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ON STATISTICALLY KÖTHE–TOEPLITZ DUALS

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Abstract. In the present paper, we introduce the concept of Δ^m-statistical boundedness of real (or complex) numbers sequences by using generalized difference operator Δ^m and examine relationships between Δ^m-statistical convergence, Δ^m-statistical Cauchiness and Δ^m-statistical boundedness. In addition to that we compute the Köthe-Toeplitz and generalized Köthe-Toeplitz duals of the set of all Δ^m-statistical bounded sequences. Moreover, we come up with the idea of statistical α and β duals of the sets of sequence which makes us capable of creating statistical equivalents of the notions of normality and perfectness of sequence spaces.

1. Introduction, Definitions and Preliminaries

Let ω be the set of all sequences of real (or complex) numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x=(x_k)$ with the usual norm $\|x\|_\infty = \sup|x_k|$, where $k\in\mathbb{N}=\{1,2,\ldots\}$, the set of positive integers. Also by bs, cs, ℓ_1, ℓ_π and ℓ_p ; we denote the spaces of *all bounded series*, *convergent series*, *absolutely convergent series*, *absolutely convergent series with respect to a permutation of* $\mathbb{N}$ and $p-$*absolutely convergent series*, respectively.

A sequence space E with a linear topology is called a $K-$space provided each of the maps $p_i : E \to \mathbb{C}$ defined by $p_i(x)=x_i$ is continuous for each $i\in\mathbb{N}$, where $\mathbb{C}$ denotes the complex field. A $K-$space E is called an $FK-$space provided E is a complete linear metric space. An $FK-$space whose topology is normable is called a $BK-$space.

The *continuous dual* of a normed space X is defined as the space all bounded linear functionals on X and denoted by X'.

By $e=(e_k)$ and $e^{(n)}=(e_k^{(n)})$ $(n=0,1,\ldots)$ we denote the sequences such that $e_k=1$ for all $k=0,1,\ldots$ and $e_k^{(n)}=1$ $(k=n)$ and $e_k^{(n)}=0$ $(k\neq n)$.

Let x and y be complex sequences and $X,Y\subset\omega$. We put $xy=(x_ky_k)_{k=0}^{\infty}$ and $x^{-1}*Y=\{y\in\omega : xy\in Y\}$, then we write

$$\begin{aligned} M(X,Y) &= \bigcap_{x\in X} x^{-1}*Y \\ &= \{a\in\omega : ax\in Y, \quad \forall x\in X\}. \end{aligned}$$

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In particular, the sets

$$X^{\delta} = M(X, \ell_{\pi})$$
$$X^{\alpha} = M(X, \ell_1);$$
$$X^{\beta} = M(X, cs);$$
$$X^{\gamma} = M(X, bs);$$
$$X^{N} = M(X, c_0)$$

are called the $\delta-, \alpha-, \beta-, \gamma-$ and $N-$ (or null) duals of X, respectively. It is well-known that $\phi \subset X^{\delta} \subset X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$ and $X^{\beta} \subset X^{N}$. If $X \subset Y$, then $Y^{\dagger} \subset X^{\dagger}$ $(\dagger \in \{\delta, \alpha, \beta, \gamma, N\})$. X^{α} and X^{β} are also called Köthe-Toeplitz and generalized Köthe-Toeplitz dual space of X respectively. It is clear that $X \subset (X^{\alpha})^{\alpha} = X^{\alpha\alpha}$. If $X = X^{\alpha\alpha}$ then X is called a perfect sequence space, one may refer to ([3], [13]).

Let X be a sequence space. Then X is called

i) *Solid* (or *normal*), if $(\alpha_k x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \leqslant 1$ for all $k \in \mathbb{N}$, whenever $(x_k) \in X$,

ii) *Symmetric*, if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$,

iii) *Sequence algebra* if $x.y \in X$, whenever $x, y \in X$, where $x.y = (x_k y_k)$ for all $k \in \mathbb{N}$.

If X is solid then $X^{\alpha} = X^{\beta} = X^{\gamma}$.

Study of difference sequence spaces is a recent development in the summability theory. Sometimes a situation may arise that we have a sequence at hand and we are interested in sequences formed by its successive differences and in the structure of these new sequences. The notion of difference sequence spaces was introduced by Kızmaz [14] and generalized by Et and Çolak [6]. Later on Et and Nuray [7] improved it in order to mainly generalize statistical convergence with respect to Δ^m difference operator as follows

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\}$$

where X is any sequence space, $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and so $\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$.

