

BRAUER–TYPE UPPER BOUNDS FOR Z -SPECTRAL RADIUS OF WEAKLY SYMMETRIC NONNEGATIVE TENSORS

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Abstract. In this paper, we establish four Brauer-type upper bounds and a lower bound for Z -spectral radius of weakly symmetric nonnegative tensors without irreducible assumption. These bounds are shown to be sharper than the existing bounds via running examples. As an application, an upper bound on the largest Z -eigenvalues of the adjacency tensors for uniform hypergraphs is provided.

1. Introduction

Let $\mathbb{C}(\mathbb{R})$ be the set of complex (real) numbers, $\mathbb{C}^n(\mathbb{R}^n)$ be the set of n -dimensional complex (real) vectors, and $[n] = \{1, 2, \dots, n\}$. Consider an m -order n -dimensional tensor \mathcal{A} consisting of n^m entries in \mathbb{R} :

$$a_{i_1 i_2 \dots i_m} \in \mathbb{R}, \quad i_j \in [n], \quad j = 1, 2, \dots, m.$$

\mathcal{A} is called nonnegative (positive) if $a_{i_1 i_2 \dots i_m} \geq 0$ ($a_{i_1 i_2 \dots i_m} > 0$).

Tensors are widely used in signal and image processing, continuum physics, higher-order statistics, blind source separation and multi-way data analysis [21]. Generally, tensor is a higher-order extension of matrix, and hence many concepts and the corresponding conclusions for matrices such as determinant, eigenvalue and singular value theory are extended to higher order tensors by exploring their multilinear algebra properties [3, 4, 10, 16, 11, 19, 26, 28, 30].

Based on matrix eigenvalue, two types of eigenvalue, called H -eigenvalue and Z -eigenvalue are developed for tensors [14, 19], and they are widely used in such as medical resonance imaging [1, 20], data analysis [9], higher-order Markov chains [12, 17], positive definiteness of even-order multivariate forms in automatical control [7, 11, 18]. However, the research on Z -eigenvalue problem is more complex than that for H -eigenvalue problem due to its nonhomogeneity [3]. For example, we obtained the largest H -eigenvalue of weakly irreducible nonnegative tensors by the iterative algorithm proposed in [6, 17], but the largest Z -eigenvalue can not be easily obtained

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by the iterative scheme in such as [3, 9, 20], even for supersymmetric irreducible non-negative tensors. Hence, researchers turn to investigating the tensor Z -eigenvalue inclusion set and making an estimation of the largest Z -eigenvalue. For this, based on weakly symmetric condition, Chang et al. [3] established the equivalent relation between the largest Z -eigenvalue and Z -spectral radius of nonnegative tensors. By characterizing the ratio of the smallest and largest values of a Perron vector, He et al. [8] derived the Ledermann-like upper bound for the largest Z -eigenvalue of the weakly symmetric and positive tensors. Furthermore, Li et al. [13] not only put forward the improved Ostrowski-like upper bound, but also gave the lower bounds for the largest Z -eigenvalue. Since then, the Z -spectral radius problems were deeply investigated [15, 22, 24, 25, 27]. Recently, Bu et al. [2] proposed Brauer-type eigenvalue inclusion sets and obtained bounds on the H -spectral radius for general tensors by the number of positive entries of a nonnegative eigenvector. To the best of our knowledge, Brauer-type bounds of Z -spectral radius are still underdeveloped because of its nonhomogeneity. In this paper, inspired by the articles [2, 3, 8, 13, 15, 22, 29], we establish four Brauer-type upper bounds and a lower bound for Z -spectral radius of weakly symmetric nonnegative tensors without irreducible assumption, which are easy to compute and have simple expressions.

This paper is organized as follows. In Section 2, we introduce important notation and recall preliminary results. In Section 3, we establish Brauer-type lower and upper bounds for the Z -spectral radius of weakly symmetric nonnegative tensors without irreducible condition. By numerical examples, we show they are tighter than existing bounds in some cases. As an application, we give an upper bound on the largest Z -eigenvalues of the adjacency tensors for uniform hypergraphs and show that the bounds are better than existing bounds by a running example in Section 4.

2. Notation and preliminaries

In this section, we introduce some definitions and give important properties on the tensor eigenvalues needed in the subsequent analysis.

DEFINITION 2.1. Let \mathcal{A} be an m -order n -dimensional tensor.

