SHARP BOUNDS FOR SÁNDOR–YANG MEANS IN TERMS
OF ONE–PARAMETER FAMILY OF BIVARIATE MEANS

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Abstract. In the article, we present the best possible parameters α_1, α_2, α_3, α_4, β_1, β_2, β_3 and β_4 on the interval $(0,1)$ such that the double inequalities

$$G_{\alpha_1}(x,y) < R_{GQ}(x,y) < G_{\beta_1}(x,y), \quad Q_{\alpha_2}(x,y) < R_{QG}(x,y) < Q_{\beta_2}(x,y),$$

$$H_{\alpha_3}(x,y) < R_{GQ}(x,y) < H_{\beta_3}(x,y), \quad C_{\alpha_4}(x,y) < R_{QG}(x,y) < C_{\beta_4}(x,y)$$

hold for all $x,y > 0$ with $x \neq y$, where $R_{GQ}(x,y)$ and $R_{QG}(x,y)$ are the Sándor-Yang means, $H_p(x,y)$, $G_p(x,y)$, $Q_p(x,y)$ and $C_p(x,y)$ are the one-parameter means.

1. Introduction

Let $r \in \mathbb{R}$ and $x,y > 0$ with $x \neq y$. Then the rth power mean $A_r(x,y)$ [1, 2, 3, 4] and Schwab-Borchardt mean $SB(x,y)$ [5, 6] of x and y are given by

$$A_r(x,y) = \left(\frac{x^r + y^r}{2} \right)^{1/r} \quad (r \neq 0), \quad A_0(x,y) = \sqrt{xy}$$

and

$$SB(x,y) = \begin{cases} \frac{\sqrt{y^2 - x^2}}{\arccos(x/y)}, & x < y, \\ \frac{\sqrt{x^2 - y^2}}{\cosh^{-1}(x/y)}, & x > y, \end{cases}$$

respectively, where $\cosh^{-1}(t) = \log(t + \sqrt{t^2 - 1})$ is the inverse hyperbolic cosine function.

It is well known that the power mean $A_r(x,y)$ is continuous and strictly increasing with respect to $r \in \mathbb{R}$ for fixed $x,y > 0$ with $x \neq y$, and the Schwab-Borchardt mean $SB(x,y)$ is non-symmetric and homogeneous of degree one with respect to its variables

x and y. Many bivariate means are the special cases of the power mean and Schwab-Borchardt mean, for example,

$$H(x,y) = \frac{2xy}{x+y} = A_{-1}(x,y), \quad G(x,y) = \sqrt{xy} = A_0(x,y), \tag{1}$$

$$A(x,y) = \frac{x+y}{2} = A_1(x,y), \quad Q(x,y) = \sqrt{\frac{x^2+y^2}{2}} = A_2(x,y), \tag{2}$$

$$T(x,y) = \frac{x-y}{2\arctan\left(\frac{x-y}{x+y}\right)} = SB[A(x,y),Q(x,y)],$$

$$NS(x,y) = \frac{x-y}{2\sinh^{-1}\left(\frac{x-y}{x+y}\right)} = SB[Q(x,y),A(x,y)],$$

$$U(x,y) = \frac{x-y}{\sqrt{2}\arctan\left(\frac{x-y}{\sqrt{2xy}}\right)} = SB[G(x,y),Q(x,y)], \tag{3}$$

and

$$V(x,y) = \frac{x-y}{\sqrt{2}\sinh^{-1}\left(\frac{x-y}{\sqrt{2xy}}\right)} = SB[Q(x,y),G(x,y)] \tag{4}$$

are respectively the harmonic [7], geometric [8], arithmetic [9], quadratic [10], second Seiffert [11], Neuman-Sánodor [12], first and second Yang means [13], where $\sinh^{-1}(t) = \log(t + \sqrt{t^2+1})$ is the inverse hyperbolic sine function.

Recently, the bivariate means have attracted the attention of many researchers due to they are closely related to the special functions [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30], they have wide applications in pure and applied mathematics, physics, mechanics, statistics, economics and other natural sciences [31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64]. In particular, many remarkable properties and inequalities involving the bivariate mean can be found in the literature [65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80].

Let $X(x,y)$ and $Y(x,y)$ be the symmetric bivariate means of x and y. Then the Sándor-Yang mean $R_{XY}(x,y)$ [81] is defined by

$$R_{XY}(x,y) = Y(x,y)e^{\frac{X(x,y)}{SB[X(x,y),Y(x,y)]}-1}$$

and Yang [81] provided the explicit formulas for $R_{AQ}(x,y)$, $R_{QA}(x,y)$, $R_{GQ}(x,y)$ and $R_{QG}(x,y)$ as follows

$$R_{AQ}(x,y) = Q(x,y)e^{\frac{A(x,y)}{T(x,y)}-1}, \quad R_{QA}(x,y) = A(x,y)^{\frac{Q(x,y)}{NS(x,y)}-1},$$

$$R_{GQ}(x,y) = Q(x,y)e^{\frac{G(x,y)}{U(x,y)}-1}, \quad R_{QG}(x,y) = G(x,y)^{\frac{Q(x,y)}{V(x,y)}-1}. \tag{5}$$

Let $x > y > 0$, $v = (x-y)/(x+y) \in (0,1)$, $p \in [0,1]$ and $N(x,y)$ be a symmetric bivariate mean. Then the one-parameter bivariate mean $N_p(x,y)$ [82] is given by

$$N_p(x,y) = N\left[\frac{(1+p)x+(1-p)y}{2}, \frac{(1+p)y+(1-p)x}{2}\right]. \qquad (6)$$

From (1), (2) and (6) we clearly see that

$$H_p(x,y) = A(x,y)\left[1-(pv)^2\right], \quad G_p(x,y) = A(x,y)\sqrt{1-(pv)^2}, \qquad (7)$$

$$Q_p(x,y) = A(x,y)\sqrt{1+(pv)^2}, \quad C_p(x,y) = A(x,y)\left[1+(pv)^2\right], \qquad (8)$$

where $C(x,y) = (x^2+y^2)/(x+y)$ is the contra-harmonic mean.

In [83, 84], the authors proved that the double inequalities

$$A_{\lambda_1} < R_{QA}(x,y) < A_{\mu_1}(x,y), \quad A_{\lambda_2} < R_{AQ}(x,y) < A_{\mu_2}(x,y),$$

$$A_{\lambda_3} < R_{QG}(x,y) < A_{\mu_3}(x,y), \quad A_{\lambda_4} < R_{GQ}(x,y) < A_{\mu_4}(x,y),$$

hold for all $x,y > 0$ with $x \neq y$ if and only if $\lambda_1 \leqslant \log 2/[1+\log 2 - \sqrt{2}\log(1+\sqrt{2})] = 1.5517\cdots$, $\mu_1 \geqslant 5/3$, $\lambda_2 \leqslant 4\log 2/(4+2\log 2 - \pi) = 1.2351\cdots$, $\mu_2 \geqslant 4/3$, $\lambda_3 \leqslant 2\log 2(2-\log 2) = 1.0607\cdots$, $\mu_3 \geqslant 4/3$, $\lambda_4 \leqslant 2\log 2/(2+\log 2) = 0.5147\cdots$ and $\mu_4 \geqslant 2/3$.