If $x \in \Delta^m(X)$ then there exists one and only one $y = (y_k) \in X$ such that $y_k = \Delta^m x_k$ and

$$x_k = \sum_{v=1}^{k-m} (-1)^m \binom{k-v-1}{m-1} y_v = \sum_{v=1}^{k} (-1)^m \binom{k+m-v-1}{m-1} y_{v-m},$$
$$y_{1-m} = y_{2-m} = \ldots = y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. Recently, a large amount of work has been carried out by many mathematicians regarding various generalizations of difference sequence spaces. For a detailed account of difference sequence spaces one may refer to ([1],[8],[9],[16]).

The idea of statistical convergence which is, in fact, a generalization of the usual notion of convergence was introduced by Fast [10] and Steinhaus [15] independently in the same year 1951 and since then several generalizations and applications of this concept have been investigated by various authors namely Connor ([4], [5]), Fridy [11], Temizsu *et al.* [16].

The concept of statistical boundedness of sequences has made its initial appearance in the frequently cited paper of Fridy and Orhan [12]. Unlike statistical convergence the notion of statistical boundedness has not found that much places in the literature since then. That being said Bhardwaj and Gupta [2] made some generalizations of statistical boundedness by introducing counterparts of the notions of statistical convergence of order α and λ-statistical convergence. Besides they showed that the Köthe-Toeplitz and generalized Köthe-Toeplitz duals of statistical bounded sequences are ϕ, the space of finitely non-zero scalar sequences.

Let K be any subset of $\mathbb{N}$ and $\kappa(n) = |K \cap \{1,2,3,...n\}|$ denote the cardinality of the enclosed set. If $\lim\limits_{n\to\infty} \frac{\kappa(n)}{n}$ exists then it is said to be the (natural) density of K and denoted by $\delta(K)$. Sets having zero density will be called *null* in this work. By the definiton it is obvious that finite sets and empty set are null. We also would like to state that sets having 1 density are called *dense*. Clearly $\mathbb{N}$ is dense.

If $x = (x_k)$ is a sequence satisfying property P for all k in a dense set then it is said that x_k holds property P for "*almost all* k" and this is abbreviated by "$a.a.$ k." [11]. A sequence $x = (x_k)$ is said to be statistically convergent to L provided that $\{k : |x_k - L| \geqslant \varepsilon\}$ is a null set for each $\varepsilon > 0$, i.e., $|x_k - L| < \varepsilon$ $a.a.$ k. It is written by $st - \lim x_k = L$. If $L = 0$ then x is a statistically null sequence. The set of all statistically null sequences and the set of all statistically convergent sequences will be denoted by Sc_0 and Sc respectively. It was shown that Sc_0 is solid in [2].

The real number sequence x is statistically bounded if there exists a number $B \geqslant 0$ such that $\delta(\{k : |x_k| > B\}) = 0$ [12]. S_b will denote the set of all statistically bounded sequences.

The main object of this work is to create another type of generalization of statistical boundedness by using Δ^m difference operator and examine some inclusion properties regarding the new concept. More importantly we focus on computing Köthe-Toeplitz and generalised Köthe-Toeplitz duals of Δ^m statistically bounded sequences and after seeing both are ϕ we generalize the notions of Köthe-Toeplitz and generalised Köthe-Toeplitz duals with *statistical sense*. This makes us capable of obtaining statistical equivalents of the notions of normality and perfectness of sequence spaces.

2. Δ^m-Statistical Boundedness of Sequencess

In the present section we shall give the definition of Δ^m-statistical boundedness and examine some inclusion properties regarding the concept.

DEFINITION 2.1. A sequence of real numbers $x = (x_k)$ is called Δ^m-statistically bounded if there exists a nonnegative real number L such that

$$\delta\{k : |\Delta^m x_k| > L\} = 0 \; i.e., \; |\Delta^m x_k| \leqslant L \; a.a.k.$$

$\Delta^m(S_b)$ will denote the set of all Δ^m-statistically bounded sequences.

The following example illustrates that there are some sequences which are not Δ^m-statistically bounded.

EXAMPLE 2.2. Let's take $m = 1$ and define x as follows

$$x_k = \begin{cases} 0 & k = 1 \\ -n(2n+1) & k = 2n+1 \\ -n(2n-1) & k = 2n \end{cases}$$

Since $\{k : |\Delta x_k| \leqslant M\}$ is a finite set, which makes it null, $\{k : |\Delta x_k| > M\}$ is dense for any $M \geqslant 0$. Hence x is not $\Delta-$statistically bounded.