- (i) We say that $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is an eigenvalue-eigenvector of \mathcal{A} if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $(\mathcal{A}x^{m-1})_i = \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} x_{i_2} \dots x_{i_m}$ and $x^{[m-1]} = [x_1^{m-1}, x_2^{m-1}, \dots, x_n^{m-1}]^T$. (λ, x) is called an H -eigenpair if both of them are real.

- (ii) We say that $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is an E -eigenpair of \mathcal{A} if

$$\mathcal{A}x^{m-1} = \lambda x \text{ and } x^T x = 1.$$

(λ, x) is called a Z -eigenpair if both of them are real.

(iii) We denote the set of Z -eigenvalues for \mathcal{A} by $\sigma(\mathcal{A})$. Assume $\sigma(\mathcal{A}) \neq \emptyset$, then the Z -spectral radius of \mathcal{A} is defined as

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

DEFINITION 2.2. [3] Let \mathcal{A} be an m -order n -dimensional tensor. \mathcal{A} is weakly symmetric if the associated homogeneous polynomial $\mathcal{A}x^m$ satisfies $\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}$.

The following lemmas play an important role in Z -spectral analysis of the nonnegative tensors [3].

LEMMA 2.1. Let \mathcal{A} be an m -order and n -dimensional weakly symmetric nonnegative tensor. Then

$$\bar{\lambda} = \lambda^* = \rho(\mathcal{A}),$$

where $\bar{\lambda} = \{\max \mathcal{A}x^m : x^T x = 1, x \in \mathbb{R}^n\}$, λ^* denotes the largest Z -eigenvalue. Moreover, if $(\bar{\lambda}, \bar{x})$ is a Z -eigenpair of \mathcal{A} , then $(\bar{\lambda}, |\bar{x}|)$ is also a Z -eigenpair of \mathcal{A} .

LEMMA 2.2. Let $\mathcal{A} = (a_{i_1 \dots i_m})$ be an m -order and n -dimensional weakly symmetric nonnegative tensor. Then

$$\rho(\mathcal{A}) \geq \max \left\{ \max_{i \in [n]} a_{i \dots i}, \frac{n \min_{i \in [n]} R_i(\mathcal{A})}{n^{\frac{m}{2}}} \right\},$$

where $R_i(\mathcal{A}) = \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m}$.

3. Brauer-type upper bounds for the Z -spectral radius

In this section, we shall establish four Brauer-type upper bounds and a lower bound for the Z -spectral radius of weakly symmetric nonnegative tensors without irreducible assumption, which improves the corresponding result in [3, 8, 15, 13, 22, 29] in some cases.

THEOREM 3.1. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ be a weakly symmetric nonnegative tensor with order $m \geq 2$ and dimension $n \geq 2$ and $R_i(\mathcal{A}) \neq 0$. Then

$$\max \left\{ \max_{i \in [n]} a_{i \dots i}, \frac{\sum_{i=1}^n R_i(\mathcal{A})}{n^{\frac{m}{2}}} \right\} \leq \rho(\mathcal{A}) \leq \max_{a_{i_1 \dots i_m} \neq 0} \prod_{j=1}^m R_{i_j}^{\frac{1}{m}}(\mathcal{A}). \tag{3.1}$$

Proof. By Lemma 2.1, we assume that $\rho(\mathcal{A})$ is the Z -spectral radius of \mathcal{A} with corresponding nonnegative eigenvector $x = (x_1, x_2, \dots, x_n)^T$, i.e.

$$\rho(\mathcal{A})x = \mathcal{A}x^{m-1},$$

that is

$$\rho(\mathcal{A})x_i = \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} x_{i_2} \dots x_{i_m}. \tag{3.2}$$

Multiplying by x_i on the both side of the above equality, one has

$$\rho(\mathcal{A})x_i^2 = \sum_{a_{ii_2 \dots i_m} \neq 0} a_{ii_2 \dots i_m} x_i x_{i_2} \dots x_{i_m}.$$

Letting $x_\beta = \max \{x_{i_1} x_{i_2} \dots x_{i_m} : a_{i_1 i_2 \dots i_m} \neq 0, i_1, \dots, i_m \in [n]\}$, and noting that $0 \leq x_i \leq 1$, we have

$$\rho(\mathcal{A})x_i^2 \leq R_i(\mathcal{A})x_\beta, \forall i \in [n]. \tag{3.3}$$

The following argument is divided into two parts.

