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SOME IMPROVEMENTS ON THE L_p INEQUALITIES FOR DIFFUSION PROCESSES

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Abstract. In this paper, we give some improvements on the $L_p(0 < p < \alpha)$ inequalities for diffusion processes. We obtain smaller constants in the L_p inequalities and derive that the growth rates of the constants, as $p \to 0^+$, grows like $O\left(\frac{1}{p^\alpha}\right)$, instead of the exponential of $\frac{1}{p}$. Finally, we apply the improved inequalities to the Ornstein-Uhlenbeck processes, Bessel processes and reflected Brownian motion with drift and get better constants.

1. Introduction

Diffusion processes are a class of stochastic processes with wide applications. Many mathematical models in engineering and finance are related to diffusion processes. In the applications, the L_p inequalities for diffusion processes are basic tools and the constants in the inequalities are also important when the estimations should be exact. L_p inequalities and Davis-type inequalities for diffusion processes have been extensively studied for a long time. Gordon[5], Burkholder[2], Rosenkrantz and Sawyer[13] and DeBlassie[3, 4] studied L_p inequalities for Bessel processes. Graversen and Peskir [6, 7], Peskir and Shiryaev[10] and Botnikov[1] established L_p inequalities and Davis-type inequalities for Ornstein-Uhlenbeck processes and reflected Brownian motion with drift. Yan and Zhu[15] introduced the condition $S(\gamma, K_1, K_2)$ and established L_p inequalities for diffusion processes satisfying the condition $S(\gamma, K_1, K_2)$. The $L_p(0 < p < \alpha)$ inequalities established before are all based on the well-known domination inequalities or domination principles, which established by Lenglart[8] and improved by Yor [12, 16].

Diffusion processes satisfying the condition $S(\gamma, K_1, K_2)$ are more general. They contain Bessel processes, Ornstein-Uhlenbeck processes and reflected Brownian motion with drift and others as special cases. In this paper, we study the constants in the $L_p(0 < p < \alpha)$ inequalities for diffusion processes satisfying the condition $S(\gamma, K_1, K_2)$. We obtain smaller constants in the L_p inequalities and derive that the growth rates of the constants, as $p \to 0^+$, grows like $O\left(\frac{1}{p^\alpha}\right)$, instead of the exponential of $\frac{1}{p}$.

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Let $X=(X_t,\mathscr{F}_t)_{t\geqslant 0}$ be a diffusion process given by

$$dX_t=\mu(X_t)\,dt+\sigma(X_t)\,dB_t,\ X_0=x_0, \tag{1}$$

where $B=(B_t)_{t\geqslant 0}$ is a standard Brownian motion starting at zero and $\mu(x),\sigma(x)$ are continuous functions.

If $\sigma(x)>0$, for $x\neq 0$ and there exist constants $\gamma,\ K_2\geqslant K_1>0$ such that

$$K_1|x|^\gamma\sigma^2(x)\leqslant|\mu(x)|\leqslant K_2|x|^\gamma\sigma^2(x),\ -\infty<x<\infty, \tag{2}$$

we say that the diffusion process X satisfies the condition $S(\gamma,K_1,K_2)$ or $\mu(x)$ and $\sigma(x)$ satisfy the condition $S(\gamma,K_1,K_2)$.

Let $f(x)$ be a positive function and $g(x)$ be a positive increasing function. We denote $f(x)=O(g(x))$, as $x\to 0$, if there exists a constant $C>0$ such that

$$\limsup_{x\to 0}\frac{f(x)}{g(x)}\leqslant C.$$

For a stochastic process $X=(X_t)_{t\geqslant 0}$ and a stopping time τ, we write

$$X_\tau^*=\sup_{s\leqslant\tau}|X_s|,\qquad X_\infty^*=\sup_{s<\infty}|X_s|.$$

Let $F(x)$ be the solution of the equation

$$\sigma^2(x)\frac{d^2y}{d^2x}+2\mu(x)\frac{dy}{dx}=2\varphi(x) \tag{3}$$

such that $y(0)=0,y'(0)=0$, where $\mu(x),\sigma(x),\varphi(x)$ are continuous functions on R and $\varphi(x)\geqslant 0$.

For a nonnegative continuous function φ, define

$$J_t=\int_0^t\varphi(X_s)\,ds,\quad t\geqslant 0,$$

then $(J_t)_{t\geqslant 0}$ is a continuous increasing process.

One of the main L_p inequalities established by Yan and Zhu[15] is the following. Let X be a diffusion process given by (1), starting at zero such that the condition $S(\gamma,K_1,K_2)$ be satisfied and $\mu(x)\leqslant 0$, for $x\geqslant 0$. Assume that φ satisfies the following condition

$$N_1|x|^{\gamma-1}\sigma^2(x)\leqslant|\varphi(x)|\leqslant N_2|x|^{\gamma-1}\sigma^2(x)e^{N_3|x|^{\gamma+1}/\gamma+1},$$

with constants $N_i>0,\ (i=1,2,3)$. If either $X_t\geqslant 0$ or the function $F(x)$ is an even function, then the following inequality

$$c_{p,\gamma}\left\|\ln^{\frac{1}{1+\gamma}}(1+J_\tau)\right\|_p\leqslant\|X_\tau^*\|_p\leqslant C_{p,\gamma}\left\|\ln^{\frac{1}{1+\gamma}}(1+J_\tau)\right\|_p \tag{4}$$

holds, for $0<p<\gamma+1$ and any stopping time τ.

