ON A HILBERT–TYPE INTEGRAL INEQUALITY WITH NON–HOMOGENEOUS KERNEL OF MIXED HYPERBOLIC FUNCTIONS

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(*Communicated by M. Krnić*)

Abstract. In this paper, by constructing a new non-homogeneous kernel of mixed hyperbolic functions, we establish a new Hilbert-type integral inequality with the best constant factor. We also consider the equivalent form of the obtained inequality. Moreover, by using the rational fraction expansion of cotangent function and cosecant function, some special Hilbert's type inequalities with the constant factors related to the higher derivatives of cotangent function and cosecant function are presented.

1. Introduction

Suppose E is a measurable set, $f(x)$ and $\mu(x)(>0)$ are two measurable functions defined on E, $p > 1$,

$$L^p(E) := \left\{ f : \|f\|_p := \left(\int_E f^p(x)dm \right)^{\frac{1}{p}} < \infty \right\},$$

and

$$L^p_\mu(E) := \left\{ f : \|f\|_{p,\mu} := \left(\int_E \mu(x)f^p(x)dm \right)^{\frac{1}{p}} < \infty \right\}.$$

Let $f, g \geqslant 0$, $f, g \in L^p(0, \infty)$, then we have the well known Hilbert inequality [2]:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi\|f\|_p\|g\|_q, \quad (1.1)$$

where the constant factor π in (1.1) is the best possible.

In the past 20 years, by introducing various new kernel functions, multiple parameters, special functions such as β- function and Γ- function, and using the skills of real or complex analysis, researchers have established a large number of new Hilbert-type inequalities(see[1,3,5-10,12-14]).

In this paper, we will establish the following Hilbert-type inequalities with non-homogeneous kernel of mixed hyperbolic functions

$$\int_0^\infty \int_0^\infty \sinh(xy)\operatorname{csch}(3xy)f(x)g(y)dxdy < \frac{\sqrt{3}\pi}{18}\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{1.2}$$

$$\int_0^\infty \int_0^\infty \operatorname{sech}(xy)\operatorname{sech}(2xy)f(x)g(y)dxdy < \frac{(\sqrt{2}-1)\pi}{2}\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{1.3}$$

$$\int_0^\infty \int_0^\infty \operatorname{csch}(xy)\tanh(2xy)f(x)g(y)dxdy < \frac{\sqrt{2}\pi}{2}\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{1.4}$$

where $\mu(x) = x^{-1}, \nu(y) = y^{-1}$.

More generally, by introducing multiple parameters, we construct the following non-homogeneous kernel

$$K(x,y) := \frac{e^{\beta_2 xy} + \eta e^{\beta_3 xy}}{e^{\beta_1 xy} + \delta e^{-\beta_1 xy}}, \tag{1.5}$$

where $\eta, \delta = \pm 1$, $\beta_1 > 0$, $\beta_3 \leqslant \beta_2 < \beta_1$ ($\beta_2 \neq \beta_3$ for $\eta = -1$), $\beta > 0$ for $\eta\delta = -1$, $\beta \geqslant 0$ for $\eta\delta = 1$. By using the some skills of analysis, we will establish a new Hilbert-type inequality with the kernel $K(x,y)$. We will show that the new obtained inequality is the unified generalization of (1.2)-(1.4) and some other inequalities. Most importantly, by the rational fraction expansion of cotangent function and cosecant function, some special inequality with the constant factor related to the higher derivatives of cotangent function and cosecant function are obtained at the end of the paper.

2. Some Lemmas

DEFINITION 2.1[11]. *Let* $z > 0$,

$$\Gamma(z) := \int_0^\infty x^{z-1}e^{-x}dx$$

is the Γ*-function. Specially, if* $z \in N$*, we have* $\Gamma(z) = (z-1)!$.

LEMMA 2.2. *Let* $a, b > 0, a + b = 1$, $\varphi(x) = \cot x$, $n \in N \cup \{0\}$, *then*

$$\varphi^{(2n)}(a\pi) = \frac{(2n)!}{\pi^{2n+1}} \sum_{k=0}^\infty \left(\frac{1}{(k+a)^{2n+1}} - \frac{1}{(k+b)^{2n+1}} \right), \tag{2.1}$$

$$\varphi^{(2n+1)}(a\pi) = -\frac{(2n+1)!}{\pi^{2n+2}} \sum_{k=0}^\infty \left(\frac{1}{(k+a)^{2n+2}} + \frac{1}{(k+b)^{2n+2}} \right). \tag{2.2}$$

Proof. We have the rational fraction expansion of $\varphi(x) = \cot x$ as follow[11]:

$$\varphi(x) = \frac{1}{x} + \sum_{k=1}^\infty \left(\frac{1}{x+k\pi} + \frac{1}{x-k\pi} \right).$$

Find the $2n$th derivative of $\varphi(x)$, then

$$\varphi^{(2n)}(x) = (2n)! \left(\frac{1}{x^{2n+1}} + \sum_{k=1}^{\infty} \left(\frac{1}{(x+k\pi)^{2n+1}} + \frac{1}{(x-k\pi)^{2n+1}} \right) \right). \qquad (2.3)$$