PROPOSITION 2.3. *If a sequence x is Δ^m-statistically convergent to a number L then it is Δ^m-statistically bounded, but the converse is not true.*

Proof. Let x be a Δ^m-statistically convergent sequence to L. Then $|\Delta^m x_k - L| < \varepsilon$ for $a.a.$ k. for each $\varepsilon > 0$. By reverse triangle inequality we have $|\Delta^m x_k| - |L| \leqslant |\Delta^m x_k - L| < \varepsilon$ which yields $|\Delta^m x_k| \leqslant |L| + \varepsilon$ for $a.a.$ k. Since ε is arbitrary x is Δ^m-statistically bounded.

For the converse, take $m = 1$ and define a sequence $x = (x_k)$ as follows

$$x = (x_k) = (0, -1, -2, -2, -4, -4, -5, -5, -6, -9, -10, -10, \\ -11, -11, -12, -12, -16, -16, ...).$$

Then we calculate Δx as follows:

$$\Delta x_k = \begin{cases} \sqrt{k} & k \text{ is square} \\ 0 & k \text{ is odd non-square} \\ 1 & k \text{ is even non-square} \end{cases}.$$

Now let $\varepsilon > 0$. Since $\delta(\{k : \Delta x_k = 1\}) = \frac{1}{2}$,

$$\{k : \Delta x_k = 1\} \subset \{k : \Delta x_k > 1 - \varepsilon\}$$

implies that $\delta(\{k : \Delta x_k > 1 - \varepsilon\}) \neq 0$. Besides,

$$\{k : \Delta x_k > 1 + \varepsilon\} \subset \left\{k^2 : k \in \mathbb{N}\right\}$$

which is null, so $\delta\{k : \Delta x_k > 1 + \varepsilon\} = 0$. Hence st-$\limsup \Delta x = 1$ by Theorem 1 in [12]. Moreover $\delta(\{k : \Delta x_k = 0\}) = \frac{1}{2}$ and $\{k : \Delta x_k = 0\} \subset \{k : \Delta x_k < \varepsilon\}$ means that $\delta\{k : \Delta x_k < \varepsilon\} \neq 0$. Also $\{k : \Delta x_k < -\varepsilon\} = \emptyset$ is null. This yields st-$\liminf \Delta x = 0$ by Theorem $1'$ in [12]. Thus x is not $\Delta-$statistically convergent due to st-$\limsup \Delta x \neq$ st-$\liminf \Delta x$. However it is clear that $\delta\{k : |\Delta x_k| > 1\} = 0$ and so x is $\Delta-$statistically bounded.

PROPOSITION 2.4. *Every Δ^m-bounded sequence is Δ^m-statistically bounded, but the converse is not true.*

Proof. Let $x \in \Delta^m(\ell_\infty)$. Then there exists some $L \geqslant 0$ such that $|\Delta^m x_k| \leqslant L$ for every $k \in \mathbb{N}$. This yields $|\Delta^m x_k| \leqslant L$ for *a.a. k.* since $\{k : |\Delta^m x_k| > L\} = \emptyset$ and $\delta(\emptyset) = 0$. Hence x is Δ^m-statistically bounded. For the converse take $m = 2$ and

$$x = (x_k) = (0,0,0,1,2,5,8,11,14,20,26,32,38,44,50,56,62,72,82,92,102,112,...).$$

Observe that

$$\Delta^2 x_k = \begin{cases} n, \; k = 2^n \\ 0, \quad else \end{cases} \quad n = 1,2,3,...$$

and $\left\{k : \Delta^2 x_k \neq 0\right\} = \{2^n : n = 1,2,3,...\}$ which is null. Therefore x is a Δ^2-statistically null sequence which yields that $x \in \Delta^2(S_b)$. But it is obvious that $\Delta^2 x$ is not bounded and so $x \notin \Delta^2(\ell_\infty)$. Thus $x \in \Delta^2(S_b) \setminus \Delta^2(\ell_\infty)$.

