{2:2}}(x)(b_1 - b_2)[\psi(a_1, \frac{x}{b_1}) - \psi(a_2, \frac{x}{b_2})].$$

From Lemma 2.2, it follows that $\psi(a, \frac{x}{b})$ is decreasing with respect to a for fixed $\frac{x}{b}$, and is decreasing with respect to $\frac{x}{b}$ for fixed a. Therefore, we can conclude that

$$\psi(a_1, \frac{x}{b_1}) \leqslant \psi(a_1, \frac{x}{b_2}) \leqslant \psi(a_2, \frac{x}{b_2}),$$

which in turn implies

$$\varphi_2(\alpha, \lambda) \leqslant 0.$$

So we get $\varphi(a,b) \leqslant 0$, condition (ii) of Theorem 1.1 is satisfied, and this completes the proof of the theorem. □

THEOREM 2.3. *Let X_1, X_2 be independent random variables with $X_i \sim PII(a_i, b_i)$, $i = 1,2$. Further, let Y_1, Y_2 be another set of independent random variables with $Y_i \sim PII(c_i, d_i), i = 1,2$. Then, for* $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in \mathscr{S}_2$, *we have*

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \gg \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} \Rightarrow Y_{1:2} \geqslant_{hr} X_{1:2}.$$

Proof. The hazard rate function of $X_{1:2}$ is, for $x > 0$,

$$h_{X_{1:2}}(x) = \sum_{i=1}^{2} \frac{a_i}{x + b_i}.$$

So we can know that the function $h_{X_{1:2}}(x)$ is permutation invariant in $(a_i, b_i)'st$, and so condition (i) of Theorem 1.1 is satisfied. Next, we have to show that condition (ii) of Theorem 1.1 also holds.

Obviously, the partial derivatives of $h_{X_{1:2}}(x)$ with respect to a_i and b_i are,

$$\frac{\partial h_{X_{1:2}}(x)}{\partial a_i} = \frac{1}{x+b_i},$$

and

$$\frac{\partial h_{X_{1:2}}(x)}{\partial b_i} = -\frac{a_i}{(x+b_i)^2},$$

respectively.

For fixed $x>0$, let us define the function $\varphi(a,b)$ as follows,

$$\phi(a,b) = (a_1-a_2)(\frac{1}{x+b_1} - \frac{1}{x+b_2}) + (b_1-b_2)\left(-\frac{a_1}{(x+b_1)^2} + \frac{a_2}{(x+b_2)^2}\right).$$

The assumption that $(a,b)\in\mathscr{S}_2$ implies that $(a_1-a_2)(b_1-b_2)\leqslant 0$. This means that $a_1\geqslant a_2>0$ and $0<b_1\leqslant b_2$, or $0<a_1\leqslant a_2$ and $b_1\geqslant b_2>0$. We present the proof only for the case when $a_1\geqslant a_2>0$ and $0<b_1\leqslant b_2$, since the proof for the other case is quite similar.

According to the assumption $a_1\geqslant a_2>0$ and $0<b_1\leqslant b_2$, for $x>0$, we have

$$\frac{1}{x+b_1} \geqslant \frac{1}{x+b_2},$$

and

$$\frac{a_1}{(x+b_1)^2} \geqslant \frac{a_2}{(x+b_2)^2}.$$

we can easily obtain that $\phi(a,b)\geqslant 0$. By Theorem 1.1, $h_{X_{1:2}}(x)\geqslant h_{Y_{1:2}}(x)$ is hold, that is, $Y_{1:2}(x)\geqslant_{hr} X_{1:2}(x)$. □

We can illustrate the results in Theorem 2.1, 2.2 and 2.3 by the following examples.

EXAMPLE 2.1. Let X_1,X_2 be independent random variables with $X_i\sim PI(a_i,b_i)$, $i=1,2$. Further, let Y_1,Y_2 be another set of independent random variables with $Y_i\sim PI(c_i,d_i), i=1,2$. Then,

(i) *Set* $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix} \in \mathscr{S}_2$ and $\begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} 4.44 & 4.56 \\ 4.68 & 4.32 \end{pmatrix}$.