Let $x = a\pi$ in (2.3), in view of $a + b = 1$, we have

$$\varphi^{(2n)}(a\pi) = \frac{(2n)!}{\pi^{2n+1}} \left(\sum_{k=0}^{\infty} \frac{1}{(k+a)^{2n+1}} - \sum_{k=1}^{\infty} \frac{1}{(k-a)^{2n+1}} \right)$$

$$= \frac{(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \left(\frac{1}{(k+a)^{2n+1}} - \frac{1}{(k+b)^{2n+1}} \right).$$

(2.1) is proved. Similarly, find the $(2n+1)$th derivative of $\varphi(x)$, we can prove (2.2).

LEMMA 2.3. *Let* $a, b > 0, a + b = 1$, $\psi(x) = \csc x$, $n \in N \cup \{0\}$, *then*

$$\psi^{(2n)}(a\pi) = \frac{(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{(k+a)^{2n+1}} + \frac{1}{(k+b)^{2n+1}} \right), \qquad (2.4)$$

$$\psi^{(2n+1)}(a\pi) = -\frac{(2n+1)!}{\pi^{2n+2}} \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{(k+a)^{2n+2}} - \frac{1}{(k+b)^{2n+2}} \right). \qquad (2.5)$$

Proof. We have the following rational fraction expansion of $\psi(x) = \csc x$[11]:

$$\psi(x) = \frac{1}{x} + \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{x+k\pi} + \frac{1}{x-k\pi} \right).$$

Find the $2n$th derivative of $\psi(x)$, then

$$\psi^{(2n)}(x) = (2n)! \left(\frac{1}{x^{2n+1}} + \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{(x+k\pi)^{2n+1}} + \frac{1}{(x-k\pi)^{2n+1}} \right) \right). \qquad (2.6)$$

Let $x = a\pi$ in (2.6), then

$$\psi^{(2n)}(a\pi) = \frac{(2n)!}{\pi^{2n+1}} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^{2n+1}} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-a)^{2n+1}} \right)$$

$$= \frac{(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{(k+a)^{2n+1}} + \frac{1}{(k+b)^{2n+1}} \right).$$

Therefore, (2.4) is proved. Similarly, (2.5) can be proved.

LEMMA 2.4. *Let* $\eta, \delta = \pm 1$, $\beta_1 > 0$, $\beta_3 \leqslant \beta_2 < \beta_1$ ($\beta_2 \neq \beta_3$ *for* $\eta = -1$), $\beta > 0$ *for* $\eta\delta = -1$, $\beta \geqslant 0$ *for* $\eta\delta = 1$. $K(x,y)$ *is defined by (1.5), and*

$$C_{\eta,\delta}(\beta_1,\beta_2,\beta_3,\beta) := \sum_{k=0}^{\infty} \frac{(-\delta)^k}{(2\beta_1 k + \beta_1 - \beta_2)^{\beta+1}} + \eta \sum_{k=0}^{\infty} \frac{(-\delta)^k}{(2\beta_1 k + \beta_1 - \beta_3)^{\beta+1}}, \quad (2.7)$$

then

$$\omega(x) := \int_0^{\infty} K(x,y)y^{\beta}\,dy = x^{-\beta-1}\Gamma(\beta+1)C_{\eta,\delta}(\beta_1,\beta_2,\beta_3,\beta), \quad (2.8)$$

$$\varpi(y) := \int_0^{\infty} K(x,y)x^{\beta}\,dx = y^{-\beta-1}\Gamma(\beta+1)C_{\eta,\delta}(\beta_1,\beta_2,\beta_3,\beta). \quad (2.9)$$

Proof. Setting $xy = t$, we have

$$\omega(x) = x^{-\beta-1}\int_0^{\infty} K(1,t)t^{\beta}\,dt. \quad (2.10)$$

Since $t \in (0,+\infty)$, $\beta_1 > 0$, $\delta = \pm 1$, we find $\frac{1}{1+\delta e^{-2\beta_1 t}} = \sum_{k=0}^{\infty}(-\delta)^k e^{-2\beta_1 kt}$. Therefore

$$\int_0^{\infty} K(1,t)t^{\beta}\,dt = \sum_{k=0}^{\infty}(-\delta)^k \int_0^{\infty} e^{-(2\beta_1 k + \beta_1 - \beta_2)t}t^{\beta}\,dt$$

$$+\eta\sum_{k=0}^{\infty}(-\delta)^k \int_0^{\infty} e^{-(2\beta_1 k + \beta_1 - \beta_3)t}t^{\beta}\,dt := I_1 + \eta I_2. \quad (2.11)$$

Setting $u = (2\beta_1 k + \beta_1 - \beta_2)t$, we have

$$I_1 = \Gamma(\beta+1)\sum_{k=0}^{\infty}\frac{(-\delta)^k}{(2\beta_1 k + \beta_1 - \beta_2)^{\beta+1}}. \quad (2.12)$$