PROPOSITION 2.5. *Every Δ^m-statistically Cauchy sequence is Δ^m-statistically bounded, but the converse is not true.*

Proof. Let x be a Δ^m-statistically Cauchy sequence. Then there exsists a number $N(=N(\varepsilon)) \in \mathbb{N}$ such that $\delta\{k : |\Delta^m x_k - \Delta^m x_N| \geqslant \varepsilon\} = 0$ for all $\varepsilon > 0$. It is obtained that $\delta\{k : |\Delta^m x_k| \geqslant \Delta^m x_N + \varepsilon\} = 0$. Thus x is Δ^m-statistically bounded. Considering Δ^m-statistically Cauchiness and Δ^m-statistically convergence are equivalent the sequence in Proposition 2.3 can be reused to prove that the converse is not true.

Now we wish to give a useful characterization for a sequence x to be Δ^m-statistically bounded. Before doing that it is necessary to present a Lemma.

LEMMA 2.6. *If $E \subseteq \mathbb{N}$ such that $|E| = \infty$ and $|E^c| = \infty$ then there exist increasing sequences of natural numbers (m_j) and (n_j) such that $E = \bigcup\limits_{j=1}^{\infty}([m_j,n_j) \cap \mathbb{N})$ where $m_j < n_j < m_{j+1}$ for all j.*

Proof. Define (m_j) and (n_j) as follows

$$m_j = \begin{cases} \min E, & j = 1 \\ \min\left\{k > n_{j-1} : k \in E\right\}, & j \geqslant 2 \end{cases}$$

$$n_j = \min\left\{k > m_j : k \in E^c\right\}.$$

It is clear that (m_j) and (n_j) both are strictly increasing and $m_j < n_j < m_{j+1}$ for all j. We claim $E = \bigcup\limits_{j=1}^{\infty}([m_j,n_j) \cap \mathbb{N})$. If $k \in E$ then there exists a $j_0 \in \mathbb{N}$ such that $k \in \left[m_{j_0}, n_{j_0}\right) \cap \mathbb{N}$ by construction of (m_j) and (n_j). This implies $k \in \bigcup\limits_{j=1}^{\infty}([m_j,n_j) \cap \mathbb{N})$. Hence $E \subseteq \bigcup\limits_{j=1}^{\infty}([m_j,n_j) \cap \mathbb{N})$. Now suppose the converse is not true. Then there is some $k \in \left[m_{j_0}, n_{j_0}\right) \cap \mathbb{N}$ for $j_0 \in \mathbb{N}$ whereas $k \notin E$. This contradicts with the definition of n_{j_0} since $m_{j_0} < k < n_{j_0}$ and $k \notin E$. Therefore $\bigcup\limits_{j=1}^{\infty}([m_j,n_j) \cap \mathbb{N}) \subseteq E$ and this completes the proof.

THEOREM 2.7. *x is Δ^m-statistically bounded if and only if there is a Δ^m-bounded sequence y such that $\Delta^m x_k = \Delta^m y_k$ a.a. k.* .

Proof. Let x be Δ^m-statistically bounded. If x is Δ^m-bounded proof is clear. Take $x \in \Delta^m(S_b) \setminus \Delta^m(\ell_\infty)$. We shall use induction method wrt m.

Let $m = 1$. Then there exists $L \geqslant 0$ such that $E = \{k : |\Delta x_k| > L\}$ is null, $|E| = \infty$ and $|E^c| = \infty$. We obtain sequences of natural numbers (m_j) and (n_j) such that $E = \bigcup_{j=1}^{\infty} ([m_j, n_j) \cap \mathbb{N})$ by Lemma 2.6. Now set $y = (y_k)$ as follows

$$y_k = \begin{cases} x_1 & k = 1 \\ x_1 - \sum_{j=1}^{k-1} \Delta x_j & k = 2, 3, ..., m_1 \\ y_{m_1} & k = m_1 + 1, m_1 + 2, ..., n_1 \\ y_{m_1} - \sum_{j=n_1}^{k-1} \Delta x_j & k = n_1 + 1, n_1 + 2, ..., m_2 \\ y_{m_2} & k = m_2 + 1, m_2 + 2, ..., n_2 \\ y_{m_2} - \sum_{j=n_2}^{k-1} \Delta x_j & k = n_2 + 1, n_2 + 2, ..., m_3 \\ y_{m_3} & k = m_3 + 1, m_3 + 2, ..., n_3 \\ . & . \\ . & . \end{cases}$$