Consider the following T-transform matrices:

$$T_{0.2} = 0.2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.8\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } T_{0.6} = 0.6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.4\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So, we have $\begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} T_{0.2}T_{0.6}$, implies that

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \gg \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix}.$$

Therefore from Theorem 2.1, we obtain that $Y_{1:2}\geqslant_{rh} X_{1:2}$.

(ii) *Set* $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 3 & 5 \end{pmatrix} \in \mathscr{T}_2$ and $\begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} 4.7 & 4.3 \\ 3.92 & 4.08 \end{pmatrix}$.

Consider the following T-transform matrices:

$T_{0.3} = 0.3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.7 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $T_{0.4} = 0.4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.6 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

So, we have $\begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} T_{0.3}T_{0.4}$, implies that

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \gg \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix}.$$

Therefore from Theorem 2.2, we obtain that $X_{2:2} \geqslant_{st} Y_{2:2}$.

EXAMPLE 2.2. Let X_1, X_2 be independent random variables with $X_i \sim PII(a_i, b_i)$, $i = 1,2$. Further, let Y_1, Y_2 be another set of independent random variables with $Y_i \sim PII(c_i, d_i), i = 1,2$. Then, *Set* $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 2 & 0.8 \\ 0.6 & 1 \end{pmatrix} \in \mathscr{S}_2$ and
$\begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} 1.208 & 1.592 \\ 0.864 & 0.736 \end{pmatrix}$.

Consider the following T-transform matrices:

$T_{0.9} = 0.9 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $T_{0.3} = 0.3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.7 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

So, we have $\begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} T_{0.9}T_{0.3}$, implies that

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \gg \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix}.$$

Therefore from Theorem 2.3, we obtain that $Y_{1:2} \geqslant_{hr} X_{1:2}$.

In the following examples, we will show that the results in Theorem 2.1 and 2.3 may not hold if both matrices of parameters are not in the corresponding conditions.

EXAMPLE 2.3. Let X_1, X_2 be independent random variables with $X_i \sim PI(a_i, b_i)$, $i = 1,2$. Further, let Y_1, Y_2 be another set of independent random variables with $Y_i \sim PI(c_i, d_i), i = 1,2$. Then,
Set $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 6 & 12 \end{pmatrix}$ and $\begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} 1.7 & 1.3 \\ 10.2 & 7.8 \end{pmatrix}$.

We find that above both matrices are not in $\mathscr{S}_2$. Considering the following T-transform matrices:

$$T_{0.3} = 0.3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.7\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ we have } \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} T_{0.3}, \text{ implies that}$$

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \gg \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix}.$$

On the other hand, by calculating, we have

$$\widetilde{r}_{X_{1:2}}(15) = 0.069, \quad \widetilde{r}_{Y_{1:2}}(15) = 0.057.$$

Thus, which means that $Y_{1:2} \ngeq_{rh} X_{1:2}$.

EXAMPLE 2.4. Let X_1, X_2 be independent random variables with $X_i \sim PII(a_i, b_i)$, $i = 1, 2$. Further, let Y_1, Y_2 be another set of independent random variables with $Y_i \sim PII(c_i, d_i), i = 1, 2$. Let
$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$ and $\begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} 2.4 & 2.6 \\ 2.2 & 2.8 \end{pmatrix}$.

We find that above both matrices are not in $\mathscr{S}_2$. Considering the following T-transform matrices:

$$T_{0.4} = 0.4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.6\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ we have } \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} T_{0.4}, \text{ implies that}$$

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \gg \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix}.$$

On the other hand, by calculating, we have

$$h_{X_{1:2}}(6) = 0.5857, \quad h_{Y_{1:2}}(6) = 0.5881,$$

Thus, which means that $Y_{1:2} \ngeq_{hr} X_{1:2}$.

Now, we can extend the special form of Theorems 2.1-2.3 to the case of $n > 2$.