Similarly, setting $u = (2\beta_1 k + \beta_1 - \beta_3)t$, then

$$I_2 = \Gamma(\beta+1)\sum_{k=0}^{\infty}\frac{(-\delta)^k}{(2\beta_1 k + \beta_1 - \beta_3)^{\beta+1}}. \quad (2.13)$$

Combining(2.11), (2.12) and (2.13), and using (2.7), we obtain

$$\int_0^{\infty} K(1,t)t^{\beta}\,dt = \Gamma(\beta+1)C_{\eta,\delta}(\beta_1,\beta_2,\beta_3,\beta). \quad (2.14)$$

Applying (2.14) to (2.10), we obtain (2.8). Similarly, it can be proved that (2.9) also holds.

REMARK2.5. *If one of the following conditions is satisfied:* (1) $\eta = 1$, $\delta = -1$, $\beta > 0$, (2) $\eta = -1$, $\delta = 1$, $\beta > 0$, (3) $\eta = 1$, $\delta = 1$, $\beta \geqslant 0$, (4) $\eta = -1$, $\delta = -1$, $\beta > 0$, *it can be easy to show that the series on the right side of (2.7) is convergent. For* $\eta = -1$, $\delta = -1$, $\beta = 0$, *since* $\beta_3 < \beta_2 < \beta_1$

$$\sum_{k=0}^{\infty}\frac{1}{2\beta_1 k + \beta_1 - \beta_2} - \sum_{k=0}^{\infty}\frac{1}{2\beta_1 k + \beta_1 - \beta_3} = \sum_{k=0}^{\infty}\frac{\beta_2 - \beta_3}{(2\beta_1 k + \beta_1 - \beta_2)(2\beta_1 k + \beta_1 - \beta_3)}$$

converges to a positive number obviously. Therefore the series on the right-hand side of (2.7) is convergent under the conditions of Lemma 2.4.

LEMMA 2.6. *Let* $\eta, \delta = \pm 1$, $\beta_1 > 0$, $\beta_3 \leqslant \beta_2 < \beta_1$ ($\beta_2 \neq \beta_3$ *for* $\eta = -1$), $\beta > 0$ *for* $\eta\delta = -1$, $\beta \geqslant 0$ *for* $\eta\delta = 1$. $K(x,y)$ *and* $C_{\eta,\delta}(\beta_1, \beta_2, \beta_3, \beta)$ *are defined by (1.5) and (2.7) respectively,* ε *is a sufficiently small positive number,* $f_\varepsilon(x)$, $g_\varepsilon(x)$ *are defined as follows :*

$$f_\varepsilon(x) := \begin{cases} x^{\frac{p\beta+\varepsilon}{p}}, & x \in (0,1], \\ 0, & x \in (1,\infty), \end{cases}$$

$$g_\varepsilon(x) := \begin{cases} 0, & x \in (0,1], \\ x^{\frac{q\beta-\varepsilon}{q}}, & x \in (1,\infty), \end{cases}$$

then

$$\varepsilon J := \varepsilon \int_0^\infty \int_0^\infty K(x,y) f_\varepsilon(x) g_\varepsilon(y) dx dy = \Gamma(\beta+1) C_{\eta,\delta}(\beta_1, \beta_2, \beta_3, \beta) + o(1). \quad (2.15)$$

Proof. Setting $xy = t$, by Fubini's theorem, we have

$$\varepsilon J = \varepsilon \int_1^\infty y^{\frac{q\beta-\varepsilon}{q}} \left(\int_0^1 K(x,y) x^{\frac{p\beta+\varepsilon}{p}} dx \right) dy = \varepsilon \int_1^\infty y^{-\varepsilon-1} \left(\int_0^y K(1,t) t^{\frac{p\beta+\varepsilon}{p}} dt \right) dy$$

$$= \varepsilon \int_1^\infty y^{-\varepsilon-1} \left(\int_0^1 K(1,t) t^{\frac{p\beta+\varepsilon}{p}} dt \right) dy + \varepsilon \int_1^\infty y^{-\varepsilon-1} \left(\int_1^y K(1,t) t^{\frac{p\beta+\varepsilon}{p}} dt \right) dy$$

$$= \int_0^1 K(1,t) t^{\frac{p\beta+\varepsilon}{p}} dt + \varepsilon \int_1^\infty K(1,t) t^{\frac{p\beta+\varepsilon}{p}} \left(\int_t^\infty y^{-\varepsilon-1} dy \right) dt$$

$$= \int_0^1 K(1,t) t^{\frac{p\beta+\varepsilon}{p}} dt + \int_1^\infty K(1,t) t^{\frac{q\beta-\varepsilon}{q}} dt. \quad (2.16)$$

Let $\varepsilon \to 0^+$ in (2.16), and using (2.14), then (2.15) is proved. $\quad\square$