Case 1. $x_\beta = 0$. Noting that $x \neq 0$, we can suppose that $x_t \neq 0, t \in [n]$. From $R_t(\mathcal{A}) \neq 0$, there exists $a_{t i_2 \dots i_m} \neq 0$ and $x_t x_{i_2} \dots x_{i_m} = x_\beta = 0$. By (3.3), we have $\rho(\mathcal{A})x_t^2 \leq 0$, that is $\rho(\mathcal{A}) = 0$. Thus

$$\rho(\mathcal{A})^m \leq \prod_{\substack{j=1 \\ a_{i_1 \dots i_m} \neq 0}}^m R_{i_j}(\mathcal{A}). \tag{3.4}$$

Case 2. $x_\beta \neq 0$. Suppose that $x_\beta = x_{j_1} x_{j_2} \dots x_{j_m}$. It follows from (3.3) that

$$\begin{aligned} \rho(\mathcal{A})x_{j_1}^2 &\leq R_{j_1}(\mathcal{A})x_\beta, \\ &\vdots \\ \rho(\mathcal{A})x_{j_m}^2 &\leq R_{j_m}(\mathcal{A})x_\beta. \end{aligned}$$

Moreover,

$$\prod_{k=1}^m \rho(\mathcal{A})^m x_{j_k}^2 \leq \prod_{k=1}^m R_{j_k}(\mathcal{A}) x_\beta^m.$$

Hence,

$$\rho(\mathcal{A})^m \leq \prod_{\substack{j=1 \\ a_{i_1 \dots i_m} \neq 0}}^m R_{i_j}(\mathcal{A}) x_\beta^{m-2} \leq \prod_{\substack{j=1 \\ a_{i_1 \dots i_m} \neq 0}}^m R_{i_j}(\mathcal{A}) \leq \max_{a_{i_1 \dots i_m} \neq 0} \prod_{j=1}^m R_{i_j}(\mathcal{A}). \tag{3.5}$$

Combining (3.4) with (3.5) yields

$$\rho(\mathcal{A}) \leq \max_{a_{i_1 \dots i_m} \neq 0} \prod_{j=1}^m R_{i_j}^{\frac{1}{m}}(\mathcal{A}).$$

Now, we make an estimation of the lower bound of $\rho(\mathcal{A})$. Suppose $e = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$ is the n order vector. It follows from Lemma 2.1 that

$$\rho(\mathcal{A}) \geq \mathcal{A}e^m = \sum_{i_1, i_2, \dots, i_m=1}^n a_{i_1 i_2 \dots i_m} \frac{1}{\sqrt{n}} \dots \frac{1}{\sqrt{n}} = \frac{\sum_{i_1, i_2, \dots, i_m=1}^n a_{i_1 i_2 \dots i_m}}{n^{\frac{m}{2}}}. \tag{3.6}$$

So the results hold. \square

Taking the principal diagonal element $a_{i\dots i}$ out of $R_i(\mathcal{A})$, we establish another Brauer-type upper bound for the Z-spectral radius.

THEOREM 3.2. *Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ be a weakly symmetric nonnegative tensor with order $m \geq 2$ and dimension $n \geq 2$ and $r_i(\mathcal{A}) \neq 0$. Then,*

$$\prod_{j=1}^m (\rho(\mathcal{A}) - a_{i_j i_j \dots i_j}) \leq \max_{\substack{a_{i_1 \dots i_m} \neq 0 \\ (i_2, \dots, i_m) \neq (i_1, \dots, i_1)}} \prod_{j=1}^m r_{i_j}(\mathcal{A}), \tag{3.7}$$

where $r_{i_j}(\mathcal{A}) = R_{i_j}(\mathcal{A}) - a_{i_j \dots i_j}$.