THEOREM 2.4. *Let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim PI(a_i, b_i)$, $i = 1, \ldots, n$. Further, let $Y_1, \ldots, Y_n$ be independent random variables with $Y_i \sim PI(c_i, d_i)$, $i = 1, \ldots, n$.*
Assume that

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix} \in \mathscr{S}_n$$

and

$$\begin{pmatrix} c_1 & c_2 & \ldots & c_n \\ d_1 & d_2 & \ldots & d_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix} T_w.$$

Then, we have $Y_{1:n} \geqslant_{st} X_{1:n}$.

Proof. For fixed $x \geqslant \max(b_1, \ldots b_n)$, let $\phi_n(a,b) = \overline{F}_{X_{1:n}}(x)$ and $\psi(a,b) = (\frac{x}{b_l})^{-a_l}$. Then, we have $\phi_n(a,b) = \prod_{l=1}^{n} \psi(a_l, b_l)$. According to Theorem 3.1, we know $\phi_2(a,b)$ is satisfied in (1.1) and according to the fact that reversed hazard rate order can imply usual stochastic order. Now, the desired result follows from Theorem 1.2. □

THEOREM 2.5. *Let $X_1,\dots,X_n$ be independent random variables with $X_i \sim PI(a_i,b_i)$, $i=1,\dots,n$. Further, let $Y_1,\dots,Y_n$ be independent random variables with $Y_i \sim PI(c_i,d_i)$, $i=1,\dots,n$. Assume that*

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in \mathscr{T}_n$$

and

$$\begin{pmatrix} c_1 & c_2 & \dots & c_n \\ d_1 & d_2 & \dots & d_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} T_w.$$

Then, we have $X_{n:n} \geqslant_{st} Y_{n:n}$.

Proof. For fixed $x \geqslant \max(b_1,\dots b_n)$, let $\phi_n(a,b) = F_{X_{n:n}}(x)$ and $\psi(a,b) = 1 - (\frac{x}{b_l})^{-a_l}$. Then, we have $\phi_n(a,b) = \prod_{l=1}^{n} \psi(a_l,b_l)$. According to Theorem 2.2, we know $\phi_2'(a,b)$ is satisfied in (1.1). Now, the desired result follows from Theorem 1.2. □

THEOREM 2.6. *Let $X_1,\dots,X_n$ be independent random variables with $X_i \sim PI(a_i,b_i)$, $i=1,\dots,n$. Further, let $Y_1,\dots,Y_n$ be independent random variables with $Y_i \sim PII(c_i,d_i)$, $i=1,\dots,n$. Assume that*

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in \mathscr{S}_n$$

and

$$\begin{pmatrix} c_1 & c_2 & \dots & c_n \\ d_1 & d_2 & \dots & d_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} T_w.$$

Then, we have $Y_{1:n} \geqslant_{hr} X_{1:n}$.

Proof. For fixed $x > 0$, let $\phi_n'(a,b) = h_{X_{1:n}}(x)$ and $\psi(a,b) = \frac{a_l}{x+b_l}$. Then, we have $\phi_n'(a,b) = \sum_{l=1}^{n} \psi(a_l,b_l)$. According to Theorem 2.3, we know $\phi_2(a,b)$ is satisfied in (1.1). Now, the desired result follows from Theorem 1.3. □

Since the finite product of the T-transform matrices with the same structures is also a T-transform matrix. So, we can obtain the following results.

COROLLARY 2.1. *Let $X_1,\dots,X_n$ be independent random variables with $X_i \sim PI(a_i,b_i), i=1,\dots,n$. Further, let $Y_1,\dots,Y_n$ be another set of independent random variables with $Y_i \sim PI(c_i,d_i), i=1,\dots,n$. Assume that*

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in \mathscr{S}_n$$

and

$$\begin{pmatrix} c_1 & c_2 & \dots & c_n \\ d_1 & d_2 & \dots & d_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} T_{w_1} \dots T_{w_k},$$

where T_{w_i}, $i=1,\dots,k$ have the same structures. Then, we have $Y_{1:n} \geqslant_{st} X_{1:n}$.