3. Main Results

THEOREM 3.1. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\eta, \delta = \pm 1$, $\beta_1 > 0$, $\beta_3 \leqslant \beta_2 < \beta_1$ ($\beta_2 \neq \beta_3$ *for* $\eta = -1$). *Let* $\beta > 0$ *for* $\eta\delta = -1$ *and* $\beta \geqslant 0$ *for* $\eta\delta = 1$. *Let* $\mu(x) = x^{-(p\beta+1)}$, $\nu(y) = y^{-(q\beta+1)}$, *and define* $f(x)$, $g(x) \geqslant 0$, *such that* $f(x) \in L_\mu^p(0,\infty)$, $g(x) \in L_\nu^q(0,\infty)$. *Further more, define* $K(x,y)$ *and* $C_{\eta,\delta}(\beta_1, \beta_2, \beta_3, \beta)$ *via (1.5) and (2.7) respectively, then*

$$\int_0^\infty \int_0^\infty K(x,y) f(x) g(y) dx dy < \Gamma(\beta+1) C_{\eta,\delta}(\beta_1, \beta_2, \beta_3, \beta) \|f\|_{p,\mu} \|g\|_{q,\nu}, \quad (3.1)$$

where the constant factor $\Gamma(\beta+1) C_{\eta,\delta}(\beta_1, \beta_2, \beta_3, \beta)$ *is the best possible.*

Proof. We start the proof of Theorem 3.1 from the following inequality (see [4]) which provides a unified treatment of Hilbert's type inequalities, that is

$$\int_{\Omega\times\Omega} K(x,y) f(x) g(y) d\mu_1(x) d\mu_2(y)$$

$$\leqslant \left(\int_\Omega \varphi^p(x) F(x) f^p(x) d\mu_1(x) \right)^{\frac{1}{p}} \left(\int_\Omega \psi^q(y) G(y) g^q(y) d\mu_2(y) \right)^{\frac{1}{q}}, \qquad (3.2)$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, μ_1, μ_2 are positive σ-finite measures, $K : \Omega \times \Omega \to \mathbb{R}$, $f, g, \varphi, \psi : \Omega \to \mathbb{R}$ are measurable, non-negative functions and

$$F(x) = \int_\Omega \frac{K(x,y)}{\psi^p(y)} d\mu_2(y), \quad G(y) = \int_\Omega \frac{K(x,y)}{\varphi^q(x)} d\mu_1(x).$$

(3.2) takes the form of equality, if and only if $f^p(x) = K_1 \varphi^{-(p+q)}(x)$, and $g^q(y) = K_2 \psi^{-(p+q)}(y)$ for arbitrary constants K_1 and K_2.

Let $\Omega = (0, \infty)$, $\varphi(x) = x^{-\frac{\beta}{q}}$, $\psi(y) = y^{-\frac{\beta}{p}}$, and let $K(x,y)$ be defined by Lemma 2.4. Then $F(x) = \omega(x)$, $G(y) = \varpi(y)$, where $\omega(x)$ and $\varpi(y)$ are defined by Lemma 2.4. Combining (3.2), (2.8) and (2.9), we have

$$\int_0^\infty \int_0^\infty K(x,y) f(x) g(y) dx dy$$

$$\leqslant \left(\int_0^\infty x^{-\frac{p\beta}{q}} \omega(x) f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{-\frac{q\beta}{p}} \varpi(y) g^q(y) dy \right)^{\frac{1}{q}}$$

$$\leqslant \Gamma(\beta + 1) C_{\eta,\delta}(\beta_1, \beta_2, \beta_3, \beta) \|f\|_{p,\mu} \|g\|_{q,\nu}. \qquad (3.3)$$

If (3.3) takes the form of equality, we obtain $x^{-(p\beta+1)} f^p(x) = \frac{K_1}{x}$, and $y^{-(q\beta+1)} g^q(y) = \frac{K_2}{y}$, which contradict the facts $f(x) \in L_\mu^p(0,\infty)$, and $g(x) \in L_\nu^q(0,\infty)$. Therefore, (3.3) keeps the form of strict inequality, and (3.1) is proved.

Next, It should be proved that the constant factor $\Gamma(\beta + 1) C_{\eta,\delta}(\beta_1, \beta_2, \beta_3, \beta)$ in (3.1) is the best possible. To do that, we suppose that there exists a positive constant k $\left(0 < k < \Gamma(\beta + 1) C_{\eta,\delta}(\beta_1, \beta_2, \beta_3, \beta) \right)$, such that (3.1) is still valid if we replace $\Gamma(\beta + 1) C_{\eta,\delta}(\beta_1, \beta_2, \beta_3, \beta)$ by k. That is

$$\int_0^\infty \int_0^\infty K(x,y) f(x) g(y) dx dy < k \|f\|_{p,\mu} \|g\|_{q,\nu}, \qquad (3.4)$$

Replacing f and g in (3.4) by f_ε and g_ε defined in Lemma 2.6 respectively, we have

$$\varepsilon \int_0^\infty \int_0^\infty K(x,y) f_\varepsilon(x) g_\varepsilon(y) dx dy < \varepsilon k \left(\int_0^1 x^{\varepsilon-1} dx \right)^{\frac{1}{p}} \left(\int_1^\infty x^{-\varepsilon-1} dx \right)^{\frac{1}{q}} = k.$$