Xu and Qian [85] found the best possible parameters α_1, α_2, β_1 and β_2 on the interval $[0,1]$, and α_3, α_4, β_3 and β_4 on the interval $[1/2,1]$ such that the double inequalities

$$Q^{\alpha_1}(x,y)A^{1-\alpha_1}(x,y) < R_{QA}(x,y) < Q^{\beta_1}(x,y)A^{1-\beta_1}(x,y),$$

$$Q^{\alpha_2}(x,y)A^{1-\alpha_2}(x,y) < R_{AQ}(x,y) < Q^{\beta_2}(x,y)A^{1-\beta_2}(x,y),$$

$$Q[\alpha_3 x+(1-\alpha_3)y, \alpha_3 y+(1-\alpha_3)x] < R_{QA}(x,y)$$
$$< Q[\beta_3 x+(1-\beta_3)y, \beta_3 y+(1-\beta_3)x],$$

$$Q[\alpha_4 x+(1-\alpha_4)y, \alpha_4 y+(1-\alpha_4)x] < R_{AQ}(x,y)$$
$$< Q[\beta_4 x+(1-\beta_4)y, \beta_4 y+(1-\beta_4)x]$$

hold for all $x,y > 0$ with $x \neq y$.

The main purpose of the article is to present the best possible parameters α_1, α_2, α_3, α_4, β_1, β_2, β_3 and β_4 on the interval $(0,1)$ such that the double inequalities

$$G_{\alpha_1}(x,y) < R_{GQ}(x,y) < G_{\beta_1}(x,y), \quad Q_{\alpha_2}(x,y) < R_{QG}(x,y) < Q_{\beta_2}(x,y),$$

$$H_{\alpha_3}(x,y) < R_{GQ}(x,y) < H_{\beta_3}(x,y), \quad C_{\alpha_4}(x,y) < R_{QG}(x,y) < C_{\beta_4}(x,y)$$

hold for all $x,y > 0$ with $x \neq y$.

2. Lemmas

In order to prove our main results, we need four lemmas which we present in this section.

LEMMA 1. *Let* $p \in (0,1)$ *and*

$$f(t) = \frac{2t}{(1-p^2)t^2+2} - \arctan(t). \tag{9}$$

Then the following statements are true:
    (1) *If* $p = \sqrt{3}/3$, *then* $f(t) < 0$ *for all* $t > 0$.
    (2) *If* $p = \sqrt{1-2/e^2} = 0.8540\cdots$, *then there exists* $\lambda > 0$ *such that* $f(t) > 0$ *for all* $t \in (0,\lambda)$ *and* $f(t) < 0$ *for* $t \in (\lambda,\infty)$.

Proof. It follows from (9) that

$$f(0) = 0, \quad f(\infty) = -\frac{\pi}{2}, \tag{10}$$

$$f'(t) = -\frac{t^2}{(t^2+1)[(1-p^2)t^2+2]^2} f_1(t), \tag{11}$$

where

$$f_1(t) = (p^4 - 4p^2 + 3)t^2 + 2(1-3p^2). \tag{12}$$

(1) If $p = \sqrt{3}/3$, then (12) leads to

$$f_1(t) = \frac{16}{9}t^2 > 0 \tag{13}$$

for $t \in (0,\infty)$.

Therefore, $f(t) < 0$ for all $t \in (0,\infty)$ follows easily from (10), (11) and (13).
    (2) If $p = \sqrt{1-2/e^2}$, then (12) and the numerical computations show that

$$p^4 - 4p^2 + 3 = 0.6146\cdots > 0, \tag{14}$$

$$f_1(0) = 2(1-3p^2) = -2.3759\cdots < 0, \quad f_1(\infty) = \infty. \tag{15}$$

From (11), (12), (14) and (15) we clearly see that there exists $\lambda_0 > 0$ such that $f(t)$ is strictly increasing on $(0,\lambda_0)$ and strictly decreasing on $(\lambda_0,1)$. Therefore, Lemma 1(2) follows from (10) and the piecewise monotonicity of the function $f(t)$ on the interval $(0,1)$. $\square$

LEMMA 2. *Let* $p \in (0,1)$ *and*

$$g(t) = \frac{2t\sqrt{t^2+1}}{(1+p^2)t^2+2} - \sinh^{-1}(t). \tag{16}$$

Then the following statements are true:
    (1) *If* $p = \sqrt{3}/3$, *then* $g(t) < 0$ *for all* $t > 0$.
    (2) *If* $p = \sqrt{8/e^2 - 1} = 0.2875$, *then exists* $\mu > 0$ *such that* $g(t) > 0$ *for* $t \in (0,\mu)$ *and* $g(t) < 0$ *for* $t \in (\mu,\infty)$.

Proof. It follows from (16) that

$$g(0) = 0, \quad g(\infty) = -\infty, \tag{17}$$

$$g'(t) = -\frac{t^2}{\sqrt{t^2+1}[(1+p^2)t^2+2]^2}g_1(t), \tag{18}$$

where

$$g_1(t) = (1+p^2)^2 t^2 + 2(3p^2 - 1). \tag{19}$$

(1) If $p = \sqrt{3}/3$, then equation (19) leads to

$$g_1(t) = \frac{16}{9}t^2 > 0 \tag{20}$$

for all $t > 0$.

Therefore, $g(t) < 0$ for all $t > 0$ follows from (17), (18) and (20).