Proof. Suppose $\rho(\mathcal{A})$ is the Z-spectral radius of \mathcal{A} with corresponding nonnegative eigenvector x . It follows from (3.2) and Lemma 2.1, that

$$\rho(\mathcal{A})x_i = \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} x_{i_2} \dots x_{i_m} = \sum_{\substack{a_{ii_2 \dots i_m} \neq 0 \\ (i_2, \dots, i_m) \neq (i, \dots, i)}} a_{ii_2 \dots i_m} x_{i_2} \dots x_{i_m} + a_{i \dots i} x_i^{m-1}.$$

Multiplying by x_i on the both sides of the above equality, we obtain

$$\rho(\mathcal{A})x_i^2 = \sum_{\substack{a_{ii_2 \dots i_m} \neq 0 \\ (i_2, \dots, i_m) \neq (i, \dots, i)}} a_{ii_2 \dots i_m} x_i x_{i_2} \dots x_{i_m} + a_{i \dots i} x_i^m.$$

From $0 \leq x_i^m \leq x_i^2 \leq 1$, one has

$$\begin{aligned} (\rho(\mathcal{A}) - a_{i \dots i})x_i^2 &= \sum_{\substack{a_{ii_2 \dots i_m} \neq 0 \\ (i_2, \dots, i_m) \neq (i, \dots, i)}} a_{ii_2 \dots i_m} x_i x_{i_2} \dots x_{i_m} + a_{i \dots i} (x_i^m - x_i^2) \\ &\leq \sum_{\substack{a_{ii_2 \dots i_m} \neq 0 \\ (i_2, \dots, i_m) \neq (i, \dots, i)}} a_{ii_2 \dots i_m} x_i x_{i_2} \dots x_{i_m}. \end{aligned}$$

Letting $x_\beta = \max \{x_{i_1} x_{i_2} \dots x_{i_m} : a_{i_1 i_2 \dots i_m} \neq 0, (i_2, \dots, i_m) \neq (i_1, \dots, i_1), i_1, \dots, i_m \in [n]\}$, then we get

$$(\rho(\mathcal{A}) - a_{i \dots i})x_i^2 \leq r_i(\mathcal{A})x_\beta. \tag{3.8}$$

Similar to the proof of Theorem 3.1, according to (3.8) and $\rho(\mathcal{A}) \geq \max_{i \in [n]} a_{i \dots i}$, one has

$$\prod_{j=1}^m (\rho(\mathcal{A}) - a_{i_j i_j \dots i_j}) \leq \max_{\substack{a_{i_1 i_2 \dots i_m} \neq 0 \\ (i_2, \dots, i_m) \neq (i_1, \dots, i_1)}} \prod_{j=1}^m r_{i_j}(\mathcal{A}). \quad \square$$

It is noted that the results of Theorems 3.1 and 3.2 are sharper than those of Proposition 3.3 in [3] and Corollary 4.5 in [22].

Choosing x_i as a component of x with the largest modulus and x_s as a component of x with the second largest modulus, we are at the position to establish the following theorem.

THEOREM 3.3. *Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ be a weakly symmetric nonnegative tensor with order $m \geq 2$ and dimension $n \geq 2$. Then,*

$$\rho(\mathcal{A}) \leq \max_{i,j \in [n], i \neq j} \frac{1}{2} \left\{ a_{i\dots i} + a_{j\dots j} + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A}) \right\},$$

where $\Delta_{i,j}(\mathcal{A}) = (a_{i\dots i} - a_{j\dots j})^2 + 4r_i(\mathcal{A})r_j(\mathcal{A})$.

Proof. Suppose that $\rho(\mathcal{A})$ is the Z-spectral radius of \mathcal{A} with corresponding non-negative eigenvector x . Let $x_t \geq x_s \geq \max\{x_k : k \in [n], k \neq t, k \neq s\}$. Obviously, $|x_t| > 0$. Noting that $0 \leq x_t^{m-1} \leq x_t \leq 1, 0 \leq x_s^{m-1} \leq x_s \leq 1$, from (3.2), we have

$$\begin{aligned} \rho(\mathcal{A})x_t - a_{t\dots t}x_t^{m-1} &= \sum_{\substack{i_2, \dots, i_m \in [n] \\ (t, i_2, \dots, i_m) \neq (t, \dots, t)}} a_{t i_2 \dots i_m} x_{i_2} \dots x_{i_m} \\ &\leq \sum_{\substack{i_2, \dots, i_m \in [n] \\ (t, i_2, \dots, i_m) \neq (t, \dots, t)}} a_{t i_2 \dots i_m} x_t^{m-2} x_s = r_t(\mathcal{A})x_t^{m-2} x_s \end{aligned}$$

and

$$(\rho(\mathcal{A}) - a_{t\dots t})x_t \leq \rho(\mathcal{A})x_t - a_{t\dots t}x_t^{m-1} \leq r_t(\mathcal{A})x_t^{m-2} x_s. \tag{3.9}$$