By Lemma 2.6, we obtain

$$\Gamma(\beta + 1) C_{\eta,\delta}(\beta_1, \beta_2, \beta_3, \beta) + o(1) < k$$

Let $\varepsilon \to 0^+$, It follows that $\Gamma(\beta + 1) C_{\eta,\delta}(\beta_1, \beta_2, \beta_3, \beta) \leqslant k$, which contradicts the hypothesis that $k < \Gamma(\beta + 1) C_{\eta,\delta}(\beta_1, \beta_2, \beta_3, \beta)$. Hence, the constant factor in (3.1) is the best possible. Theorem 3.1 is proved. $\square$

THEOREM 3.2. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\eta, \delta = \pm 1$, $\beta_1 > 0$, $\beta_3 \leqslant \beta_2 < \beta_1$ ($\beta_2 \neq \beta_3$ *for* $\eta = -1$), $\beta > 0$ *for* $\eta\delta = -1$, $\beta \geqslant 0$ *for* $\eta\delta = 1$. $\mu(x) = x^{-(p\beta+1)}$, $v(y) = y^{-(q\beta+1)}$, $f(x) \geqslant 0$, $f(x) \in L^p_\mu(0,\infty)$, $K(x,y)$ *and* $\Gamma(\beta+1)C_{\eta,\delta}(\beta_1,\beta_2,\beta_3,\beta)$ *are defined by Lemma 2.4, then*

$$\int_0^\infty y^{p\beta+p-1} \left(\int_0^\infty K(x,y)f(x)dx \right)^p dy$$

$$< \left(\Gamma(\beta+1)C_{\eta,\delta}(\beta_1,\beta_2,\beta_3,\beta) \right)^p \left(\|f\|_{p,\mu} \right)^p, \qquad (3.5)$$

where the constant factor $\left(\Gamma(\beta+1)C_{\eta,\delta}(\beta_1,\beta_2,\beta_3,\beta) \right)^p$ *is the best possible, and (3.5) is equivalent to (3.1).*

Proof. Setting $g(y) := y^{p\beta+p-1}(\int_0^\infty K(x,y)f(x)dx)^{p-1}$, by (3.1), we find

$$0 < (\|g\|_{q,v})^{pq} = \left(\int_0^\infty y^{-(q\beta+1)} g^q(y) dy \right)^p$$

$$= \left(\int_0^\infty y^{p\beta+p-1} \left(\int_0^\infty K(x,y)f(x)dx \right)^p dy \right)^p = \left(\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy \right)^p$$

$$\leqslant \left(\Gamma(\beta+1)C_{\eta,\delta}(\beta_1,\beta_2,\beta_3,\beta) \right)^p \left(\|f\|_{p,\mu} \right)^p \left(\|g\|_{q,v} \right)^p. \qquad (3.6)$$

Therefore

$$0 < (\|g\|_{q,v})^q = \int_0^\infty y^{p\beta+p-1} \left(\int_0^\infty K(x,y)f(x)dx \right)^p dy$$

$$\leqslant \left(\Gamma(\beta+1)C_{\eta,\delta}(\beta_1,\beta_2,\beta) \right)^p \left(\|f\|_{p,\mu} \right)^p. \qquad (3.7)$$

Since $f(x) \in L^p_\mu(0,\infty)$, by (3.7), it follows that $g(x) \in L^q_v(0,\infty)$. By using (3.1) again, both (3.6) and (3.7) keep the form of strict inequalities, then (3.5) is proved.

On the other hand, suppose that (3.5) holds, by Hölder's inequality, we have

$$\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy$$

$$= \int_0^\infty \left(y^{\beta+\frac{1}{q}} \int_0^\infty K(x,y)f(x)dx \right) \left(y^{-\left(\beta+\frac{1}{q}\right)} g(y) \right) dy$$

$$\leqslant \left(\int_0^\infty y^{p\beta+p-1} \left(\int_0^\infty K(x,y)f(x)dx \right)^p dy \right)^{\frac{1}{p}} \|g\|_{q,v}. \qquad (3.8)$$

Applying (3.5) to (3.8), we obtain (3.1). Hence, (3.1) and (3.5) are equivalent.