As in the theory of martingale inequalities, one interesting problem for L_p inequalities of diffusion processes is to find the best constants or the growth rates of the constants in the inequalities as $p \to \infty$ and $p \to 0^+$. Since the L_p inequalities established by the domination inequality are only the type of inequalities for $0 < p < \alpha$, we study the growth rates of the constants as $p \to 0^+$. Simple calculations show that all the growth rates of the constants obtained by the approach of domination inequality are

$$\frac{1}{c_{p,\gamma}} = O\left(2^{\frac{1}{p}}\right),\ C_{p,\gamma} = O\left(2^{\frac{1}{p}}\right),\ \ p \to 0^+.$$

This approach yields only the exponential of $\frac{1}{p}$ estimate, for $\frac{1}{c_{p,\gamma}}$ and $C_{p,\gamma}$.

Recently, Ren and Shen[11] established an improved domination inequality. By this inequality, we can show that the growth rates of the constants depend on the function φ and

$$\frac{1}{c_{p,\gamma}} = O\left(\frac{1}{p^{\frac{1}{1+\gamma}}}\right),\ C_{p,\gamma} = O\left(\frac{1}{p^{\frac{1}{1+\gamma}}}\right),\ \ p \to 0^+.$$

This gives us more information for the growth rates of the constants in the L_p inequalities, as $p \to 0^+$.

Throughout this paper, we shall use the standard notions of general theory of stochastic processes and thus consider stochastic processes with cadlag paths. We suppose that $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ is a filtered probability space with filtration $(\mathscr{F}_t)_{t \geqslant 0}$ satisfying the usual conditions, $B = (B_t)_{t \geqslant 0}$ is a standard Brownian motion with $B_0 = 0$. A stochastic process $A = (A_t)_{t \geqslant 0}$ is called an increasing process if it is adapted to the family $(\mathscr{F}_t)$, whose paths are positive, increasing, finite and right continuous on $[0, +\infty)$. An adapted positive cadlag process X is called dominated by an adapted increasing process A with $A_0 \geqslant 0$, if

$$E(X_\tau) \leqslant E(A_\tau), \tag{5}$$

for any bounded stopping time τ.

For a continuous increasing function $H(x)$ from R_+ to R_+ with $H(0) = 0$, set

$$\widetilde{H}(x) = H(x) + x\int_x^\infty \frac{dH(u)}{u}.$$

Ren and Shen established the following improved domination inequality.

LEMMA 1. *Let X be an adapted positive cadlag process and be dominated by a predictable increasing process A with $X_0 = A_0 = 0$, $H(x)$ be a continuous increasing function from R_+ to R_+ with $H(0) = 0$. Then for any stopping time τ and any $0 < \lambda \leqslant 1$, the following inequality holds*

$$E\left[H\left(X_\tau^*\right)\right] \leqslant E\left[\left(H + \lambda\widetilde{H}\right)\left(\frac{A_\tau}{\lambda}\right)\right]. \tag{6}$$

Using this improved domination inequality, we give some improvements on the L_p inequalities for diffusion processes and, as applications, obtain some new L_p inequalities for the Ornstein-Uhlenbeck processes, Bessel processes and the reflected Brownian motion with drift.

2. Main results and proofs

Let $F(x)$ be the solution of the equation (3)

$$\sigma^2(x)\frac{d^2y}{d^2x}+2\mu(x)\frac{dy}{dx}=2\varphi(x),\ y(0)=0,y'(0)=0.$$

Then

$$F(x)=\int_0^x e^{-\int_0^t \frac{2\mu(u)}{\sigma^2(u)}du}dt\int_0^t \frac{2\varphi(s)}{\sigma^2(s)}e^{\int_0^s \frac{2\mu(u)}{\sigma^2(u)}du}ds. \tag{7}$$

Since $\mu(x),\sigma(x),\varphi(x)$ are continuous functions on R and $\varphi(x)\geqslant 0$, $F(x)$ is a continuous increasing function on R_+ with $F(0)=0$.

If $\mu(x)$ is an odd fuction, $\sigma^2(x)$ and $\varphi(x)$ are even functions, $F(x)$ is an even function.

Denote by $H(x)=F^{-1}(x)$ the inverse of $F(x)$ on R_+. Let $H_p(x)=[H(x)]^p=[F^{-1}(x)]^p$, for $p>0$. Then $H_p(x)$ is a continuous increasing function from R_+ to R_+ with $H_p(0)=0$ and for $x\geqslant 0,\ p>0$

$$H_p[F(x)]=x^p.$$

For $x\geqslant 0$, define the function $\widetilde{H}_p(x)$ by

$$\widetilde{H}_p(x)=H_p(x)+x\int_x^\infty \frac{dH_p(u)}{u},\quad p>0.$$

We have the following inequality.