(2) If $p = \sqrt{8/e^2 - 1}$, then then (19) and the numerical computations give

$$g_1(0) = 2(3p^2 - 1) = -1.5039\cdots < 0, \quad g_1(\infty) = \infty. \tag{21}$$

From (18), (19) and (21) we clearly see that there exists $\mu_0 > 0$ such that $g(t)$ is strictly increasing on $(0, \mu_0)$ and strictly decreasing on (μ_0, ∞). Therefore, Lemma 2(2) follows easily from (17) and the piecewise monotonicity of the function $g(t)$ on the interval $(0, 1)$. $\square$

LEMMA 3. *Let $p \in (0,1)$ and*

$$h(t) = \frac{2t[(1+p^2)t^2+2]}{(t^2+2)[(1-p^2)t^2+2]} - \arctan(t). \tag{22}$$

Then the following statements are true:

(1) *If $p = \sqrt{6}/6$, then $h(t) < 0$ for all $t > 0$.*

(2) *If $p = \sqrt{1 - \sqrt{2}/e} = 0.6926\cdots$, then there exists $\sigma > 0$ such that $h(t) > 0$ for $t \in (0, \sigma)$ and $h(t) < 0$ for $t \in (\sigma, \infty)$.*

Proof. It follows from (22) that

$$h(0) = 0, \quad h(\infty) = -\frac{\pi}{2}, \tag{23}$$

$$h'(t) = \frac{t^2}{(t^2+1)(t^2+2)^2[(1-p^2)t^2+2]^2}h_1(t), \tag{24}$$

where

$$h_1(t) = (p^4 + 2p^2 - 3)t^6 - 2(3p^4 - 14p^2 + 7)t^4 \tag{25}$$
$$-4(2p^4 - 18p^2 + 5)t^2 + 8(6p^2 - 1).$$

(1) If $p = \sqrt{6}/6$, then equation (25) becomes

$$h_1(t) = -\frac{1}{36}\left(95t^4 + 342t^2 + 296\right)t^2. \tag{26}$$

Therefore, $h(t) < 0$ for all $t > 0$ follows easily from (23), (24) and (26).

(2) If $p = \sqrt{1 - \sqrt{2}/e}$, then we denote $h_2(t) = h_1'(t)/(2t)$ and $h_3(t) = h_2'(t)/(4t)$. Equation (26) and numerical computations show that

$$p^4 + 2p^2 - 3 = -1.8103\cdots < 0, \tag{27}$$

$$3p^4 - 14p^2 + 7 = 0.9740\cdots > 0, \tag{28}$$

$$2p^4 - 18p^2 + 5 = -3.1750\cdots < 0, \tag{29}$$

$$6p^2 - 1 = 1.8784\cdots > 0, \tag{30}$$

$$h_2(t) = 2(p^4 + 2p^2 - 3)t^4 - 4(3p^4 - 14p^2 + 7)t^2 - 4(2p^4 - 18p^2 + 5), \tag{31}$$

$$h_3(t) = 3(p^4 + 2p^2 - 3)t^2 - 2(3p^4 - 14p^2 + 7). \tag{32}$$

It follows from (25) and (27)-(32) that

$$h_1(0) = 8(6p^2 - 1) > 0, \quad h_1(\infty) = -\infty, \tag{33}$$

$$h_2(0) = -4(2p^4 - 18p^2 + 5) > 0, \quad h_2(\infty) = -\infty, \tag{34}$$

$$h_3(t) < 0 \tag{35}$$

for all $t > 0$.

From (34) and (35) we know that there exists $\sigma_0 > 0$ such that $h_1(t)$ is strictly increasing on $(0, \sigma_0)$ and strictly decreasing on (σ_0, ∞). Then (24) and (33) lead to the conclusion that there exists $\sigma_1 > 0$ such that $h(t)$ is strictly increasing on $(0, \sigma_1)$ and strictly decreasing on (σ_1, ∞). Therefore, Lemma 3(2) follows easily from (23) and the piecewise monotonicity of the function $h(t)$ on the interval $(0, 1)$. $\quad\square$

LEMMA 4. *Let $p \in (0, 1)$ and*

$$k(t) = \frac{2t\sqrt{t^2 + 1}[(1 - p^2)t^2 + 2]}{(t^2 + 2)[(1 + p^2)t^2 + 2]} - \sinh^{-1}(t). \tag{36}$$

Then the following statements are true:

(1) *If $p = \sqrt{6}/6$, then $k(t) < 0$ for all $t > 0$.*

(2) *If $p = \sqrt{2\sqrt{2}/e - 1} = 0.2012\cdots$, then there exists $\tau > 0$ such that $k(t) > 0$ for $t \in (0, \tau)$ and $k(t) < 0$ for $t \in (\tau, \infty)$.*

Proof. It follows from (36) that

$$k(0) = 0, \quad k(\infty) = -\infty, \tag{37}$$

$$k'(t) = -\frac{t^2}{\sqrt{1+t^2}(t^2+2)^2[(1+p^2)t^2+2]^2}k_1(t), \tag{38}$$

where

$$k_1(t) = (1+p^2)^2 t^6 + 2(5p^4 + 14p^2 + 1)t^4 + 4(2p^4 + 18p^2 - 1)t^2 + 8(6p^2 - 1). \tag{39}$$

(1) If $p = \sqrt{6}/6$, then equation (39) becomes

$$k_1(t) = \frac{1}{36}(49t^4 + 250t^2 + 296)t^2. \tag{40}$$

Therefore, $k(t) < 0$ for all $t > 0$ follows from (37), (38) and (40).

(2) If $p = \sqrt{2\sqrt{2}/e - 1}$, then we denote $k_2(t) = k_1'(t)/(2t)$ and $k_3(t) = k_2'(t)/(4t)$. Equation (39) and numerical computations lead to

$$2p^4 + 18p^2 - 1 = -0.2673\cdots < 0, \tag{41}$$

$$6p^2 - 1 = -0.7568\cdots < 0, \tag{42}$$

$$k_2(t) = 3(1+p^2)^2 t^4 + 4(5p^4 + 14p^2 + 1)t^2 + 4(2p^4 + 18p^2 - 1), \tag{43}$$

$$k_3(t) = 3(1+p^2)^2 t^2 + 2(5p^4 + 14p^2 + 1). \tag{44}$$

It follows from (39) and (41)-(44) that

$$k_1(0) = 8(6p^2 - 1) < 0, \quad k_1(\infty) = \infty, \tag{45}$$

$$k_2(0) = 4(2p^4 + 18p^2 - 1) < 0, \quad k_2(\infty) = \infty, \tag{46}$$

$$k_3(t) > 0 \tag{47}$$

for all $t > 0$.

From (46) and (47) we know that there exists $\tau_0 > 0$ such that $k_1(t)$ is strictly decreasing on $(0, \tau_0)$ and strictly increasing on (τ_0, ∞). Then (38) and (45) lead to the conclusion that exists $\tau_1 > 0$ such that $k(t)$ is strictly increasing on $(0, \tau_1)$ and strictly decreasing on (τ_1, ∞). Therefore, Lemma 4(2) follows from (37) and the piecewise monotonicity of the function $k(t)$ on the interval $(0, 1)$. $\quad\square$

3. Main Results

THEOREM 1. *Let $\alpha_1, \beta_1 \in (0, 1)$. Then the double inequality*

$$G_{\alpha_1}(x,y) < R_{GQ}(x,y) < G_{\beta_1}(x,y)$$

holds for all $x, y > 0$ with $x \neq y$ if and only if $\alpha_1 \geqslant \sqrt{1 - 2/e^2} = 0.8540\cdots$ and $\beta_1 \leqslant \sqrt{3}/3 = 0.5773\cdots$.