If the constant factor $\left(\Gamma(\beta+1)C_{\eta,\delta}(\beta_1,\beta_2,\beta_3,\beta) \right)^p$ in (3.5) is not the best possible, from the equivalence of (3.1) and (3.5), we may get a contradiction that the constant factor in (3.1) is not the best possible. Therefore the constant factor in (3.5) is the best possible. Theorem 3.2 is proved. □

4. Corollaries

Setting $\eta = -1, \delta = -1$, $\beta_3 = -\beta_2$, $\beta = 2n \, (n \in N \cup \{0\})$ in Theorem 3.1, and using (2.1) and (2.7), we can obtain the following corollary:

COROLLARY 4.1. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\beta_2 > 0$, $\beta_2 < \beta_1$, $n \in N \cup \{0\}$, $\varphi(x) = \cot x$, $\mu(x) = x^{-(2np+1)}$, $\nu(y) = y^{-(2qn+1)}$, $f(x)$, $g(x) \geqslant 0$, $f(x) \in L_\mu^p(0, \infty)$, $g(x) \in L_\nu^q(0, \infty)$, *then*

$$\int_0^\infty \int_0^\infty \sinh(\beta_2 xy)\operatorname{csch}(\beta_1 xy) f(x) g(y) dx dy$$

$$< - \left(\frac{\pi}{2\beta_1}\right)^{2n+1} \varphi^{(2n)}\left(\frac{\beta_1 + \beta_2}{2\beta_1}\pi\right) \|f\|_{p,\mu} \|g\|_{q,\nu}. \qquad (4.1)$$

Let $\beta_1 = 2\lambda$, $\beta_2 = \lambda$ *in (4.1),* $\lambda > 0$, *in view of* $\frac{e^{\lambda xy} - e^{-\lambda xy}}{e^{2\lambda xy} - e^{-2\lambda xy}} = \frac{1}{2}\operatorname{sech}(\lambda xy)$, *then*

$$\int_0^\infty \int_0^\infty \operatorname{sech}(\lambda xy) f(x) g(y) dx dy$$

$$< -\frac{1}{2^{4n+1}}\left(\frac{\pi}{\lambda}\right)^{2n+1} \varphi^{(2n)}\left(\frac{3\pi}{4}\right) \|f\|_{p,\mu} \|g\|_{q,\nu}. \qquad (4.2)$$

Particularly, setting $\lambda = 1$, $n = 0$ *in (4.2), then* $\mu(x) = x^{-1}$, $\nu(y) = y^{-1}$, *and*

$$\int_0^\infty \int_0^\infty \operatorname{sech}(xy) f(x) g(y) dx dy < \frac{\pi}{2}\|f\|_{p,\mu}\|g\|_{q,\nu}, \qquad (4.3)$$

Let $\beta_1 = 3\lambda$, $\beta_2 = \lambda$ *in (4.1),* $\lambda > 0$, *then*

$$\int_0^\infty \int_0^\infty \sinh(\lambda xy)\operatorname{csch}(3\lambda xy) f(x) g(y) dx dy$$

$$< - \left(\frac{\pi}{6\lambda}\right)^{2n+1} \varphi^{(2n)}\left(\frac{2\pi}{3}\right) \|f\|_{p,\mu} \|g\|_{q,\nu}. \qquad (4.4)$$

Particularly, setting $\lambda = 1$, $n = 0$ *in (4.4), we obtain (1.2).*
Let $\beta_1 = 3\lambda$, $\beta_2 = 2\lambda$ *in (4.1),* $\lambda > 0$, *we obtain*

$$\int_0^\infty \int_0^\infty \sinh(2\lambda xy)\operatorname{csch}(3\lambda xy) f(x) g(y) dx dy$$

$$< - \left(\frac{\pi}{6\lambda}\right)^{2n+1} \varphi^{(2n)}\left(\frac{5\pi}{6}\right) \|f\|_{p,\mu} \|g\|_{q,\nu}. \qquad (4.5)$$

In particular, setting $\lambda = 1$, $n = 0$ *in (4.5), then* $\mu(x) = x^{-1}$, $\nu(y) = y^{-1}$, *and*

$$\int_0^\infty \int_0^\infty \sinh(2xy)\operatorname{csch}(3xy) f(x) g(y) dx dy < \frac{\sqrt{3}\pi}{6}\|f\|_{p,\mu}\|g\|_{q,\nu}.$$

Let $\beta_1 = 4\lambda$, $\beta_2 = \lambda$ in (4.1), in view of $\frac{e^{\lambda xy} - e^{-\lambda xy}}{e^{4\lambda xy} - e^{-4\lambda xy}} = \frac{1}{4}\operatorname{sech}(\lambda xy)\operatorname{sech}(2\lambda xy)$, then

$$\int_0^\infty \int_0^\infty \operatorname{sech}(\lambda xy)\operatorname{sech}(2\lambda xy) f(x)g(y)dxdy$$

$$< -\frac{1}{2^{6n+1}}\left(\frac{\pi}{\lambda}\right)^{2n+1} \varphi^{(2n)}\left(\frac{5\pi}{8}\right) \|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.6}$$

In particular, setting $\lambda = 1$, $n = 0$ in (4.6), then $\mu(x) = x^{-1}, \nu(y) = y^{-1}$, since $\cot\frac{5\pi}{8} = 1 - \sqrt{2}$, we obtain (1.3).