On the other hand, it holds that

$$(\rho(\mathcal{A}) - a_{s\dots s})x_s \leq \sum_{\substack{i_2, \dots, i_m \in [n] \\ (s, i_2, \dots, i_m) \neq (s, \dots, s)}} a_{s i_2 \dots i_m} x_t^{m-1} = r_s(\mathcal{A})x_t^{m-1}. \tag{3.10}$$

From Lemma 2.2, (3.9) with (3.10), using the fact that $\rho(\mathcal{A}) \geq \max_{i \in [n]} a_{i\dots i}$, we have

$$(\rho(\mathcal{A}) - a_{t\dots t})(\rho(\mathcal{A}) - a_{s\dots s}) \leq r_t(\mathcal{A})r_s(\mathcal{A})x_t^{2m-4} \leq r_t(\mathcal{A})r_s(\mathcal{A}),$$

equivalently

$$\rho(\mathcal{A})^2 - (a_{t\dots t} + a_{s\dots s})\rho(\mathcal{A}) + a_{t\dots t}a_{s\dots s} - r_t(\mathcal{A})r_s(\mathcal{A}) \leq 0.$$

Solving for $\rho(\mathcal{A})$, we obtain

$$\rho(\mathcal{A}) \leq \frac{1}{2} \left\{ a_{t\dots t} + a_{s\dots s} + \sqrt{(a_{t\dots t} - a_{s\dots s})^2 + 4r_t(\mathcal{A})r_s(\mathcal{A})} \right\}.$$

Thus

$$\rho(\mathcal{A}) \leq \max_{i,j \in [n], i \neq j} \frac{1}{2} \left\{ a_{i\dots i} + a_{j\dots j} + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A}) \right\}. \quad \square$$

From Theorem 3.3, the eigenvalue inclusion theorem of [23] for nonnegative matrices can be extended to weakly symmetric nonnegative tensors. Further, it can be verified that

$$\max_{i,j \in [n], i \neq j} \frac{1}{2} \left\{ a_{i\dots i} + a_{j\dots j} + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A}) \right\} \leq \max_{i \in [n]} R_i(\mathcal{A}).$$

THEOREM 3.4. *Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ be a weakly symmetric nonnegative tensor with order $m \geq 2$ and dimension $n \geq 2$. Then*

$$\rho(\mathcal{A}) \leq \max_{i,j \in [n], i \neq j} \frac{1}{2} \left\{ a_{i\dots i} + a_{j\dots j} + \tilde{r}_i(\mathcal{A}) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A}) \right\},$$

where $M_i(\mathcal{A}) = \sum_{j \neq i, j \in [n]} a_{ij\dots j}$, $\tilde{r}_i(\mathcal{A}) = r_i(\mathcal{A}) - M_i(\mathcal{A})$ and $\Lambda_{i,j}(\mathcal{A}) = (a_{i\dots i} - a_{j\dots j} + \tilde{r}_i(\mathcal{A}))^2 + 4M_i(\mathcal{A})r_j(\mathcal{A})$.

Proof. Suppose that $\rho(\mathcal{A})$ is the Z-spectral radius of \mathcal{A} with corresponding nonnegative eigenvector x . Let $x_t \geq x_s \geq \{\max x_k : k \in N, k \neq t, k \neq s\}$. Note that $0 \leq x_t^{m-1} \leq x_t \leq 1$, $0 \leq x_s^{m-1} \leq x_s \leq 1$. Similar to the proof of Theorem 3.3, we have

$$\begin{aligned} (\rho(\mathcal{A}) - a_{t\dots t})x_t &= \sum_{\substack{(i_2, \dots, i_m) \neq (t, \dots, t) \\ (i_2, \dots, i_m) \neq (j, \dots, j)}} a_{ti_2 \dots i_m} x_{i_2} \dots x_{i_m} + \sum_{j \neq t, j \in [n]} a_{tj \dots j} x_j^{m-1} + a_{t\dots t}(x_t^{m-1} - x_t) \\ &\leq \tilde{r}_t(\mathcal{A})x_t^{m-1} + M_t(\mathcal{A})x_s^{m-1} \leq \tilde{r}_t(\mathcal{A})x_t + M_t(\mathcal{A})x_s. \end{aligned}$$

Moreover,

$$(\rho(\mathcal{A}) - a_{t\dots t} - \tilde{r}_t(\mathcal{A}))x_t \leq M_t(\mathcal{A})x_s. \tag{3.11}$$