Proof. Since $R_{GQ}(x,y)$ and $G_p(x,y)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $x > y > 0$. Let $p \in (0,1)$ and $t = (x-y)/\sqrt{2xy} \in (0,\infty)$. Then from (1)-(3), (5) and (7) we get

$$\log[R_{GQ}(x,y)] - \log[G_p(x,y)] \tag{48}$$

$$= \frac{1}{2}\log(1+t^2) + \frac{\arctan(t)}{t} - \frac{1}{2}\log\left[\frac{1}{2}(1-p^2)t^2 + 1\right] - 1.$$

Let

$$F(t) = \frac{1}{2}\log(1+t^2) + \frac{\arctan(t)}{t} - \frac{1}{2}\log\left[\frac{1}{2}(1-p^2)t^2 + 1\right] - 1. \tag{49}$$

Then simple computations lead to

$$F(0) = 0, \tag{50}$$

$$F(\infty) = \frac{1}{2}\log\left(\frac{2}{1-p^2}\right) - 1, \tag{51}$$

$$F'(t) = \frac{1}{t^2}f(t), \tag{52}$$

where $f(t)$ is defined by (9).

We divide the proof into four cases.

Case 1 $p = \sqrt{3}/3$. Then from Lemma 1(1), (48)-(50) and (52) we get

$$R_{GQ}(x,y) < G_{\sqrt{3}/3}(x,y).$$

Case 2 $p = \sqrt{1-2/e^2}$. Then from Lemma 1(2) and (52) we know that there exists $\lambda > 0$ such that $F(t)$ is strictly increasing on $(0,\lambda)$ and strictly decreasing on (λ,∞). Note that (51) becomes

$$F(\infty) = 0. \tag{53}$$

Therefore,

$$G_{\sqrt{1-2/e^2}}(x,y) < R_{GQ}(x,y)$$

follows easily from (48)-(50) and (53) together with the piecewise monotonicity of the function $F(t)$ on the interval $(0,\infty)$.

Case 3 $\sqrt{3}/3 < p < 1$. Let $t \to 0^+$. Then making use of (49) and power series expansion we get

$$F(t) = \frac{1}{4}\left(p^2 - \frac{1}{3}\right)t^2 + o\left(t^2\right). \tag{54}$$

Equations (48), (49) and (54) imply that there exists small enough $0 < \delta_1 < 1$ such that

$$R_{GQ}(x,y) > G_p(x,y)$$

for all $x > y > 0$ with $(x-y)/\sqrt{2xy} \in (0,\delta_1)$.

Case 4 $0 < p < \sqrt{1 - 2/e^2}$. Then equation (51) leads to

$$F(\infty) < 0. \tag{55}$$

From (48), (49) and (55) we clearly see that there exists large enough $M_1 > 0$

$$G_p(x, y) > R_{GQ}(x, y)$$

for all $x > y > 0$ with $(x - y)/\sqrt{2xy} \in (M_1, \infty)$. $\quad\square$

THEOREM 2. *Let* $\alpha_2, \beta_2 \in (0, 1)$. *Then the double inequality*

$$Q_{\alpha_2}(x, y) < R_{QG}(x, y) < Q_{\beta_2}(x, y)$$

holds for all $x, y > 0$ *with* $x \neq y$ *if and only* $\alpha_2 \leqslant \sqrt{8/e^2 - 1} = 0.2875\cdots$ *and* $\beta_2 \geqslant \sqrt{3}/3 = 0.5773\cdots$.

Proof. Since $R_{QG}(x, y)$ and $Q_p(x, y)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $x > y > 0$. Let $p \in (0, 1)$ and $t = (x - y)/\sqrt{2xy} \in (0, \infty)$. Then it follows from (1), (2), (4), (5) and (8) that

$$\log[R_{QG}(x, y)] - \log[Q_p(x, y)] \tag{56}$$

$$= \frac{\sqrt{1 + t^2}\sinh^{-1}(t)}{t} - \frac{1}{2}\log\left[\frac{1}{2}(1 + p^2)t^2 + 1\right] - 1.$$

Let

$$G(t) = \frac{\sqrt{1 + t^2}\sinh^{-1}(t)}{t} - \frac{1}{2}\log\left[\frac{1}{2}(1 + p^2)t^2 + 1\right] - 1. \tag{57}$$

Then

$$G(0) = 0, \tag{58}$$

$$G(\infty) = \frac{3}{2}\log 2 - \frac{1}{2}\log(1 + p^2) - 1, \tag{59}$$

$$G'(t) = \frac{1}{t^2\sqrt{1 + t^2}}g(t), \tag{60}$$

where $g(t)$ is defined by (16).

We divide the proof into four cases.

Case 1 $p = \sqrt{3}/3$. Then from Lemma 2(1), (56)-(58) and (60) we know that

$$R_{QG}(x, y) < Q_{\sqrt{3}/3}(x, y).$$

Case 2 $p = \sqrt{8/e^2 - 1}$. Then it follows from Lemma 2(2) and (60) that there exists $\mu > 0$ such that $G(t)$ is strictly increasing on $(0, \mu)$ and strictly decreasing on (μ, ∞). Note that (59) becomes

$$G(\infty) = 0. \tag{61}$$

Therefore,

$$R_{QG}(x,y) > Q_{\sqrt{8/e^2-1}}(x,y)$$

follows from (56)-(58) and (61) together with the piecewise monotonicity of the function $G(t)$ on the interval $(0,1)$.

Case 3 $0 < p < \sqrt{3}/3$. Let $t \to 0^+$. Then making use of (57) and the power series expansion we have

$$G(t) = \frac{1}{4}\left(\frac{1}{3} - p^2\right)t^2 + o\left(t^2\right). \tag{62}$$

Equations (56), (57) and (62) lead to the conclusion that there exists small enough $0 < \delta_2 < 1$ such that

$$R_{QG}(x,y) > Q_p(x,y)$$

for all $x > y > 0$ with $(x-y)/\sqrt{2xy} \in (0, \delta_2)$.