Observe that $\Delta y_k = \begin{cases} 0 & k \in E \\ \Delta x_k & k \in E^c \end{cases}$ and $\Delta x_k = \Delta y_k$ *a.a.* k. due to E^c is dense. Besides $y = (y_k)$ is $\Delta-$bounded since $|\Delta y_k| = |\Delta x_k| \leqslant L$ for all $k \in E^c$. Now assume the assertion holds for $m - 1$, i.e. if x is $\Delta^{m-1}-$statistically bounded then there is a $\Delta^{m-1}-$bounded sequence y such that $\Delta^{m-1} x_k = \Delta^{m-1} y_k$ *a.a.* k.. We shall show it is true for m as well. Let x be Δ^m-statistically bounded. Then it is derived that $z = \Delta x$ is $\Delta^{m-1}-$statistically bounded as $\Delta^m x = \Delta^{m-1}(\Delta x)$. Then there exists $y \in \Delta^{m-1}(\ell_\infty)$ such that $\Delta^{m-1} z_k = \Delta^{m-1} y_k$ *a.a.* k. by assumption for $m - 1$. Also, there is a sequence s such that $\Delta s = y$ which implies $\Delta^{m-1}(\Delta s) \in \ell_\infty$ and so $s \in \Delta^m(\ell_\infty)$. We can write

$$\Delta^m s_k = \Delta^{m-1}(\Delta s_k) = \Delta^{m-1} y_k = \Delta^{m-1} z_k = \Delta^{m-1}(\Delta x_k) = \Delta^m x_k \ a.a.\ k.$$

This completes the proof of necessity part.

Conversely, assume $\exists y \in \Delta^m(\ell_\infty)$ such that $\Delta^m x_k = \Delta^m y_k$ *a.a.* k.. Then there is $L \geqslant 0$ such that $|\Delta^m y_k| \leqslant L$ for all k and $\delta \{k : \Delta^m x_k \neq \Delta^m y_k\} = 0$. Since

$$\{k : |\Delta^m x_k| > L\} \subset \{k : \Delta^m x_k \neq \Delta^m y_k\}$$

we get $|\Delta^m x_k| \leqslant L$ *a.a.* k.. Hence x is Δ^m-statistically bounded.

3. $\alpha-, \beta-, \gamma-, \delta-$ and $N-$ Duals of $\Delta^m(S_b)$

In this section we show that $\alpha-, \beta-, \gamma-, \delta-$ and $N-$ duals of $\Delta^m(S_b)$ are ϕ, the space of finitely non-zero scalar suquences. We will benefit some well-known fundamental properties regarding the concept of duals.

We shall initially give the duals of the space of all statistically null sequences.

THEOREM 3.1. *$Sc_0^{\dagger} = \phi$ for $\dagger \in \{\alpha,\beta,\gamma,\delta,N\}$.*

Proof. It is clear that $\phi \subseteq Sc_0^{\dagger}$ for $\dagger \in \{\alpha,\beta,\gamma,\delta,N\}$. Now we claim $Sc_0^{\beta} \subseteq \phi$ or equivalently $\phi^c \subseteq (Sc_0^{\beta})^c$. If $a=(a_k) \in \phi^c$, then there exists $K \subseteq \mathbb{N}$ such that $|K| = \infty$ and $k \in K$ implies $a_k \neq 0$. We may pick a subset $J \subseteq K$ with $\delta(J) = 0$ and $|J| = \infty$. Now let $x = (x_k)$ be defined as follows

$$x_k = \begin{cases} \frac{1}{a_k} & k \in J \\ 0 & else \end{cases}$$

It is easy to see that $\{k : |x_k| > \varepsilon\} \subseteq J$ for all $\varepsilon > 0$ which implies $\{k : |x_k| > \varepsilon\}$ is null and so $st-\lim x = 0$, i.e. $x \in Sc_0$. On the other hand $\sum_{k=1}^{\infty} a_k x_k = \sum_{k\in J} a_k \frac{1}{a_k} = \sum_{k\in J} 1 = \infty$ which yields $a \notin Sc_0^{\beta}$, i.e. $a \in (Sc_0^{\beta})^c$. Thus $Sc_0^{\beta} \subseteq \phi$. Since Sc_0 is a solid space $Sc_0^{\alpha} = Sc_0^{\beta} = Sc_0^{\gamma} = \phi$. Besides $Sc_0^{\delta} = \phi$ by $Sc_0^{\delta} \subseteq Sc_0^{\beta}$. Now observe $a_k x_k = 1$ when $k \in J$ and $a_k x_k = 0$ otherwise. So it is clear that $\lim_{k\to\infty} a_k x_k \neq 0$ which implies $a \notin Sc_0^N$ i.e. $a \in (Sc_0^N)^c$ as well. Therefore $Sc_0^N \subseteq \phi$ which yields $Sc_0^N = \phi$.