On the other hand, we get

$$(\rho(\mathcal{A}) - a_{s\dots s})x_s = \sum_{(i_2, \dots, i_m) \neq (s, \dots, s)} a_{si_2 \dots i_m} x_{i_2} \dots x_{i_m} + a_{s\dots s}(x_s^{m-1} - x_s) \leq r_s(\mathcal{A})x_t. \tag{3.12}$$

Multiplying (3.11) with (3.12) yields

$$(\rho(\mathcal{A}) - a_{t\dots t} - \tilde{r}_t(\mathcal{A}))(\rho(\mathcal{A}) - a_{s\dots s}) \leq M_t(\mathcal{A})r_s(\mathcal{A}).$$

Solving for $\rho(\mathcal{A})$ we obtain

$$\rho(\mathcal{A}) \leq \frac{1}{2} \left\{ a_{t\dots t} + a_{s\dots s} + \tilde{r}_t(\mathcal{A}) + \Lambda_{t,s}^{\frac{1}{2}}(\mathcal{A}) \right\},$$

which implies

$$\rho(\mathcal{A}) \leq \max_{i,j \in [n], i \neq j} \frac{1}{2} \left\{ a_{i\dots i} + a_{j\dots j} + \tilde{r}_i(\mathcal{A}) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A}) \right\}. \quad \square$$

Now, we shall give numerical comparisons among our results and existing bounds.

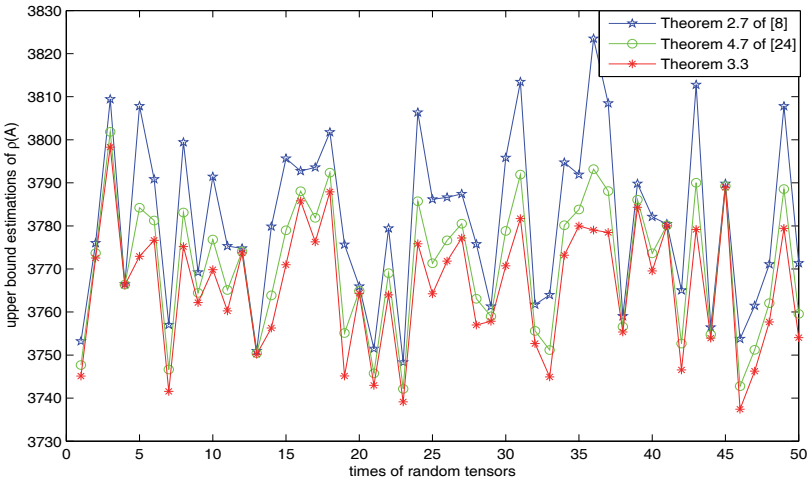
EXAMPLE 3.1. Consider 3 order 3 dimensional tensor $\mathcal{A} = (a_{ijk})$ defined by $a_{111} = \frac{1}{2}, a_{222} = 1, a_{333} = 3$ and all other entries are $\frac{1}{3}$. A direct computing gives that $(\rho(\mathcal{A}), x) = (3.1970, (0.1927, 0.1990, 0.9609))$. For this tensor, the bounds via different estimations given in the literature are shown in Table 1. It follows from Table 1 that our results are tighter than some existing results.

Table 1: Comparison estimations of the Z -spectral radius with different methods

Reference	bounds
Proposition 3.3 of [3]	$\rho(\mathcal{A}) \leq 9.8150$
Theorem 2.7 of [8]	$\rho(\mathcal{A}) \leq 5.6079$
Theorem 3.3 of [13]	$1.8070 \leq \rho(\mathcal{A}) \leq 5.5494$
Theorem 2.4 of [15]	$1.0863 \leq \rho(\mathcal{A}) \leq 5.2694$
Corollary 4.5 of [22]	$\rho(\mathcal{A}) \leq 5.6667$
Theorem 4.7 of [24]	$\rho(\mathcal{A}) \leq 5.2624$
Theorem 3.1	$2.4056 \leq \rho(\mathcal{A}) \leq 5.6667$
Theorem 3.1 and Theorem 3.2	$2.4056 \leq \rho(\mathcal{A}) \leq 5.1402$
Theorem 3.1 and Theorem 3.3	$2.4056 \leq \rho(\mathcal{A}) \leq 4.8480$
Theorem 3.1 and Theorem 3.4	$2.4056 \leq \rho(\mathcal{A}) \leq 5.4037$

EXAMPLE 3.2. Consider 3 order 60 dimensional tensor $\mathcal{A} = (a_{ijk})$ defined by all entries are randomly generated in $[0.1, 1]$. By randomly generating the above 50 tensors, we draw upper bounds of the spectral radius with different methods in Figure 1. We can verify that our results are sharper than some existing results from Figure 1.