Case 4 $\sqrt{8/e^2 - 1} < p < 1$. Then equation (59) leads to

$$G(\infty) < 0. \tag{63}$$

Equations (56), (57) and (63) imply that there exists large enough $M_2 > 0$

$$Q_p(x,y) > R_{GQ}(x,y)$$

for all $x > y > 0$ with $(x-y)/\sqrt{2xy} \in (M_2, \infty)$. $\quad\square$

THEOREM 3. *Let $\alpha_3, \beta_3 \in (0,1)$. Then the double inequality*

$$H_{\alpha_2}(x,y) < R_{GQ}(x,y) < H_{\beta_3}(x,y)$$

holds for all $x,y > 0$ with $x \neq y$ if and only if $\alpha_3 \geqslant \sqrt{1 - \sqrt{2}/e} = 0.6926\cdots$ and $\beta_3 \leqslant \sqrt{6}/6 = 0.4082\cdots$.

Proof. Since $R_{GQ}(x,y)$ and $H_p(x,y)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $x > y > 0$. Let $p \in (0,1)$ and $t = (x-y)/\sqrt{2xy} \in (0,\infty)$. Then it follows from (1)-(3), (5) and (7) that

$$\log[R_{GQ}(x,y)] - \log[H_p(x,y)] \tag{64}$$

$$= \frac{1}{2}\log\left(\frac{t^4 + 3t^2 + 2}{2}\right) + \frac{\arctan(t)}{t} - \log\left[\frac{1}{2}(1 - p^2)t^2 + 1\right] - 1.$$

Let

$$H(t) = \frac{1}{2}\log\left(\frac{t^4 + 3t^2 + 2}{2}\right) + \frac{\arctan(t)}{t} - \log\left[\frac{1}{2}(1 - p^2)t^2 + 1\right] - 1. \tag{65}$$

Then

$$H(0) = 0, \tag{66}$$

$$H(\infty) = \frac{1}{2}\log 2 - \log(1-p^2) - 1, \tag{67}$$

$$H'(t) = \frac{h(t)}{t^2}, \tag{68}$$

where $h(t)$ is defined by (22).

We divide the proof into four cases.

Case 1 $p = \sqrt{6}/6$. Then from Lemma 3(1), (64)-(66) and (68) we know that

$$R_{GQ}(x,y) < H_{\sqrt{6}/6}(x,y).$$

Case 2 $p = \sqrt{1 - \sqrt{2}/e}$. Then it follows from Lemma 3(2) and (68) that there exists $\sigma > 0$ such that $H(t)$ is strictly increasing on $(0, \sigma)$ and strictly decreasing on (σ, ∞). Note that (67) becomes

$$H(\infty) = 0. \tag{69}$$

Therefore,

$$R_{GQ}(x,y) > Q_{\sqrt{1-\sqrt{2}/e}}(x,y)$$

follows from (64)-(66) and (69) together with the piecewise monotonicity of the function $H(t)$ on the interval $(0,1)$.

Case 3 $\sqrt{6}/6 < p < 1$. Let $t \to 0^+$. Then making use of (65) and the power series expansion we get

$$H(t) = \frac{1}{2}\left(p^2 - \frac{1}{6}\right)t^2 + o\left(t^2\right). \tag{70}$$

Equations (64), (65) and (70) lead to the conclusion that there exists small enough $0 < \delta_3 < 1$ such that

$$R_{GQ}(x,y) > H_p(x,y)$$

for all $x > y > 0$ with $(x-y)/\sqrt{2xy} \in (0, \delta_3)$.

Case 4 $0 < p < \sqrt{1 - \sqrt{2}/e}$. Then equation (67) leads to

$$H(\infty) < 0. \tag{71}$$

Equations (64), (65) and (71) imply that there exists large enough $M_3 > 0$

$$R_{GQ}(x,y) < H_p(x,y)$$

for all $x > y > 0$ with $(x-y)/\sqrt{2xy} \in (M_3, \infty)$. $\square$

THEOREM 4. *Let $\alpha_4, \beta_4 \in (0,1)$. Then the double inequality*

$$C_{\alpha_2}(x,y) < R_{QG}(x,y) < C_{\beta_4}(x,y)$$

holds for all $x,y > 0$ with $x \neq y$ if and only $\alpha_4 \leqslant \sqrt{2\sqrt{2}/e - 1} = 0.2012\cdots$ and $\beta_4 \geqslant \sqrt{6}/6 = 0.4082\cdots$.

Proof. Since $R_{QG}(x,y)$ and $C_p(x,y)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $x > y > 0$. Let $p \in (0,1)$ and $t = (x-y)/\sqrt{2xy} \in (0,\infty)$. Then it follows from (1), (2), (4), (5) and (8) that

$$\log[R_{QG}(x,y)] - \log[C_p(x,y)] \tag{72}$$

$$= \frac{1}{2}\log\left(\frac{1}{2}t^2 + 1\right) + \frac{\sqrt{1+t^2}\sinh^{-1}(t)}{t} - \log\left[\frac{1}{2}(1+p^2)t^2 + 1\right] - 1.$$

Let

$$K(t) = \frac{1}{2}\log\left(\frac{1}{2}t^2 + 1\right) + \frac{\sqrt{1+t^2}\sinh^{-1}(t)}{t} - \log\left[\frac{1}{2}(1+p^2)t^2 + 1\right] - 1. \tag{73}$$

Then

$$K(0) = 0, \tag{74}$$

$$K(\infty) = \frac{3}{2}\log 2 - \log(1+p^2) - 1, \tag{75}$$

$$K'(t) = \frac{1}{t^2\sqrt{1+t^2}}k(t), \tag{76}$$

where $k(t)$ is defined by (36).

We divide the proof into four cases.

Case 1 $p = \sqrt{6}/6$. Then from Lemma 4(1), (72)-(74) and (76) we know that

$$R_{QG}(x,y) < C_{\sqrt{6}/6}(x,y).$$

Case 2 $p = \sqrt{2\sqrt{2}/e - 1}$. Then it follows from Lemma 4(2) and (76) that there exists $\tau > 0$ such that $K(t)$ is strictly increasing on $(0,\tau)$ and strictly decreasing on (τ,∞). Note that equation (75) leads to

$$K(\infty) = 0. \tag{77}$$

Therefore,

$$R_{QG}(x,y) > C_{\sqrt{2\sqrt{2}/e-1}}(x,y)$$

follows from (72)-(74) and (77) together with the piecewise monotonicity of the function $K(t)$ on the interval $(0,1)$.

Case 3 $0 < p < \sqrt{6}/6$. Let $t \to 0^+$. Then making use of (73) and the power series expansion we get

$$K(t) = \frac{1}{2}\left(\frac{1}{6} - p^2\right)t^2 + o\left(t^2\right). \tag{78}$$

Equations (72), (73) and (78) lead to the conclusion that there exists small enough $0 < \delta_4 < 1$ such that

$$R_{QG}(x,y) > C_p(x,y)$$

for all $x > y > 0$ with $(x-y)/\sqrt{2xy} \in (0,\delta_4)$.