COROLLARY 3.2. *$Sc^{\dagger} = S_b^{\dagger} = \phi$ for $\dagger \in \{\alpha,\beta,\gamma,\delta,N\}$.*

In order to fulfill what we have promised about $\Delta^m(S_b)$ in introduction we will discuss a basic property regarding densities of subsets of natural numbers and base some handy theorems on it. We believe it has an important role in the following theorems.

PROPOSITION 3.3. *Let $K \subseteq \mathbb{N}$ and $L = \{k-1 : k \in K\}$. If K is null then so is L.*

Proof. Assume $1 \notin K$. We denote $|L \cap \{1,2,3,...,n\}|$ by $\ell(n)$ and $|K \cap \{1,2,3,...,n\}|$ by $\kappa(n)$. It can be seen that $\ell(n) \leqslant \kappa(n)+1$ for all n and so

$$\delta(L) = \lim \frac{\ell(n)}{n} \leqslant \lim \frac{\kappa(n)+1}{n} = \lim \frac{\kappa(n)}{n} + \lim \frac{1}{n} = \delta(K) + \lim \frac{1}{n} = 0$$

Hence $\delta(L) = 0$.

THEOREM 3.4. *If x is statistically bounded then it is $\Delta-$statistically bounded.*

Proof. Let $x = (x_k) \in S_b$. Then there is a $\mu \geqslant 0$ such that $K = \{k : |x_k| > \mu\}$ is null. It is clear $L = \{k : |x_{k+1}| > \mu\} = \{k-1 : k \in K\}$. Then L is null by Proposition 3.3. Thus (x_{k+1}) is statistically bounded as well. Since S_b is a subspace of ω, $\Delta x = (\Delta x_k) = (x_k - x_{k+1}) = (x_k) - (x_{k+1})$ is statistically bounded which yields that $x = (x_k) \in \Delta(S_b)$.

The following results are easily derivable from Theorem 3.4.

COROLLARY 3.5. *a) $\Delta^{m-1}(S_b) \subseteq \Delta^m(S_b)$ for $m \geqslant 1$.*
b) $S_b \subseteq \Delta^m(S_b)$ for $m \geqslant 1$.

THEOREM 3.6. *$(\Delta^m(S_b))^\dagger = \phi$ for $\dagger \in \{\alpha,\beta,\gamma,\delta,N\}$ and $m \geqslant 0$.*

Proof. The inclusion $S_b \subseteq \Delta^m(S_b)$ implies $(\Delta^m(S_b))^\dagger \subseteq S_b^\dagger$. Considering $S_b^\dagger = \phi$ by Corollary 3.2 we conclude $(\Delta^m(S_b))^\dagger = \phi$.

As is seen, the usual duals of Sc_0, Sc, S_b and $\Delta^m(S_b)$ are all ϕ which we do not find so interesting. Therefore we wish to introduce a new type of duals of sequence spaces with respect to statistical sense. We begin introducing statistical summability of series.

REMARK 3.7. We would like to note that the first definition of a statistical convergent series seems to have been given by B. C. Tripathy ([17], [18]). It is based on searching statistical limit of the sequence of partial sums of the related series. However we attempt to provide a more convenient approach in a quite different way which has a direct link to the notion of density of sets.

DEFINITION 3.8. Let $x = (x_k) \in \omega$ and $L \in \mathbb{R}$, $\sum_{k=1}^{\infty} x_k$ is said to be statistically summable to L, if there exists some $E \subseteq \mathbb{N}$ such that $\delta(E) = 1$ and $\sum_{k \in E} x_k = L$.

DEFINITION 3.9. Let X be any subspace of ω. The statistical-α dual of X and the statistical-β dual of X are defined respectively as follows:

$$X^{st-\alpha} = \left\{ x \in \omega : \sum_{k=1}^{\infty} |x_k y_k| \text{ is statistically summable } \forall y \in X \right\}$$

$$X^{st-\beta} = \left\{ x \in \omega : \sum_{k=1}^{\infty} x_k y_k \text{ is statistically summable } \forall y \in X \right\}.$$

Let X, $Y \subseteq \omega$ and $\dagger \in \{\alpha,\beta\}$. It can be shown that $X^{st-\alpha} \subseteq X^{st-\beta}$, $Y^{st-\dagger} \subset X^{st-\dagger}$ for $X \subseteq Y$ and $X \subset (X^{st-\dagger})^{st-\dagger}$.