THEOREM 1. *Let X be a diffusion process given by (1), starting at zero , $F(x)$ be the solution of the equation (3), $\mu(x),\sigma(x),\varphi(x)$ be continuous functions and $\varphi(x)\geqslant 0$, $J_t=\int_0^t\varphi(X_s)\,ds$. If either $X_t\geqslant 0$ or the function $F(x)$ is an even function and*

$$\widetilde{H}_p(x)\leqslant C_pH_p(x),\quad x\geqslant 0, \tag{8}$$

for some $p>0$, then for $0<\lambda\leqslant 1$

$$\frac{1}{\lambda C_p+1}E\left[H_p(\lambda J_\tau)\right]\leqslant E\left[(X_\tau^*)^p\right]\leqslant(\lambda C_p+1)E\left[H_p\left(\frac{J_\tau}{\lambda}\right)\right] \tag{9}$$

hold for any stopping time τ, where C_p is a constant.

Proof. Since $F(x)$ is the solution of the equation (3), $X_t\geqslant 0$ or the function $F(x)$ is an even function, by Itô formula we have

$$F(|X_t|)=F(X_t)=F(X_0)+\int_0^t\varphi(X_s)\,ds+\int_0^t\sigma(X_s)F'(X_s)\,dB_s.$$

By the optional sampling theorem of martingale theory, we get

$$E\left[F(|X_\tau|)\right]=E\left[\int_0^\tau\varphi(X_s)\,ds\right]=E(J_\tau), \tag{10}$$

for any bounded stopping time τ. This shows that $F(|X_t|)$ is dominated by J_t and J_t is dominated by $F(X_t^*)$. By Lemma 1, we have

$$E\left[(X_\tau^*)^p\right] = E\left[H_p\left(F(X_\tau^*)\right)^p\right] \leqslant E\left[\left(H_p + \lambda \widetilde{H}_p\right)\left(\frac{J_\tau}{\lambda}\right)\right] \leqslant (\lambda C_p + 1) E\left[H_p\left(\frac{J_\tau}{\lambda}\right)\right],$$

for $0 < \lambda \leqslant 1$. By the same approach for $X_t = \lambda J_t$ and $A_t = \lambda F(X_t^*)$ in Lemma 1, we have

$$E\left[H_p(\lambda J_\tau)\right] \leqslant E\left[\left(H_p + \lambda \widetilde{H}_p\right)(F(X_\tau^*))\right] \leqslant (\lambda C_p + 1) E\left[(X_\tau^*)^p\right].$$

This completes the proof of Theorem 1. □

By Theorem 1, we can establish some L_p inequalities for diffusion processes satisfying the inequality (8) and study the constants in these L_p inequalities as $p \to 0^+$.

2.1. $\varphi(x) = N x^\nu \sigma^2(x)$

THEOREM 2. *Let X be a diffusion process given by (1), starting at zero, $\mu(x)$ and $\sigma(x)$ be continuous functions with $x|\mu(x)| = K\sigma^2(x)$, $\varphi(x) = N x^\nu \sigma^2(x)(\nu \geqslant -1)$, for some constants $K > 0$, $N > 0$ and $\mu(x) \geqslant 0$ for $x \geqslant 0$ or $\mu(x) \leqslant 0$ and $\nu + 1 > 2K$. $J_t = \int_0^t \varphi(X_s)\,ds$. If $X_t \geqslant 0$ or $F(x)$ is even, then for $0 < p < 2+\nu$ and any stopping time τ, we have*

$$\frac{a}{c_{p,\nu}} \left\| J_\tau^{\frac{1}{2+\nu}} \right\|_p \leqslant \|X_\tau^*\|_p \leqslant a c_{p,\nu} \left\| J_\tau^{\frac{1}{2+\nu}} \right\|_p, \tag{11}$$

where a is an absolute constant and

$$c_{p,\nu} = \left(\frac{2+\nu}{2+\nu-p}\right)^{\frac{1}{p}} \cdot \left(\frac{2+\nu}{p}\right)^{\frac{1}{2+\nu}} = O\left(\frac{1}{p^{\frac{1}{2+\nu}}}\right), \quad p \to 0^+.$$

Proof. Let $F(x)$ be the solution of the equation (3)

$$\sigma^2(x)\frac{d^2y}{d^2x} + 2\mu(x)\frac{dy}{dx} = 2\varphi(x), y(0) = y'(0) = 0.$$

For $\mu(x) \geqslant 0$, from (7) we have

$$\begin{aligned} F(x) &= \frac{2N}{(2K+\nu+1)(2+\nu)} x^{2+\nu}, \\ H_p &= \left(\frac{(2K+\nu+1)(2+\nu)}{2N}\right)^{\frac{p}{2+\nu}} x^{\frac{p}{2+\nu}}, \quad p > 0, \\ \widetilde{H}_p(x) &= \frac{2+\nu}{2+\nu-p} H_p(x), \quad 0 < p < 2+\nu. \end{aligned}$$