Case 4 $\sqrt{2\sqrt{2}/e-1} < p < 1$. Then equation (75) leads to

$$K(\infty) < 0. \tag{79}$$

Equations (72), (73) and (79) imply that there exists large enough $M_4 > 0$

$$R_{QG}(x,y) < C_p(x,y)$$

for all $x > y > 0$ with $(x-y)/\sqrt{2xy} \in (M_4, \infty)$. $\quad\square$

REFERENCES

[1] T.-H. ZHAO, B.-C. ZHOU, M.-K. WANG AND Y.-M. CHU, *On approximating the quasi-arithmetic mean*, J. Inequal. Appl., **2019** (2019), Article 42, 12 pages.

[2] J.-L. WANG, W.-M. QIAN, Z.-Y. HE AND Y.-M. CHU, *On approximating the Toader mean by other bivariate means*, J. Funct. Spaces, **2019** (2019), Article ID 6082413, 7 pages.

[3] Y.-M. CHU AND M.-K. WANG, *Optimal Lehmer mean bounds for the Toader mean*, Results Math., **61**, 3-4 (2012), 223–229.

[4] S.-H. WU AND Y.-M. CHU, *Schur m-power convexity of generalized geometric Bonferroni mean involving theree parameters*, J. Inequal. Appl., **2019** (2019), Article 57, 11 pages.

[5] W.-M. QIAN, H.-Z. XU AND Y.-M. CHU, *Improvements of bounds for the Sándor-Yang means*, J. Inequal. Appl., **2019** (2019), Article 73, 8 pages.

[6] W.-M. QIAN, Z.-Y. HE, H.-W. ZHANG AND Y.-M. CHU, *Sharp bounds for Neuman means in terms of two-parameter contraharmonic and arithmetic mean*, J. Inequal. Appl., **2019** (2019), Article 168, 13 pages.

[7] W.-M. QIAN, X.-H. ZHANG AND Y.-M. CHU, *Sharp bounds for the Toader-Qi mean in terms of harmonic and geometric means*, J. Math. Inequal., **11**, 1 (2017), 121–127.

[8] W.-M. QIAN AND Y.-M. CHU, *Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters*, J. Inequal. Appl., **2017** (2017), Article 274, 10 pages.

[9] Y.-M. CHU, C. ZONG AND G.-D. WANG, *Optimal convex combination bounds of Seiffert and geometric means for the arithmetic mean*, J. Math. Inequal., **5**, 3 (2011), 429–434.

[10] Y.-M. CHU, M.-K. WANG AND S.-L. QIU, *Optimal combinations bounds of root-square and arithmetic means for Toader mean*, Proc. Indian Acad. Sci. Math. Sci., **122**, 1 (2012), 41–51.

[11] Y.-M. CHU, H. WANG AND T.-H. ZHAO, *Sharp bounds for the Neuman mean in terms of the quadratic and second Seiffert means*, J. Inequal. Appl., **2014** (2014), Article 299, 14 pages.

[12] Y.-M. CHU AND B.-Y. LONG, *Bounds of the Neuman-Sándor mean using power and identric means*, Abstr. Appl. Anal., **2013** (2013), Article ID 832591, 6 pages.

[13] W.-M. QIAN AND Y.-M. CHU, *Best possible bounds for Yang mean using generalized logarithmic mean*, Math. Probl. Eng., **2016** (2016), Article ID 8901258, 7 pages.

[14] C.-X. HUANG, L.-H. HUANG AND Y.-G. HE, *Mean square exponential stability of stochastic Cohen-Grossberg neural networks with unbounded distributed delays*, Discrete Dyn. Nat. Soc., **2010** (2010), Article ID 513218, 15 pages.

[15] C.-X. HUANG AND L.-Z. LIU, *Boundedness of multilinear singular integral operator with a non-smooth kernel and mean oscillation*, Quaest. Math., **40**, 3 (2017), 295–312.

[16] M.-K. WANG, Y.-M. CHU AND Y.-P. JIANG, *Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions*, Rocky Mountain J. Math., **46**, 2 (2016), 679–691.

[17] Y.-M. CHU, M. ADIL KHAN, T. ALI AND S. S. DRAGOMIR, *Inequalities for α-fractional differentiable functions*, J. Inequal. Appl., **2017** (2017), Article 93, 12 pages.

[18] M. ADIL KHAN, Y.-M. CHU, T. U. KHAN AND J. KHAN, *Some new inequalities of Hermite-Hadamard type for s-convex functions with applications*, Open Math., **15**, 1 (2017), 1414–1430.

[19] T.-R. HUAN, B.-W. HAN, X.-Y. MA AND Y.-M. CHU, *Optimal bounds for the generalized Euler-Mascheroni constant*, J. Inequal. Appl., **2018** (2018), Article 118, 9 pages.

[20] H.-Z. XU, Y.-M. CHU AND W.-M. QIAN, *Sharp bounds for the Sándor-Yang means in terms of arithmetic and contra-harmonic means*, J. Inequal. Appl., **2018** (2018), Article 127, 13 pages.

[21] M. ADIL KHAN, Y.-M. CHU, A. KASHURI, R. LIKO AND G. ALI, *Conformable fractional integrals versions of Hermite-Hadamard inequalities and their generalizations*, J. Funct. Spaces, **2018** (2018), Article ID 6928130, 9 pages.

[22] Y.-Q. SONG, M. ADIL KHAN, S. ZAHEER ULLAH AND Y.-M. CHU, *Integral inequalities involving strongly convex functions*, J. Funct. Spaces, **2018** (2018), Article ID 6595921, 8 pages.

[23] M. ADIL KHAN, A. IQBAL, M. SULEMAN AND Y.-M. CHU, *Hermite-Hadamard type inequalities for fractional integrals via Green's function*, J. Inequal. Appl., **2018** (2018), Article 161, 15 pages.

[24] M. ADIL KHAN, Y. KHURSHID, T.-S. DU AND Y.-M. CHU, *Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals*, J. Funct. Spaces, **2018** (2018), Article ID 5357463, 12 pages.

[25] T.-R. HUANG, S.-Y. TAN, X.-Y. MA AND Y.-M. CHU, *Monotonicity properties and bounds for the complete p-elliptic integrals*, J. Inequal. Appl., **2018** (2018), Article 239, 11 pages.

[26] S.-L. QIU, X.-Y. MA AND Y.-M. CHU, *Sharp Landen transformation inequalities for hypergeometric functions, with applications*, J. Math. Anal. Appl., **474**, 2 (2019), 1306–1337.

[27] Z.-H. YANG, W.-M. QIAN, Y.-M. CHU AND W. ZHANG, *Monotonicity rule for the quotient of two functions and its application*, J. Inequal. Appl., **2017** (2017), Article 106, 13 pages.