PROPOSTION 3.10. *Let $\{X_i\}_{i\in I}$ be any collection of sequence spaces where I is an index set. Then $\left(\bigcup_{i\in I} X_i\right)^{st-\dagger} = \bigcap_{i\in I} X_i^{st-\dagger}$ for $\dagger \in \{\alpha,\beta\}$.*

Proof. Take $\dagger = \alpha$. If $a = (a_k) \in \left(\bigcup_{i\in I} X_i\right)^{st-\alpha}$ then there is an $E \subseteq \mathbb{N}$ such that $\delta(E) = 1$ and $\sum_{k\in E} |a_k x_k| < \infty$ for all $x = (x_k) \in \bigcup_{i\in I} X_i$. If $y = (y_k) \in X_i \subset \bigcup_{i\in I} X_i$ then there exists some $F \subseteq \mathbb{N}$ such that $\delta(F) = 1$ and $\sum_{k\in F} |a_k y_k| < \infty$ which implies $a \in X_i^{st-\alpha}$ for all $i \in I$. So $a \in \bigcap_{i\in I} X_i^{st-\alpha}$ and $\left(\bigcup_{i\in I} X_i\right)^{st-\alpha} \subseteq \bigcap_{i\in I} X_i^{st-\alpha}$.

Now, if $a \in \bigcap_{i\in I} X_i^{st-\alpha}$ then $a \in X_i^{st-\alpha}$ for all $i \in I$. Let $y \in \bigcup_{i\in I} X_i$. Then there is an $i \in I$ such that $y = (y_k) \in X_i$. It follows that there is a set $G \subseteq \mathbb{N}$ such that $\delta(G) = 1$ and $\sum_{k\in G} |a_k y_k| < \infty$ since $a \in X_i^{st-\alpha}$. This yields $a \in \left(\bigcup_{i\in I} X_i\right)^{st-\alpha}$ as y is arbitrary.

Hence $\bigcap_{i\in I} X_i^{st-\alpha} \subseteq \left(\bigcup_{i\in I} X_i\right)^{st-\alpha}$. This completes the proof. The case $\dagger=\beta$ can be seen analogously.

THEROEM 3.11. *If $X\in\{c_0,c,\ell_\infty,Sc_0,Sc,S_b\}$ and $\dagger\in\{\alpha,\beta\}$ then $X^{st-\dagger}=\ell_b$, where*

$$\ell_b=\left\{x\in\omega:\sum_{k\in E}|x_k|<\infty \text{ for some } E\subseteq\mathbb{N} \text{ with } \delta(E)=1\right\}.$$

Proof. Let $X=S_b$ and $\dagger=\alpha$. If $x\in\ell_b$ and $y\in S_b$ then there exist $E,F\subseteq\mathbb{N}$ such that $\delta(E)=\delta(F)=1$, $\sum_{k\in E}|x_k|<\infty$ and $y.\chi_F\in\ell_\infty$ where χ_F is the characteristic sequence of F. Note $\delta(E\cap F)=1$ and there is an $M\geqslant 0$ such that $|y_k|\leqslant M$ for all $k\in F$. It follows $\sum_{k\in E\cap F}|x_ky_k|\leqslant M\sum_{k\in E\cap F}|x_k|<\infty$ which yields $\sum_{k=1}^{\infty}|x_ky_k|$ is statistically summable and so $x\in S_b^{st-\alpha}$. Hence

$$\ell_b\subseteq S_b^{st-\alpha} \tag{1}$$

Now we wish to take $X=c_0$ and $\dagger=\beta$. We claim $c_0^{st-\beta}\subseteq\ell_b$. Let $x\notin\ell_b$. Then we write $\sum_{k\in E}|x_k|=\infty$ for all $E\subseteq\mathbb{N}$ with $\delta(E)=1$. So there exists a strictly increasing sequence of positive integers $(k(j))_{j=0}^{\infty}$ such that $k(0)=1$ and $\sum_{k\in N_j}|x_k|\geqslant j+1$ for $j=0,1,2,...$ where $N_j=E\cap[k(j),k(j+1)-1]$. Now define the sequence $z=(z_k)$ as follows

$$z_k=\begin{cases} 0 & k\notin N_j\vee x_k=0\\ \frac{|x_k|}{(j+1)x_k} & k\in N_j\wedge x_k\neq 0\end{cases}.$$