For $\mu(x) \leqslant 0$ and $\nu+1>2K$,

$$F(x)=\frac{2N}{(\nu-2K+1)(2+\nu)}x^{2+\nu},$$
$$H_p=\left(\frac{(\nu-2K+1)(2+\nu)}{2N}\right)^{\frac{p}{2+\nu}}x^{\frac{p}{2+\nu}}, \quad p>0,$$
$$\widetilde{H}_p(x)=\frac{2+\nu}{2+\nu-p}H_p(x), \quad 0<p<2+\nu.$$

Let

$$a=\max\left\{\left(\frac{(2K+\nu+1)(2+\nu)}{2N}\right)^{\frac{1}{2+\nu}}, \left(\frac{(\nu-2K+1)(2+\nu)}{2N}\right)^{\frac{1}{2+\nu}}\right\}.$$

By Theorem 1, we obtain

$$\begin{aligned} E\left[(X_\tau^*)^p\right] &\leqslant E\left[\left(\lambda\frac{2+\nu}{2+\nu-p}+1\right)\cdot a^p\cdot\left(\frac{J_\tau}{\lambda}\right)^{\frac{p}{2+\nu}}\right] \\ &= a^p\cdot\left(\lambda\frac{2+\nu}{2+\nu-p}+1\right)\cdot\lambda^{\frac{-p}{2+\nu}}E\left(J_\tau^{\frac{p}{2+\nu}}\right). \end{aligned}$$

Let

$$\phi_{p,\nu}(\lambda)=\lambda^{\frac{-p}{2+\nu}}\left(\lambda\frac{2+\nu}{2+\nu-p}+1\right),$$

then $\phi_{p,\nu}$ is strictly decreasing in $(0,\frac{p}{2+\nu})$, and strictly increasing in $(\frac{p}{2+\nu},1]$. $\phi_{p,\nu}$ takes its minimum at $\lambda=\frac{p}{2+\nu}$ and yields the desired inequality

$$E\left[(X_\tau^*)^p\right]\leqslant a^p\cdot\frac{2+\nu}{2+\nu-p}\cdot\left(\frac{2+\nu}{p}\right)^{\frac{p}{2+\nu}}E\left(J_\tau^{\frac{p}{2+\nu}}\right).$$

For the left hand, by Theorem 1

$$E\left(a^p\cdot J_\tau^{\frac{p}{2+\nu}}\right)\leqslant\lambda^{\frac{-p}{2+\nu}}\left(\lambda\frac{2+\nu}{2+\nu-p}+1\right)E\left[(X_\tau^*)^p\right].$$

Take $\lambda=\frac{p}{2+\nu}$, we get

$$a^pE\left(J_\tau^{\frac{p}{2+\nu}}\right)\leqslant\frac{2+\nu}{2+\nu-p}\cdot\left(\frac{2+\nu}{p}\right)^{\frac{p}{2+\nu}}E\left[(X_\tau^*)^p\right].$$

This completes the proof of Theorem 2. □

REMARK 1. Since

$$\frac{2+\nu}{2+\nu-p}\cdot\left(\frac{2+\nu}{p}\right)^{\frac{p}{2+\nu}}<\phi_{p,\nu}(1)=\frac{4+2\nu-p}{2+\nu-p}$$

and when $\nu = 0$, we have

$$\frac{2}{2-p} \cdot \left(\frac{2}{p}\right)^{\frac{p}{2}} < \frac{4-p}{2-p}.$$

We obtain some constants smaller than the constants widely used.

2.2. $N_1|x|^\gamma \sigma^2(x) \leqslant \varphi(x) \leqslant N_2|x|^\gamma \sigma^2(x) e^{N_3|x|^{\gamma+1}/\gamma+1}$

THEOREM 3. *Let X be a diffusion process given by (1), starting at zero, $\mu(x)$ and $\sigma(x)$ be continuous functions satisfying the condition $S(\gamma, K_1, K_2), N_1|x|^\gamma \sigma^2(x) \leqslant \varphi(x) \leqslant N_2|x|^\gamma \sigma^2(x) e^{N_3|x|^{\gamma+1}/\gamma+1}$, for some constants $\gamma > -1, K_i > 0 (i = 1,2), N_i > 0 (i = 1,2,3)$ and $\mu(x) \leqslant 0$, for $x \geqslant 0$. $J_t = \int_0^t \varphi(X_s)\,ds$. If $X_t \geqslant 0$ or $F(x)$ is even, then for $0 < p < 2+\gamma$ and any stopping time τ, we have*

$$\frac{a_1}{c_{p,\gamma}} \left\| \ln^{\frac{1}{1+\gamma}} \left(1 + J_\tau^{\frac{1+\gamma}{2+\gamma}}\right) \right\|_p \leqslant \|X_\tau^*\|_p \leqslant a_2 c_{p,\gamma} \left\| \ln^{\frac{1}{1+\gamma}} \left(1 + J_\tau^{\frac{1+\gamma}{2+\gamma}}\right) \right\|_p, \tag{12}$$

where a_1 and a_2 are absolute constants and

$$c_{p,\gamma} = \left(\frac{2+\gamma}{2+\gamma-p}\right)^{\frac{1}{p}} \cdot \left(\frac{2+\gamma}{p}\right)^{\frac{1}{2+\gamma}} = O\left(\frac{1}{p^{\frac{1}{2+\gamma}}}\right), \quad p \to 0^+.$$