Similarly, setting $\eta = 1, \delta = -1, \beta_3 = -\beta_2$, $\beta = 2n+1(n \in N \cup \{0\})$ in Theorem 3.1, and using (2.2) and (2.7), the following corollary holds:

COROLLARY 4.2. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\beta_2 \geqslant 0$, $\beta_2 < \beta_1$, $n \in N \cup \{0\}, \varphi(x) = \cot x$, $\mu(x) = x^{-(2np+p+1)}$, $\nu(y) = y^{-(2qn+q+1)}$, $f(x), g(x) \geqslant 0$, $f(x) \in L^p_\mu(0,\infty)$, $g(x) \in L^q_\nu(0,\infty)$, then*

$$\int_0^\infty \int_0^\infty \cosh(\beta_2 xy)\operatorname{csch}(\beta_1 xy) f(x)g(y)dxdy$$

$$< -\left(\frac{\pi}{2\beta_1}\right)^{2n+2} \varphi^{(2n+1)}\left(\frac{\beta_1 + \beta_2}{2\beta_1}\pi\right) \|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.7}$$

Let $\beta_1 = \lambda$, $\beta_2 = 0$ in (4.7), $\lambda > 0$, we have

$$\int_0^\infty \int_0^\infty \operatorname{csch}(\lambda xy) f(x)g(y)dxdy < -\left(\frac{\pi}{2\lambda}\right)^{2n+2} \varphi^{(2n+1)}\left(\frac{\pi}{2}\right) \|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{4.8}$$

Let $\beta_1 = 2\lambda$, $\beta_2 = \lambda$ in (4.7), $\lambda > 0$. Because of $\frac{e^{\lambda xy} + e^{-\lambda xy}}{e^{2\lambda xy} - e^{-2\lambda xy}} = \frac{1}{2}\operatorname{csch}(\lambda xy)$, we obtain a inequality similar to (4.8):

$$\int_0^\infty \int_0^\infty \operatorname{csch}(\lambda xy) f(x)g(y)dxdy$$

$$< -\frac{1}{2^{4n+3}}\left(\frac{\pi}{\lambda}\right)^{2n+2} \varphi^{(2n+1)}\left(\frac{3\pi}{4}\right) \|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.9}$$

REMARK 4.3. *It should be noted that (4.8) is equivalent to (4.9). In fact, by using (2.2) and the following obvious equality*

$$\sum_{k=0}^\infty \left(\frac{1}{(4k+1)^{2n+2}} + \frac{1}{(4k+3)^{2n+2}}\right) = \sum_{k=0}^\infty \frac{1}{(2k+1)^{2n+2}},$$

it can be easy to show $\varphi^{(2n+1)}\left(\frac{3\pi}{4}\right) = 2^{2n+1}\varphi^{(2n+1)}\left(\frac{\pi}{2}\right)$. Hence, (4.8) and (4.9) are equivalent.

Let $\beta_1 = 3\lambda$, $\beta_2 = \lambda$ or $\beta_1 = 3\lambda$, $\beta_2 = 2\lambda$ in (4.7), $\lambda > 0$, then the following two inequalities hold:

$$\int_0^\infty \int_0^\infty \cosh(\lambda xy)\operatorname{csch}(3\lambda xy) f(x)g(y)dxdy$$

$$< -\left(\frac{\pi}{6\lambda}\right)^{2n+2}\varphi^{(2n+1)}\left(\frac{2\pi}{3}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.10}$$

$$\int_0^\infty\int_0^\infty \cosh(2\lambda xy)\operatorname{csch}(3\lambda xy)f(x)g(y)dxdy$$

$$< -\left(\frac{\pi}{6\lambda}\right)^{2n+2}\varphi^{(2n+1)}\left(\frac{5\pi}{6}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.11}$$

Let $\beta_1 = 4\lambda$, $\beta_2 = \lambda$ in (4.7), $\lambda > 0$, we have

$$\int_0^\infty\int_0^\infty \operatorname{csch}(\lambda xy)\operatorname{sech}(2\lambda xy)f(x)g(y)dxdy$$

$$< -\frac{1}{4^{3n+2}}\left(\frac{\pi}{\lambda}\right)^{2n+2}\varphi^{(2n+1)}\left(\frac{5\pi}{8}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.12}$$

Setting $\eta = 1, \delta = 1$, $\beta_3 = -\beta_2$, $\beta = 2n(n \in N \cup \{0\})$ in Theorem 3.1, and using (2.4) and (2.7), we obtain another corollary:

COROLLARY 4.4. *Let* $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $\beta_2 \geqslant 0$, $\beta_2 < \beta_1$, $n \in N \cup \{0\}, \psi(x) = \csc x$, $\mu(x) = x^{-(2np+1)}$, $\nu(y) = y^{-(2qn+1)}$, $f(x)$, $g(x) \geqslant 0$, $f(x) \in L_\mu^p(0,\infty)$, $g(x) \in L_\nu^q(0,\infty)$, then*

$$\int_0^\infty\int_0^\infty \cosh(\beta_2 xy)\operatorname{sech}(\beta_1 xy)f(x)g(y)dxdy$$

$$< \left(\frac{\pi}{2\beta_1}\right)^{2n+1}\psi^{(2n)}\left(\frac{\beta_1+\beta_2}{2\beta_1}\pi\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.13}$$