[28] M. ADIL KHAN, S. BEGUM, Y. KHURSHID AND Y.-M. CHU, *Ostrowski type inequalities involving conformable fractional integrals*, J. Inequal. Appl., **2018** (2018), Article 70, 14 pages.

[29] T.-H. ZHAO, M.-K. WANG, W. ZHANG, AND Y.-M. CHU, *Quadratic transformation inequalities for Gaussian hypergeometric function*, J. Inequal. Appl., **2018** (2018), Article 251, 15 pages.

[30] Z.-H. YANG, W.-M. QIAN, Y.-M. CHU AND W. ZHANG, *On rational bounds for the gamma function*, J. Inequal. Appl., **2017** (2017), Article 210, 17 pages.

[31] C.-X. HUANG, H. ZHANG, J.-D. CAO AND H.-J. HU, *Stability and Hopf bifurcation of a delayed prey-predator model with disease in the predator*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **29**, 6 (2019), Article ID 1950099, 23 pages.

[32] B. WANG, C.-L. LUO, S.-H. LI AND Y.-M. CHU, *Sharp one-parameter geometric and quadratic means bounds for the Sándor-Yang means*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **114**, 2 (2020), DOI: 10.1007/s13398-019-00734-0.

[33] C.-X. HUANG, H. ZHANG AND L.-H. HUANG, *Almost periodicity analysis for a delayed Nicholson's blowflies model with nonlinear density-dependent mortality term*, Commun. Pure Appl. Anal., **18**, 6 (2019), 3337–3349.

[34] C.-X. HUANG AND B.-W. LIU, *New studies on dynamic analysis of inertial neural networks involving non-reduced order method*, Neurocomputing, **325**, 24 (2019), 283–287.

[35] J.-F. WANG, C.-X. HUANG AND L.-H. HUANG, *Discontinuity-induced limit cycles in a general planar piecewise linear system of saddle-focus type*, Nonlinear Anal. Hybrid Syst., **33** (2019), 162–178.

[36] J.-F. WANG, X.-Y. CHEN AND L.-H. HUANG, *The number and stability of limit cycles for planar piecewise linear systems of node-saddle type*, J. Math. Anal. Appl., **469**, 1 (2019), 405–427.

[37] Z.-W. CAI, J.-H. HUANG AND L.-H. HUANG, *Periodic orbit analysis for the delayed Filippov system*, Proc. Amer. Math. Soc., **146**, 11 (2019), 4667–4682.

[38] W.-M. QIAN, Y.-Y. YANG, H.-W. ZHANG AND Y.-M. CHU, *Optimal two-parameter geometric and arithmetic mean bounds for the Sándor-Yang mean*, J. Inequal., **2019** (2019), Article 287, 12 pages.

[39] Z.-W. CAI, J.-H. HUANG AND L.-H. HUANG, *Generalized Lyapunov-Razumikhin method for retarded differential inclusions: applications to discontinuous neural networks*, Discrete Contin. Dyn. Syst., **22B**, 9 (2017), 3591–3614.

[40] L. DUAN, L.-H. HUANG, Z.-Y. GUO AND X.-W. FANG, *Periodic attractor for reaction-diffusion high-order Hopfield neural networks with time-varying delays*, Comput. Math. Appl., **73**, 2 (2017), 233–245.

[41] S. ZAHEER ULLAH, M. ADIL KHAN AND Y.-M. CHU, *A note on generalized convex functions*, J. Inequal. Appl., **2019** (2019), Article 291, 10 pages.

[42] C.-X. HUANG, Y.-C. QIAN, L.-H. HUANG AND R. P. AGARWAL, *Dynamical behaviors of a food-chain model with stage structure and time delays*, Adv. Difference Equ., **2018**, (2018), Article 186, 26 pages.

[43] L. DUAN, X.-W. FANG AND C.-X. HUANG, *Global exponential convergence in a delayed almost periodic Nicholson's blowflies model with discontinuous harvesting*, Math. Methods Appl. Sci., **41**, 5 (2018), 1954–1965.

[44] Y.-X. TAN, C.-X. HUANG, B. SUN AND T. WANG, *Dynamics of a class of delayed reaction-diffusion systems with Neumann boundary condition*, J. Math. Anal. Appl., **458**, 2 (2018), 1115–1130.

[45] I. ABBAS BALOCH AND Y.-M. CHU, *Petrović-type inequalities for harmonic h-convex functions*, J. Funct. Spaces, **2020** (2020), Article ID 3075390, 7 pages.

[46] L. DUAN AND C.-X. HUANG, *Existence and global attractivity of almost periodic solutions for a delayed differential neoclassical growth model*, Math. Methods Appl. Sci., **40**, 3 (2017), 814–822.

[47] C.-X. HUANG, S. GUO AND L.-Z. LIU, *Boundedness on Morrey space for Toeplitz type operator associated to singular integral operator with variable Calderón-Zygmund kernel*, J. Math. Inequal., **8**, 3 (2014), 453–464.

[48] C.-X. HUANG, Z.-C. YANG, T.-S. YI AND X.-F. ZOU, *On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities*, J. Differential Equations, **256**, 7 (2014), 2101–2114.

[49] C.-X. HUANG, H.-F. KUANG, X.-H. CHEN AND F.-H. WEN, *An LMI approach for dynamics of switched cellular neural networks with mixed delays*, Abstr. Appl. Anal., **213** (2013), Article ID 870486, 8 pages.

[50] C.-X. HUANG, C.-L. PENG, X.-H. CHEN AND F.-H. WEN, *Dynamics analysis of a class of delayed economic model*, Abstr. Appl. Anal., **213** (2013), Article ID 962738, 12 pages.

[51] Z.-L. TIAN, Y. LIU, Y. ZHANG, Z.-Y. LIU AND M.-.Y. TIAN, *The general inner-outer iteration method based on regular splittings for the PageRank problem*, Appl. Math. Comput., **356** (2019), 479–501.

[52] J. PENG AND Y. ZHANG, *Heron triangles with figurate number sides*, Acta Math. Hungar., **157**, 2 (2019), 478–488.

[53] Y.-J. JIANG AND X.-J. XU, *A monotone finite volume method for time fractional Fokker-Planck equations*, Sci. China Math., **62**, 4 (2019), 783–794.

[54] M. A. LATIF, S. RASHID, S. S. DRAGOMIR AND Y.-M. CHU, *Hermite-Hadamard type inequalities for co-ordinated convex and quasi-convex functions and their applications*, J. Inequal. Appl., **2019** (2019), Article 317, 33 pages.