Proof. Let $F(x)$ be the solution of the equation (3). Since $\mu(x)$ and $\sigma(x)$ satisfy condition $S(\gamma, K_1, K_2)$ and $\mu(x) \leqslant 0$, for $x \geqslant 0$,

$$-K_2 x^\gamma \sigma^2(x) \leqslant \mu(x) \leqslant -K_1 x^\gamma \sigma^2(x).$$

Yan and Zhu[15] proved that there exist constants $c_i > 0, d_i > 0 (i = 1,2)$ depending only on γ, K_i, N_i such that

$$c_1 \int_0^x t^{1+\gamma} e^{d_1 t^{1+\gamma}} dt \leqslant F(x) \leqslant c_2 \int_0^x t^{1+\gamma} e^{d_2 t^{1+\gamma}} dt \ (x \geqslant 0).$$

From this inequality, we can show that there exist constants $\eta_2 \geqslant \eta_1 > 0$ depending only on γ such that

$$\left(e^{\eta_1 t^{1+\gamma}} - 1\right)^{\frac{1+\gamma}{2+\gamma}} \leqslant \int_0^x t^{1+\gamma} e^{d_1 t^{1+\gamma}} dt \leqslant \left(e^{\eta_2 t^{1+\gamma}} - 1\right)^{\frac{1+\gamma}{2+\gamma}}$$

holds, for all $x \geqslant 0$. Thus there exist constants $b_2 \geqslant b_1 > 0$ depending only on γ, K_i, N_i such that

$$\left(e^{b_1 t^{1+\gamma}} - 1\right)^{\frac{1+\gamma}{2+\gamma}} \leqslant F(x) \leqslant \left(e^{b_2 t^{1+\gamma}} - 1\right)^{\frac{1+\gamma}{2+\gamma}},$$

for $x \geqslant 0$. Hence, for $x \geqslant 0$ and $0 < p < \infty$, we have

$$\left(\frac{1}{b_2}\right)^{\frac{p}{1+\gamma}} \ln^{\frac{p}{1+\gamma}} \left(1 + x^{\frac{1+\gamma}{2+\gamma}}\right) \leqslant H_p(x) \leqslant \left(\frac{1}{b_1}\right)^{\frac{p}{1+\gamma}} \ln^{\frac{p}{1+\gamma}} \left(1 + x^{\frac{1+\gamma}{2+\gamma}}\right). \tag{13}$$

For the functions of type

$$H_p(x) = A^{\frac{p}{1+\gamma}} \ln^{\frac{p}{1+\gamma}}\left(1 + Bx^{\frac{1+\gamma}{2+\gamma}}\right),$$

for some constants $A > 0, B > 0$ and $0 < p < \infty$. Let

$$G_p(x) = \frac{x}{H_p(x)} \int_x^\infty \frac{dH_p(u)}{u}.$$

Elementary calculations show that

$$0 \leqslant G_p(x) \leqslant \frac{p}{2+\gamma-p},$$

for all $x \geqslant 0$ and $0 < p < 2+\gamma$. Hence, we get

$$\widetilde{H}_p(x) = H_p(x) + x\int_x^\infty \frac{dH_p(u)}{u} \leqslant \frac{2+\gamma}{2+\gamma-p} H_p(x), \tag{14}$$

for all $x \geqslant 0$ and $0 < p < 2+\gamma$. By Theorem 1, for $0 < \lambda \leqslant 1$, we have

$$\begin{aligned} E\left[(X_\tau^*)^p\right] &\leqslant (\lambda C_p + 1) E\left[H_p\left(\frac{J_\tau}{\lambda}\right)\right] \\ &\leqslant \left(\lambda \frac{2+\gamma}{2+\gamma-p} + 1\right) \cdot \left(\frac{1}{b_1}\right)^{\frac{p}{1+\gamma}} E\left[\ln^{\frac{p}{1+\gamma}}\left(1 + \left(\frac{J_\tau}{\lambda}\right)^{\frac{1+\gamma}{2+\gamma}}\right)\right] \\ &= \lambda^{\frac{-p}{2+\gamma}}\left(\frac{2+\gamma}{2+\gamma-p} + 1\right) \cdot \left(\frac{1}{b_1}\right)^{\frac{p}{1+\gamma}} E\left[\ln^{\frac{p}{1+\gamma}}\left(1 + J_\tau^{\frac{1+\gamma}{2+\gamma}}\right)\right]. \end{aligned}$$