Figure 1: Comparison estimations of the Z -spectral radius with random tensors



EXAMPLE 3.3. Consider 3 order n dimensional tensor $\mathcal{A} = (a_{ijk})$ defined by all entries are randomly generated in $[0.1, 1]$. The upper bound estimations of $\rho(\mathcal{A})$ are given with different variable dimensions in Table 2. From Table 2, the superiority of our conclusion is also shown.

Table 2: Comparison estimations of the Z-spectral radius with dimension change tensors

n	<i>Theorem 2.7 of [8]</i>	<i>Theorem 4.7 of [24]</i>	<i>Theorem 3.3</i>
30	$\rho(\mathcal{A}) \leq 1.3995e + 03$	$\rho(\mathcal{A}) \leq 1.3910e + 03$	$\rho(\mathcal{A}) \leq 1.3827e + 03$
60	$\rho(\mathcal{A}) \leq 3.8417e + 03$	$\rho(\mathcal{A}) \leq 3.8029e + 03$	$\rho(\mathcal{A}) \leq 3.7841e + 03$
90	$\rho(\mathcal{A}) \leq 7.1148e + 03$	$\rho(\mathcal{A}) \leq 7.1147e + 03$	$\rho(\mathcal{A}) \leq 7.1012e + 03$
120	$\rho(\mathcal{A}) \leq 1.1566e + 04$	$\rho(\mathcal{A}) \leq 1.1533e + 04$	$\rho(\mathcal{A}) \leq 1.1523e + 04$
150	$\rho(\mathcal{A}) \leq 1.6919e + 04$	$\rho(\mathcal{A}) \leq 1.6891e + 04$	$\rho(\mathcal{A}) \leq 1.6884e + 04$
180	$\rho(\mathcal{A}) \leq 2.3286e + 04$	$\rho(\mathcal{A}) \leq 2.3245e + 04$	$\rho(\mathcal{A}) \leq 2.3236e + 04$

4. The Z-spectral radius of uniform hypergraphs

It is well known that eigenvalues and eigenvectors of directly weighted hypergraphs are widely used in practice problem, such as complex network, image representation and so on. Some results on the Z-eigenvalues or E-eigenvalues of tensors of an even uniform hypergraph can be found in [5, 31]. Next, we use our bounds to estimate Z-spectral radius of direct hypergraphs.

Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ be a hypergraph with edge set $\mathbb{E} = \{\mathbb{E}_1, \mathbb{E}_2, \dots, \mathbb{E}_m\}$ and vertex set $\mathbb{V} = [n]$. If every edge of \mathcal{G} has cardinality k , then we call \mathcal{G} a k -uniform hypergraph. In the following, we consider k -uniform hypergraphs on n vertices with $3 \leq k \leq n$. The degree d_i of vertex i is defined as $d_i = |\{\mathbb{E}_p \in \mathbb{E} \mid i \in \mathbb{E}_p\}|$. For a k -uniform hypergraph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, its adjacency tensor $\mathcal{A}(\mathcal{G})$ corresponds to the following form:

$$\mathcal{A}(\mathcal{G})x^k = \sum_{\mathbb{E}_p \in \mathbb{E}} \mathcal{A}(\mathbb{E}_p)x^k, \forall x \in \mathbb{R}^n,$$

where $\mathcal{A}(\mathbb{E}_p)x^k = kx_{p_1} \dots x_{p_k}$ and $\mathbb{E}_p = \{p_1, \dots, p_k\} \subseteq \mathbb{V}$. $\mathcal{A}(\mathcal{G})$ is a k -order n -dimensional nonnegative symmetric tensor whose (i_1, \dots, i_k) -entry is

$$\mathcal{A}(\mathcal{G})_{i_1, \dots, i_k} = \begin{cases} \frac{1}{(k-1)!}, & \{i_1, i_2, \dots, i_k\} \in \mathbb{E}, \\ 0, & \text{otherwise.} \end{cases}$$

Xie et al. [31] considered the adjacency tensor and its Z-eigenvalues for a uniform hypergraph, and obtained some bounds on the Z-spectral radius of the adjacency tensors for uniform hypergraphs.