COROLLARY 2.2. *Let $X_1,\dots,X_n$ be independent random variables with $X_i \sim PI(a_i,b_i), i=1,\dots,n$. Further, let $Y_1,\dots,Y_n$ be another set of independent random variables with $Y_i \sim PI(c_i,d_i), i=1,\dots,n$. Assume that*

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in \mathscr{T}_n$$

and

$$\begin{pmatrix} c_1 & c_2 & \dots & c_n \\ d_1 & d_2 & \dots & d_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} T_{w_1} \dots T_{w_k},$$

where $T_{w_i},\ i=1,\dots,k$ have the same structures. Then, we have $X_{n:n} \geqslant_{st} Y_{n:n}$.

COROLLARY 2.3. *Let $X_1,\dots,X_n$ be independent random variables with $X_i \sim PII(a_i,b_i), i=1,\dots,n$. Further, let $Y_1,\dots,Y_n$ be another set of independent random variables with $Y_i \sim PII(c_i,d_i), i=1,\dots,n$. Assume that*

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in \mathscr{S}_n$$

and

$$\begin{pmatrix} c_1 & c_2 & \dots & c_n \\ d_1 & d_2 & \dots & d_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} T_{w_1} \dots T_{w_k},$$

where $T_{w_i}, i=1,\dots,k$ have the same structures. Then, we have $Y_{1:n} \geqslant_{hr} X_{1:n}$.

Since the finite product of the T-transform matrices with different structures may not be a T-transform matrix. For this case, we also have discussed and given the results.

THEOREM 2.7. *Let $X_1,\dots,X_n$ be independent random variables with $X_i \sim PI(a_i,b_i)$, $i=1,\dots,n$. Further, let $Y_1,\dots,Y_n$ be another set of independent random variables with $Y_i \sim PI(c_i,d_i), i=1,\dots,n$. Assume that*

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in \mathscr{S}_n,$$

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} T_{w_1} \dots T_{w_i} \in \mathscr{S}_n, \text{ for } i=1,\dots,k-1,$$

and

$$\begin{pmatrix} c_1 & c_2 & \dots & c_n \\ d_1 & d_2 & \dots & d_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} T_{w_1} \dots T_{w_k}.$$

Then, we have $Y_{1:n} \geqslant_{st} X_{1:n}$.

Proof. Set $\begin{pmatrix} a_1^{(j)} & \dots & a_n^{(j)} \\ b_1^{(j)} & \dots & b_n^{(j)} \end{pmatrix} = \begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix} T_{w_1} \dots T_{w_j},\ j=1,\dots,k-1$. Let $Z_1^{(j)},\dots,Z_n^{(j)}, j=1,\dots,k-1$, be the sets of independent random variables with $Y_i^{(j)} \sim$

$PI(a_i^{(j)}, b_i^{(j)})$, $i = 1, \ldots, n$ and $j = 1, \ldots, k-1$. From the assumption of the theorem, it follows that

$$\begin{pmatrix} a_1^{(j)} & \ldots & a_n^{(j)} \\ b_1^{(j)} & \ldots & b_n^{(j)} \end{pmatrix} \in \mathscr{S}_n$$

for $j = 1, \ldots, k-1$.
Using these observations and the results of Theorem 2.4, it follows that $Y_{1:n} \geqslant_{st} Z_{1:n}^{(k-1)} \geqslant_{st} \ldots \geqslant_{st} Z_{1:n}^{(1)} \geqslant_{st} X_{1:n}$. □

THEOREM 2.8. *Let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim PI(a_i, b_i)$, $i = 1, \ldots, n$. Further, let $Y_1, \ldots, Y_n$ be another set of independent random variables with $Y_i \sim PI(c_i, d_i), i = 1, \ldots, n$. Assume that*

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix} \in \mathscr{T}_n,$$

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix} T_{w_1} \ldots T_{w_i} \in \mathscr{T}_n, \ \textit{for} \ i = 1, \ldots, k-1,$$

and

$$\begin{pmatrix} c_1 & c_2 & \ldots & c_n \\ d_1 & d_2 & \ldots & d_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix} T_{w_1} \ldots T_{w_k}.$$

Then, we have $X_{n:n} \geqslant_{st} Y_{n:n}$.