Let

$$a_2 = \left(\frac{1}{b_1}\right)^{\frac{1}{1+\gamma}}, \quad \phi_{p,\gamma}(\lambda) = \lambda^{\frac{-p}{2+\gamma}}\left(\lambda \frac{2+\gamma}{2+\gamma-p} + 1\right),$$

then $\phi_{p,\gamma}$ takes its minimum at $\lambda = \frac{p}{2+\gamma}$ and yields the desired inequality

$$E\left[(X_\tau^*)^p\right] \leqslant a_2^p \cdot \frac{2+\gamma}{2+\gamma-p} \cdot \left(\frac{2+\gamma}{p}\right)^{\frac{p}{2+\gamma}} E\left[\ln^{\frac{p}{1+\gamma}}\left(1 + J_\tau^{\frac{1+\gamma}{2+\gamma}}\right)\right].$$

For the left hand, by Theorem 1

$$E\left[H_p(\lambda J_\tau)\right] \leqslant (\lambda C_p + 1) E\left[(X_\tau^*)^p\right].$$

From (13), we have

$$\left(\frac{1}{b_2}\right)^{\frac{p}{1+\gamma}} E\left[\ln^{\frac{p}{1+\gamma}}\left(1 + J_\tau^{\frac{1+\gamma}{2+\gamma}}\right)\right] \leqslant \lambda^{\frac{-p}{2+\gamma}}\left(\frac{2+\gamma}{2+\gamma-p} + 1\right) E\left[(X_\tau^*)^p\right].$$

Let $a_1 = \left(\frac{1}{b_2}\right)^{\frac{1}{1+\gamma}}$ and take $\lambda = \frac{p}{2+\gamma}$, we get

$$a_1^p E\left[\ln^{\frac{p}{1+\gamma}}\left(1 + J_\tau^{\frac{1+\gamma}{2+\gamma}}\right)\right] \leqslant \frac{2+\gamma}{2+\gamma-p} \cdot \left(\frac{2+\gamma}{p}\right)^{\frac{p}{2+\gamma}} E\left[(X_\tau^*)^p\right].$$

This completes the proof of Theorem 3. □

2.3. $N_1|x|^{\gamma-1}\sigma^2(x) \leqslant \varphi(x) \leqslant N_2|x|^{\gamma-1}\sigma^2(x)e^{N_3|x|^{\gamma+1}/\gamma+1}$

As in the proof of Theorem 3, we can give the following theorem.

THEOREM 4. *Let X be a diffusion process given by (1), starting at zero, $\mu(x)$ and $\sigma(x)$ be continuous functions satisfying the condition $S(\gamma, K_1, K_2), N_1|x|^{\gamma-1}\sigma^2(x) \leqslant \varphi(x) \leqslant N_2|x|^{\gamma-1}\sigma^2(x)e^{N_3|x|^{\gamma+1}/\gamma+1}$, for some constants $\gamma > 0, K_i > 0 (i = 1,2), N_i > 0 (i = 1,2,3)$ and $\mu(x) \leqslant 0$, for $x \geqslant 0, J_t = \int_0^t \varphi(X_s)\,ds$. If $X_t \geqslant 0$ or $F(x)$ is even, then for $0 < p < 1+\gamma$ and any stopping time τ, we have*

$$\frac{a_1}{c_{p,\gamma}}\left\|\ln^{\frac{1}{1+\gamma}}(1+J_\tau)\right\|_p \leqslant \|X_\tau^*\|_p \leqslant a_2 c_{p,\gamma}\left\|\ln^{\frac{1}{1+\gamma}}(1+J_\tau)\right\|_p, \tag{15}$$

where a_1 and a_2 are absolute constants and

$$c_{p,\gamma} = \left(\frac{1+\gamma}{1+\gamma-p}\right)^{\frac{1}{p}} \cdot \left(\frac{1+\gamma}{p}\right)^{\frac{1}{1+\gamma}} = O\left(\frac{1}{p^{\frac{1}{1+\gamma}}}\right), \quad p \to 0^+.$$

If $F(x)$ is not even, by the method of Peskir[9], define the functions $F_1(x)$ and $F_2(x)$ on R_+ as follows

$$F_1(x) = \max\{F(-x), F(x)\}, \;\; F_2(x) = \min\{F(-x), F(x)\}$$

and let $H_i(x) = F_i^{-1}(x)$ be the inverse of $F_i(x)$, for $x \geqslant 0$, $H_{ip}(x) = [F_i^{-1}(x)]^p$, for $p > 0$ $(i = 1,2)$.

$$\widetilde{H}_{ip}(x) = H_{ip}(x) + x\int_x^\infty \frac{dH_{ip}(u)}{u}, \quad p > 0.$$

As in the proof of Theorem 1, we can give the following inequality.