LEMMA 4.1. *Let $\mathcal{A}(\mathcal{G})$ be the adjacency tensor of a n -vertex k -uniform hypergraph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$. Then*

$$\rho(\mathcal{G}) \leq \binom{n-1}{k-1} n^{\frac{2-k}{2}} \tag{4.1}$$

and

$$\rho(\mathcal{G}) \leq \Delta, \tag{4.2}$$

where $\Delta = \max_{1 \leq i \leq n} d_i$ is the maximum degree of \mathcal{G} and $\rho(\mathcal{G})$ denotes the Z-spectral radius of $\mathcal{A}(\mathcal{G})$.

Considering that adjacency tensor of a uniform hypergraph is nonnegative and weakly symmetric, we further propose the following bound.

THEOREM 4.1. *Let $\mathcal{A}(\mathcal{G}) = (a_{i_1 i_2 \dots i_k})$ be the adjacency tensor of a n -vertex k -uniform hypergraph \mathcal{G} . Then we have*

$$\rho(\mathcal{G}) \leq \max_{\{i_1, i_2, \dots, i_k\} \in \mathbb{E}} \left\{ \prod_{j=1}^k d_{i_j}^{\frac{1}{k}}(\mathcal{A}(\mathcal{G})) \right\}. \tag{4.3}$$

Proof. Suppose that $\mathcal{A}(\mathcal{G})$ is the adjacency tensor of a k -uniform hypergraph \mathcal{G} . Since the adjacency tensor is weakly symmetric, then $\rho(\mathcal{G})$ is the largest Z -eigenvalue of $\mathcal{A}(\mathcal{G})$. By Theorem 3.2, we obtain

$$\prod_{j=1}^k (\rho(\mathcal{G}) - a_{i_j i_j \dots i_j}) \leq \max_{\substack{a_{i_1 \dots i_k} \neq 0 \\ (i_1, i_2, \dots, i_k) \neq (i_1, \dots, i_1)}} \prod_{j=1}^k r_{i_j}(\mathcal{A}(\mathcal{G})).$$

Notice that $a_{i_j i_j \dots i_j} = 0$, for all $i_j \in [n]$. Hence

$$\rho(\mathcal{G}) \leq \max_{a_{i_1 \dots i_k} \neq 0} \prod_{j=1}^k r_{i_j}^{\frac{1}{k}}(\mathcal{A}(\mathcal{G})) = \max_{\{i_1, i_2, \dots, i_k\} \in \mathbb{E}} \left\{ \prod_{j=1}^k d_{i_j}^{\frac{1}{k}}(\mathcal{A}(\mathcal{G})) \right\}. \quad \square$$

Surely, if n is considerably larger than k and $|\mathbb{E}| < \binom{n}{k}$, then Lemma 4.1 is less effective. We take an example to show the efficiency of new upper bounds.

EXAMPLE 4.1. Consider 3-uniform hypergraph \mathcal{G} with vertex set $\mathbb{V} = \{1, 2, \dots, 8\}$ and edge set $\mathbb{E}(\mathcal{G}) = \{e_1, e_2, e_3, e_4\}$, where

$$e_1 = \{1, 2, 8\}, e_2 = \{3, 4, 7\}, e_3 = \{5, 6, 8\}, e_4 = \{6, 7, 8\}.$$

From Lemma 3.1, we have $d_1 = \dots = d_5 = 1, d_6 = d_7 = 2, d_8 = 3$. By (4.2), we estimate

$$\rho(\mathcal{G}) \leq \frac{21\sqrt{2}}{4} \approx 7.4246.$$

It follows from (4.3) that

$$\rho(\mathcal{G}) \leq 3.$$

Using Theorem 4.1, we obtain

$$\rho(\mathcal{G}) \leq 2.2894.$$

It is clear that the result of Theorem 4.1 is tighter than those of (4.1) and (4.2).

5. Conclusions

In this paper, we proposed four Brauer-type upper bounds for the Z -spectral radius of weakly symmetric nonnegative tensors, which all depend only on the entries to tensors itself. By running examples, we showed that the obtained results are sharper than existing results. As an application, an upper bound on the Z -spectral radius of the adjacency tensors for uniform hypergraphs was presented and a numerical example revealed the validity of the proposed bounds.

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