Let $\beta_1 = \lambda$, $\lambda > 0$, $\beta_2 = 0$ in (4.13), we obtain the following inequality

$$\int_0^\infty\int_0^\infty \operatorname{sech}(\lambda xy)f(x)g(y)dxdy < \left(\frac{\pi}{2\lambda}\right)^{2n+1}\psi^{(2n)}\left(\frac{\pi}{2}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.14}$$

REMARK 4.5. *(4.14) is the equivalent form of (4.2). In fact, by using (2.1), (2.4) and the following equality*

$$\sum_{k=0}^\infty\left(\frac{1}{(4k+1)^{2n+1}}-\frac{1}{(4k+3)^{2n+1}}\right)=\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^{2n+1}},$$

we obtain $\varphi^{(2n+1)}\left(\frac{3\pi}{4}\right)=-2^{2n}\psi^{(2n+1)}\left(\frac{\pi}{2}\right)$. *Therefore, (4.14) is equivalent to (4.2).*

Let $\beta_1 = 2\lambda$, $\beta_2 = \lambda$ in (4.13), $\lambda > 0$. Since $\frac{e^{\lambda xy}+e^{-\lambda xy}}{e^{2\lambda xy}+e^{-2\lambda xy}}=\frac{1}{2}\operatorname{csch}(\lambda xy)\tanh(2\lambda xy)$, we obtain

$$\int_0^\infty\int_0^\infty \operatorname{csch}(\lambda xy)\tanh(2\lambda xy)f(x)g(y)dxdy$$

$$< \frac{1}{2^{4n+1}}\left(\frac{\pi}{\lambda}\right)^{2n+1}\psi^{(2n)}\left(\frac{3\pi}{4}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.15}$$

In particular, setting $\lambda = 1$, $n = 0$ in (4.15), then $\mu(x) = x^{-1}$, $\nu(y) = y^{-1}$, and we obtain (1.4).

Let $\beta_1 = p$, $\beta_2 = 2 - p$ in (4.13), $1 < p \leqslant 2$, we obtain

$$\int_0^\infty \int_0^\infty \cosh((2-p)xy)\operatorname{sech}(pxy)f(x)g(y)dxdy$$

$$< \left(\frac{\pi}{2p}\right)^{2n+1} \psi^{(2n)}\left(\frac{\pi}{p}\right) \|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.16}$$

In particular, setting $n = 0$ in (4.16), then $\mu(x) = x^{-1}$, $\nu(y) = y^{-1}$, and

$$\int_0^\infty \int_0^\infty \cosh((2-p)xy)\operatorname{sech}(pxy)f(x)g(y)dxdy < \frac{\pi}{2p\sin\frac{\pi}{p}}\|f\|_{p,\mu}\|g\|_{q,\nu}.$$

Similarly, setting $\eta = -1$, $\delta = 1$, $\beta_3 = -\beta_2$, $\beta = 2n+1(n \in N \cup \{0\})$ in Theorem 3.1, we obtain

COROLLARY 4.6. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\beta_2 > 0$, $\beta_2 < \beta_1$, $n \in N \cup \{0\}$, $\psi(x) = \csc x$, $\mu(x) = x^{-(2np+p+1)}$, $\nu(y) = y^{-(2qn+q+1)}$, $f(x)$, $g(x) \geqslant 0$, $f(x) \in L_\mu^p(0,\infty)$, $g(x) \in L_\nu^q(0,\infty)$, then*

$$\int_0^\infty \int_0^\infty \sinh(\beta_2 xy)\operatorname{sech}(\beta_1 xy)f(x)g(y)dxdy$$

$$< \left(\frac{\pi}{2\beta_1}\right)^{2n+2} \psi^{(2n+1)}\left(\frac{\beta_1 + \beta_2}{2\beta_1}\pi\right) \|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.17}$$

Similarly, giving β_1, β_2 and n different values in (4.17), we can also obtain some special inequalities.

Setting $\eta = 1$, $\delta = 1$, $\beta_2 = \beta_3 = -\beta_1$, $\beta = 2n+1(n \in N \cup \{0\})$ in Theorem 3.1, we obtain

COROLLARY 4.6. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\beta_2 > 0$, $\beta_2 < \beta_1$, $n \in N \cup \{0\}$, $\psi(x) = \csc x$, $\mu(x) = x^{-(2np+p+1)}$, $\nu(y) = y^{-(2qn+q+1)}$, $f(x)$, $g(x) \geqslant 0$, $f(x) \in L_\mu^p(0,\infty)$, $g(x) \in L_\nu^q(0,\infty)$, then*

$$\int_0^\infty \int_0^\infty (1 - \tanh(\beta_1 xy))f(x)g(y)dxdy$$

$$< \left(\frac{\pi}{2\beta_1}\right)^{2n+2} \psi^{(2n+1)}\left(\frac{\beta_1 + \beta_2}{2\beta_1}\pi\right) \|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.17}$$

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