THEOREM 5. *Let X be a diffusion process given by (1), starting at zero, $F(x)$ be the solution of equation (3), $\mu(x), \sigma(x), \varphi(x)$ be continuous functions and $\varphi(x) \geqslant 0, J_t = \int_0^t \varphi(X_s)\,ds$. If*

$$\widetilde{H}_{ip}(x) \leqslant C_{ip}H_{ip}(x), \quad i = 1,2, \tag{16}$$

for some $p > 0$ and all $x \geqslant 0$. Then for $0 < \lambda \leqslant 1$, we have

$$\frac{1}{\lambda C_{1p}+1}E\left[H_{1p}(\lambda J_\tau)\right] \leqslant E\left[(X_\tau^*)^p\right] \leqslant (\lambda C_{2p}+1)E\left[H_{2p}\left(\frac{J_\tau}{\lambda}\right)\right], \tag{17}$$

for any stopping time τ, where C_{1p} and C_{2p} are constants.

As in the Theorem 2, Theorem 3 and Theorem 4, similar inequalities can also be established.

3. Applications

3.1. L_p inequalities for the Ornstein-Uhlenbeck process

Let $V=(V_t)_{t\geqslant 0}$ be an Ornstein-Uhlenbeck velocity process solving the Langevin equation

$$dV_t=-\beta V_t dt+dB_t, \tag{18}$$

with $V_0=0$, where $\beta>0$ and $B=(B_t)_{t\geqslant 0}$ is a standard Brownian motion. $\mu(x)=-\beta x$ is an odd function and $\sigma^2(x)=1$ is an even function. The Ornstein-Uhlenbeck process satisfies the condition $S(1,\beta,\beta)$. Graversen and Peskir[7] introduced the functional

$$I_t=\int_0^t e^{\beta V_r^2}dr$$

and established the following Davis-type inequality

$$\frac{A_1}{\sqrt{\beta}}E\left[\sqrt{\ln\left(1+\beta I_\tau\right)}\right]\leqslant E\left(\sup_{0\leqslant t\leqslant\tau}|V_t|\right)\leqslant\frac{A_2}{\sqrt{\beta}}E\left[\sqrt{\ln(1+\beta I_\tau)}\right], \tag{19}$$

for any stopping time τ with $A_1=\frac{1}{3}$ and $A_2=3$.

Take $\varphi(x)=e^{\beta x^2}$, then $F(x)=\frac{1}{\beta}\left(e^{\beta x^2}-1\right)$ is an even function. By Theorem 3, with more accurate calculation, for $0<p<2$ and any stopping time τ, we have

$$\frac{1}{c_p\sqrt{\beta}}\left\|\ln^{\frac{1}{2}}\left(1+\beta I_\tau\right)\right\|_p\leqslant\|V_\tau^*\|_p\leqslant\frac{c_p}{\sqrt{\beta}}\left\|\ln^{\frac{1}{2}}\left(1+\beta I_\tau\right)\right\|_p, \tag{20}$$

with $c_p=\left(\frac{2}{2-p}\right)^{\frac{1}{p}}\sqrt{\frac{2}{p}}=O\left(\frac{1}{\sqrt{p}}\right)$, as $p\to 0^+$.

If $p=1$, we get the inequality of Graversen and Peskir (19) with smaller constants $A_1=\frac{1}{2\sqrt{2}}$ and $A_2=2\sqrt{2}$.

Take $\varphi(x)=3\beta^{\frac{3}{2}}|x|e^{\beta x^2}$, $J_t=\int_0^t\varphi(X_s)ds$. Then $F(x)$ is an even function and for $x\geqslant 0$

$$F(x)=3\beta^{\frac{3}{2}}\int_0^x t^2e^{\beta t^2}dt,$$

$$\left(e^{\frac{2}{3}\beta x^2}-1\right)^{\frac{3}{2}}\leqslant F(x)\leqslant\left(e^{\beta x^2}-1\right)^{\frac{3}{2}}.$$

By Theorem 3, for $0<p<2$ and any stopping time τ, we have

$$\frac{1}{c_p\sqrt{\beta}}\left\|\ln^{\frac{1}{2}}\left(1+J_\tau^{\frac{2}{3}}\right)\right\|_p\leqslant\|V_\tau^*\|_p\leqslant\sqrt{\frac{3}{2}}\frac{c_p}{\sqrt{\beta}}\left\|\ln^{\frac{1}{2}}\left(1+J_\tau^{\frac{2}{3}}\right)\right\|_p, \tag{21}$$

with $c_p=\left(\frac{3}{3-p}\right)^{\frac{1}{p}}\left(\frac{3}{p}\right)^{\frac{1}{3}}=O\left(\frac{1}{\sqrt[3]{p}}\right)$, as $p\to 0^+$.

3.2. L_p inequalities for Bessel processes

Let $\delta \geqslant 0$ and $x \geqslant 0$. The unique strong solution of the stochastic differential equation

$$dY_t = \delta dt + 2\sqrt{|Y_t|}dB_t,\ Y_0 = x \tag{22}$$

is called a squared Bessel process of dimension δ, started at x and the process $Z = \sqrt{|Y|}$ is called a Bessel process of dimension δ.