[55] J. LI, J.-Y. YING AND D.-X. XIE, *On the analysis and application of an ion size-modified Poisson-Boltzmann equation*, Nonlinear Anal. Real World Appl., **47** (2019), 188–203.

[56] Y. ZHANG, *On products of consecutive arithmetic progressions II*, Acta Math. Hungar., **156**, 1 (2018), 240–254.

[57] K.-X. ZHU, Y.-Q. XIE AND F. ZHOU, *Pullback attractors for a damped semilinear wave equation with delays*, Acta Math. Sin., **34**, 7 (2018), 1131–1150.

[58] Z.-Y. LIU, N.-C. WU, X.-R. QIN AND Y.-L. ZHANG, *Trigonometric transform splitting methods for real symmetric Toeplitz systems*, Comput. Math. Appl., **75**, 8 (2018), 2782–2794.

[59] W.-S. TANG AND J.-J. ZHANG, *Symplecticity-preserving continuous-stage Runge-Kutta-Nyström methods*, Appl. Math. Comput., **323** (2018), 204–219.

[60] Z.-Y. LIU, X.-R. QIN, N.-C. WU AND Y.-L. ZHANG, *The shifted classical circulant and skew circulant splitting iterative methods for Toeplitz matrices*, Canad. Math. Bull., **60**, 4 (2018), 807–815.

[61] H.-J. HU AND X.-F. ZOU, *Existence of an extinction wave in the Fisher equation with a shifting habitat*, Proc. Amer. Math. Soc., **145**, 11 (2017), 4763–4771.

[62] W.-S. WANG, *On A-stable one-leg methods for solving nonlinear Volterra functional differential equations*, Appl. Math. Comput., **314** (2017), 380–390.

[63] H.-J. HU AND L.-Z. LIU, *Weighted inequalities for a general commutator associated to a singular integral operator satisfying a variant of Hörmander's condition*, Math. Notes, **101**, 5-6 (2017), 830–840.

[64] W.-S. WANG AND Y.-Z. CHEN, *Fast numerical valuation of options with jump under Merton's model*, J. Comput. Appl. Math., **318** (2017), 79–92.

[65] Z.-H. YANG, Y.-M. CHU AND W. ZHANG, *High accuracy asymptotic bounds for the complete elliptic integral of the second kind*, Appl. Math. Comput., **348** (2019), 552–564.

[66] M.-K. WANG, Y.-M. CHU AND W. ZHANG, *Precise estimates for the solution of Ramanujan's generalized modular equation*, Ramanujan J., **49**, 3 (2019), 653–668.

[67] M.-K. WANG, Y.-M. CHU AND W. ZHANG, *Monotonicity and inequalities involving zero-balanced hypergeometric function*, Math. Inequal. Appl., **22**, 2 (2019), 601–617.

[68] M. ADIL KHAN, S. ULLAH ZAHEER AND Y.-M. CHU, *The concept of coordinate strongly convex functions and related inequalities*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **113**,

3 (2019), 2235–2251.

[69] S. ZAHEER ULLAH, M. ADIL KHAN, Z. A. KHAN AND Y.-M. CHU, *Integral majorization type inequalities for the functions in the sense of strong convexity*, J. Funct. Spaces, **2019** (2019), Article ID 9487823, 11 pages.

[70] S. ZAHEER ULLAH, M. ADIL KHAN AND Y.-M. CHU, *Majorization theorems for strongly convex functions*, J. Inequal. Appl., **2019** (2019), Article 58, 13 pages.

[71] M. ADIL KHAN, S.-H. WU, H. ULLAH AND Y.-M. CHU, *Discrete majorization type inequalities for convex functions on rectangles*, J. Inequal. Appl., **2019** (2019), Article 16, 18 pages.

[72] Y. KHURSHID, M. ADIL KHAN AND Y.-M. CHU, *Conformable integral inequalities of the Hermite-Hadamard type in terms of GG- and GA-convexities*, J. Funct. Spaces, **2019** (2019), Article ID 6926107, 8 pages.

[73] Y. KHURSHID, M. ADIL KHAN, Y.-M. CHU AND Z. A. KHAN, *Hermite-Hadamard-Fejér inequalities for conformable fractional integrals via preinvex functions*, J. Funct. Spaces, **2019** (2019), Article ID 3146210, 9 pages.

[74] M. ADIL KHAN, M. HANIF, Z. ABDUL HAMEED KHAN, K. AHMAD AND Y.-M. CHU, *Association of Jensen's inequalities for s-convex function with Csiszár divergence*, J. Inequal. Appl., **2019** (2019), Article 162, 14 pages.

[75] C. E. XIAO, J.-B. LIU AND Y.-L. LIU, *An inverse pollution problem in porous media*, Appl. Math. Comput., **218**, 7 (2011), 3649–3653.

[76] T.-H. ZHAO, Y.-M. CHU AND H. WANG, *Logarithmically complete monotonicity properties relating to the gamma function*, Abstr. Appl. Anal., **2011** (2011), Article ID 896483, 13 pages.

[77] L. ZHANG AND J.-L. LI, *A new globalization technique for nonlinear conjugate gradient methods for nonconvex minimization*, Appl. Math. Comput., **217**, 24 (2011), 10295–10304.

[78] Y. CHEN AND L.-Z. LIU, *Lipschitz estimates for multilinear commutators of singular integral operators on spaces of homogeneous type*, Miskolc Math. Notes, **11**, 2 (2010), 201–220.

[79] Q. ZHANG AND L.-Z. LIU, *A good λ estimate for multilinear commutator of fractional integral on spaces of homogeneous type*, J. Math. Inequal., **4**, 3 (2010), 371–389.

[80] J. LI AND Y.-J. XU, *An inverse coefficient problem with nonlinear parabolic equation*, J. Appl. Math. Comput., **34**, 1-2 (2010), 195–206.

[81] Z.-H. YANG, *Three families of two-parameter means constructed by trigonometric functions*, J. Inequal. Appl., **2013** (2013), Article 541, 27 pages.

[82] E. NEUMAN, *A one-parameter family of bivariate means*, J. Math. Inequal., **7**, 3 (2013), 399–412.

[83] T.-H. ZHAO, W.-M. QIAN AND Y.-Q. SONG, *Optimal bounds for two Sándor-type means in terms of power means*, J. Inequal. Appl., **2016** (2016), Article 64, 10 pages.

[84] X.-H. HE, W.-M. QIAN, H.-Z. XU AND Y.-M. CHU, *Sharp power mean bounds for two Sándor-Yang means*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **113**, 3 (2019), 2627–2638.

[85] H.-Z. XU AND W.-M. QIAN, *Sharp bounds for Sándor-Yang means in terms of quadratic mean*, J. Math. Inequal., **12**, 4 (2018), 1149–1158.

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