THEOREM 2.9. *Let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim PII(a_i, b_i), i = 1, \ldots, n$. Further, let $Y_1, \ldots, Y_n$ be another set of independent random variables with $Y_i \sim PII(c_i, d_i), i = 1, \ldots, n$. Assume that*

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix} \in \mathscr{S}_n,$$

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix} T_{w_1} \ldots T_{w_i} \in \mathscr{S}_n, \ \textit{for} \ i = 1, \ldots, k-1,$$

and

$$\begin{pmatrix} c_1 & c_2 & \ldots & c_n \\ d_1 & d_2 & \ldots & d_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix} T_{w_1} \ldots T_{w_k}.$$

Then, we have $Y_{1:n} \geqslant_{hr} X_{1:n}$.

The proofs of Theorem 2.8 and Theorem 2.9 are similar to that of Theorem 3.7.

The following result we obtain usual stochastic order of the largest order statistics from type II in the condition of majorization orders.

THEOREM 2.10. *Let $X_1,\dots,X_n$ be independent random variables with $X_i \sim PII(a_i,b), i=1,\dots,n$. Further, let $Y_1,\dots,Y_n$ be independent random variables with $Y_i \sim PII(c_i,b), i=1,\dots,n$. Then $(a_1,a_2,\dots,a_n) \succeq^m (c_1,c_2,\dots,c_n)$ imples $X_{n:n} \geqslant_{st} Y_{n:n}$.*

Proof. The distribution function of $X_{n:n}$ is given by, for $x>0$,

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n}[1-(1+\frac{x}{b})^{-a_i}].$$

From the condition of $(a_1,a_2,\dots,a_n) \succeq^m (c_1,c_2,\dots,c_n)$, we only want to show $F_{X_{n:n}}(x)$ is Schur-concave in $(a_1,a_2,\dots,a_n)$. The partial derivative of $F_{X_{n:n}}(x)$ with respect to a_i is

$$\frac{\partial F_{X_{n:n}}(x)}{\partial a_i} = F_{X_{n:n}}(x)\cdot\frac{(1+\frac{x}{b})^{-a_i}\ln(1+\frac{x}{b})}{1-(1+\frac{x}{b})^{-a_i}}.$$

Let

$$\begin{aligned}\varphi(x) &= (a_i-a_j)\left(\frac{\partial F_{X_{n:n}}(x)}{\partial a_i} - \frac{\partial F_{X_{n:n}}(x)}{\partial a_j}\right)\\ &= F_{X_{n:n}}(x)(a_i-a_j)\left(\omega(a_i,1+\frac{x}{b}) - \omega(a_j,1+\frac{x}{b})\right).\end{aligned}$$

From Lemma 2.1 we complete the proof. □

Lastly, we will illustrate the results of Theorems 2.7-2.9 by three numerical examples.

Example 2.5. Let X_1,X_2,X_3 be independent random variables with $X_i \sim PI(a_i,b_i)$, $i=1,2,3$. Further, let Y_1,Y_2,Y_3 be another set of independent random variables with $Y_i \sim PI(c_i,d_i), i=1,2,3$.
Set

$$\begin{pmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix} = \begin{pmatrix} 6 & 4 & 1\\ 2 & 3 & 5\end{pmatrix} \in \mathscr{S}_3$$
$$\textit{and} \quad \begin{pmatrix} c_1 & c_2 & c_3\\ d_1 & d_2 & d_3\end{pmatrix} = \begin{pmatrix} 4.14 & 2.72 & 4.14\\ 3.04 & 3.92 & 3.04\end{pmatrix}.$$

Considering the following three T-transform matrices:

$$T_{0.3} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0.3 & 0.7\\ 0 & 0.7 & 0.3\end{pmatrix},\ T_{0.8} = \begin{pmatrix} 0.8 & 0.2 & 0\\ 0.2 & 0.8 & 0\\ 0 & 0 & 1\end{pmatrix},\ T_{0.5} = \begin{pmatrix} 0.5 & 0 & 0.5\\ 0 & 1 & 0\\ 0.5 & 0 & 0.5\end{pmatrix}.$$