For the squared Bessel process Y and Bessel process Z started at 0, Yan and Zhu[15] established the following inequalities

$$\frac{\delta}{a_p}\|\tau\|_p \leqslant \|Y_\tau^*\|_p \leqslant a_p\delta\|\tau\|_p, \quad 0<p<1,$$

with $a_p = \left(\frac{2-p}{1-p}\right)^{\frac{1}{p}} = O\left(2^{\frac{1}{p}}\right)$ as $p \to 0^+$.

$$\frac{\sqrt{\delta}}{b_p}\|\sqrt{\tau}\|_p \leqslant \|Z_\tau^*\|_p \leqslant b_p\sqrt{\delta}\|\sqrt{\tau}\|_p, \quad 0<p<2,$$

with $b_p = \left(\frac{4-p}{2-p}\right)^{\frac{1}{p}} = O\left(2^{\frac{1}{p}}\right)$ as $p \to 0^+$.

Since Y and Z are positive processes, we get the following inequalities from Theorem 2 with $\varphi(x) = 1$

$$\frac{\delta}{c_p}\delta\|\tau\|_p \leqslant \|Y_\tau^*\|_p \leqslant c_p\delta\|\tau\|_p, \quad 0<p<1, \tag{23}$$

with $c_p = \left(\frac{1}{1-p}\right)^{\frac{1}{p}}\frac{1}{p} = O\left(\frac{1}{p}\right)$ as $p \to 0^+$.

$$\frac{\sqrt{\delta}}{d_p}\|\sqrt{\tau}\|_p \leqslant \|Z_\tau^*\|_p \leqslant d_p\sqrt{\delta}\|\sqrt{\tau}\|_p, \quad 0<p<2, \tag{24}$$

$d_p = \left(\frac{2}{2-p}\right)^{\frac{1}{p}}\sqrt{\frac{2}{p}} = O\left(\frac{1}{\sqrt{p}}\right)$ as $p \to 0^+$.

REMARK 2. The constants obtained by Lenglart domination inequalities are:

$$a_p = \left(\frac{2-p}{1-p}\right)^{\frac{1}{p}} = \left(2+\frac{p}{1-p}\right)^{\frac{1}{p}} = O\left(2^{\frac{1}{p}}\right), \quad p \to 0^+,$$

$$b_p = \left(\frac{4-p}{2-p}\right)^{\frac{1}{p}} = \left(2+\frac{p}{2-p}\right)^{\frac{1}{p}} = O\left(2^{\frac{1}{p}}\right), \quad p \to 0^+.$$

The growth rate of a_p and b_p, as $p \to 0^+$, are the exponential of $\frac{1}{p}$.

REMARK 3. The constants we obtained are $O\left(\frac{1}{p}\right)$ or $O\left(\frac{1}{\sqrt{p}}\right)$. The growth rates of constants as $p \to 0^+$ are substantially improved.

3.3. L_p inequalities for reflected Brownian motion with drift

Let $X=(X_t)_{t\geqslant 0}$ be the strong solution of the SDE

$$dX_t = -\mu sgn(X_t)\,dt + dB_t,\ X_0 = 0, \tag{25}$$

where $\mu > 0$ and $B=(B_t)$ is a standard Brownian motion. $|X| = (|X_t|)_{t\geqslant 0}$ is a realization of the reflected Brownian motion with drift $-\mu$. $\mu(x) = -\mu sgn(x)$ and $\sigma(x) = 1$ satisfies the condition $S(0,\mu,\mu)$. Take $\varphi(x)=1$, then $F(x)$ is an even function and for $x \geqslant 0$

$$F(x) = \frac{1}{2\mu^2}\left(e^{2\mu x} - 2\mu x - 1\right).$$

Since

$$\frac{1}{\mu^2}\left(e^{\frac{\mu x}{2}} - 1\right)^2 \leqslant F(x) \leqslant \frac{1}{\mu^2}\left(e^{\mu x} - 1\right)^2,$$
$$\frac{1}{\mu}\ln\left(1+\mu\sqrt{x}\right) \leqslant H(x) \leqslant \frac{2}{\mu}\ln\left(1+\mu\sqrt{x}\right).$$

As in the proof of Theorem 3, we can obtain the following inequality

$$\frac{1}{\mu c_p}\left\|\ln\left(1+\mu\sqrt{\tau}\right)\right\|_p \leqslant \|X_\tau^*\|_p \leqslant \frac{2c_p}{\mu}\left\|\ln\left(1+\mu\sqrt{\tau}\right)\right\|_p, \quad 0<p<2, \tag{26}$$

for any stopping time τ of X. And $c_p = \left(\frac{2}{2-p}\right)^{\frac{1}{p}}\sqrt{\frac{2}{p}} = O\left(\frac{1}{\sqrt{p}}\right)$ as $p \to 0^+$.

For $p=1$, we get the following inequality

$$\frac{1}{2\sqrt{2}\mu}E\left[\ln\left(1+\mu\sqrt{\tau}\right)\right] \leqslant E\left(X_\tau^*\right) \leqslant \frac{4\sqrt{2}}{\mu}E\left[\ln\left(1+\mu\sqrt{\tau}\right)\right]. \tag{27}$$

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