Then z is clearly a null sequence. However,

$$\sum_{k\in E}x_kz_k=\sum_{j=0}^{\infty}\sum_{k\in N_j}\frac{1}{j+1}|x_k|=\sum_{j=0}^{\infty}\frac{1}{j+1}\sum_{k\in N_j}|x_k|\geqslant\sum_{j=0}^{\infty}\frac{1}{j+1}(j+1)=\sum_{j=0}^{\infty}1=\infty$$

for each $E\subseteq\mathbb{N}$ with $\delta(E)=1$ which yields $\sum_{k=1}^{\infty}x_kz_k$ is not statistically summable and so $x\notin c_0^{st-\beta}$. Hence

$$c_0^{st-\beta}\subseteq\ell_b \tag{2}$$

Considering $(1),(2)$ and well-known inclusions between the members of $\{c_0,c,\ell_\infty,Sc_0,Sc,S_b\}$ completes the proof.

Now we would like to introduce some new concepts regarding sequence spaces with respect to st-α and st-β duality.

DEFINITION 3.12. Let $u=(u_k)\in\omega$ and X be a sequence space. If $|u_k|\leqslant|x_k|$ for all $k\in E$ such that $\delta(E)=1$ for some $x=(x_k)\in X$ implies $u\in X$, then X is said to be statistically normal (or statistically solid) space. *i.e.* X is statistically normal if

$$\{u=(u_k)\in\omega\mid\exists(x_k)\in X,\ \exists E\subseteq\mathbb{N}\ \delta(E)=1\ \forall k\in E:|u_k|\leqslant|x_k|\}\subset X.$$

DEFINITION 3.13. Let X be a sequence space. X is called a *statistically-$\dagger$ space*, for short *st-$\dagger$ space*, if $X = (X^{st-\dagger})^{st-\dagger}$ where $\dagger \in \{\alpha, \beta\}$. In particular a st-$\alpha$ space is called *statistically perfect space* or *statistically Köthe space*.

THEROEM 3.14. *ℓ_b is a statistically perfect space.*

Proof. $\ell_b \subseteq (\ell_b^{st-\alpha})^{st-\alpha}$ holds by Definition 3.8**.** Therefore we only need to show $(\ell_b^{st-\alpha})^{st-\alpha} \subseteq \ell_b$. First we shall prove $S_b \subset \ell_b^{st-\alpha}$. If $a = (a_k) \in S_b$, then there exist a set $F \subseteq \mathbb{N}$ and a real number $L \geqslant 0$ such that $\delta(F) = 1$ and $|a_k| \leqslant L$ for all $k \in F$. Let any $x = (x_k) \in \ell_b$ be given. Then we get a set $E \subseteq \mathbb{N}$ such that $\delta(E) = 1$ and $\sum_{k\in E} |x_k| < \infty$. It follows that $\sum_{k\in E\cap F} |a_k x_k| \leqslant L \sum_{k\in E\cap F} |x_k| < \infty$ where $\delta(E\cap F) = 1$ which yields $a \in \ell_b^{st-\alpha}$ and so $S_b \subset \ell_b^{st-\alpha}$. This follows that $(\ell_b^{st-\alpha})^{st-\alpha} \subseteq S_b^{st-\alpha} = \ell_b$ and hence $\ell_b = (\ell_b^{st-\alpha})^{st-\alpha}$.

THEOREM 3.15. *Every statistically perfect space is statistically normal.*

Proof. Let X be a statistically perfect space, i.e. $X = (X^{st-\alpha})^{st-\alpha}$ and $u \in \omega$ such that $|u_k| \leqslant |x_k|$ for all $k \in E \subseteq \mathbb{N}$ with $\delta(E) = 1$ for some $x \in X$. If $y \in X^{st-\alpha}$ then there exists some $F \subseteq \mathbb{N}$ such that $\delta(F) = 1$ and $\sum_{k\in F} |y_k x_k| < \infty$. It follows that $|u_k| \leqslant |x_k|$ for all $k \in E\cap F$ implies $\sum_{k\in E\cap F} |u_k y_k| < \infty$ where $\delta(E\cap F) = 1$. This yields $u \in (X^{st-\alpha})^{st-\alpha} = X$ since y is arbitrary. Hence X is statistically normal.

Theorem 3.15 yields the following corollary.

COROLLARY 3.16. *ℓ_b is statistically normal.*

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