Then, we have

$$\begin{pmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix} T_{0.3} = \begin{pmatrix} 6 & 1.9 & 3.1\\ 2 & 4.4 & 3.6\end{pmatrix} \in \mathscr{S}_3,$$

$$\begin{pmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix} T_{0.3}T_{0.8} = \begin{pmatrix} 5.18 & 2.72 & 3.1\\ 2.48 & 3.92 & 3.6\end{pmatrix} \in \mathscr{S}_3,$$

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} T_{0.3}T_{0.8}T_{0.5}.$$

So, the condition of Theorem 2.7 is satisfied which means that $Y_{1:3} \geqslant_{st} X_{1:3}$.

EXAMPLE 2.6. Let X_1, X_2, X_3 be independent random variables with $X_i \sim PI(a_i, b_i)$, $i = 1,2,3$. Further, let Y_1, Y_2, Y_3 be another set of independent random variables with $Y_i \sim PI(c_i, d_i), i = 1,2,3.$
Set

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 2 \\ 2 & 6 & 8 \end{pmatrix} \in \mathscr{T}_3$$
$$\textit{and} \quad \begin{pmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix} = \begin{pmatrix} 3.17 & 3.17 & 4.66 \\ 6.58 & 6.58 & 2.94 \end{pmatrix}.$$

Considering the following three T-transform matrices:

$$T_{0.35} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.35 & 0.65 \\ 0 & 0.65 & 0.35 \end{pmatrix}, \; T_{0.2} = \begin{pmatrix} 0.2 & 0 & 0.8 \\ 0 & 1 & 0 \\ 0.8 & 0 & 0.2 \end{pmatrix}, \; T_{0.5} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we have

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} T_{0.35} = \begin{pmatrix} 5 & 2.7 & 3.3 \\ 2 & 7.4 & 6.7 \end{pmatrix} \in \mathscr{T}_3,$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} T_{0.35}T_{0.2} = \begin{pmatrix} 3.64 & 2.7 & 4.66 \\ 5.76 & 7.4 & 2.94 \end{pmatrix} \in \mathscr{T}_3,$$

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} T_{0.35}T_{0.2}T_{0.5}.$$

So, the condition of Theorem 2.8 is satisfied which means that $X_{3:3} \geqslant_{st} Y_{3:3}$.

EXAMPLE 2.7. Let X_1, X_2, X_3 be independent random variables with $X_i \sim PII(a_i, b_i), i = 1,2,3$. Further, let Y_1, Y_2, Y_3 be another set of independent random variables with $Y_i \sim PII(c_i, d_i), i = 1,2,3.$
Set

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 & 0.1 \\ 0.6 & 1 & 2 \end{pmatrix} \in \mathscr{S}_3$$
$$\textit{and} \quad \begin{pmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix} = \begin{pmatrix} 0.326 & 0.458 & 0.416 \\ 1.523 & 1.069 & 1.008 \end{pmatrix}.$$

Considering the following three T-transform matrices:

$$T_{0.3} = \begin{pmatrix} 0.3 & 0 & 0.7 \\ 0 & 1 & 0 \\ 0.7 & 0 & 0.3 \end{pmatrix}, \; T_{0.4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.4 & 0.6 \\ 0 & 0.6 & 0.4 \end{pmatrix}, \; T_{0.9} = \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.9 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we have

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} T_{0.3} = \begin{pmatrix} 0.31 & 0.3 & 0.59 \\ 1.58 & 1 & 1.02 \end{pmatrix} \in \mathscr{S}_3,$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} T_{0.3}T_{0.4} = \begin{pmatrix} 0.31 & 0.474 & 0.416 \\ 1.58 & 1.012 & 1.008 \end{pmatrix} \in \mathscr{S}_3,$$

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} T_{0.3}T_{0.4}T_{0.9}.$$

So, the condition of Theorem 3.9 is satisfied which means that $Y_{1:3} \geqslant_{hr} X_{1:3}$.

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