Journal of Mathematical Inequalities
Volume 13, Number 4 (2019), 1245–1247 doi:10.7153/jmi-2019-13-88

A NOTE ON A CONJECTURE ABOUT A BETTER APPROXIMATION

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(*Communicated by T. Burić*)

Abstract. In this note we give some examples which invalidate the Conjecture from [1].

The classical Bernstein operator is defined by

$$B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \; x \in [0,1], \; f \in C[0,1],$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \; k = 0, \ldots, n.$$

It is known that the Bernstein operator reproduces the affine functions.

Let $\mu \in \mathbb{R}, \; \mu > 0$. In [1] the authors studied the operator given by

$$\mathcal{G}_n(f)(x) = e^{\mu x} \sum_{k=0}^{n} p_{n,k}(a_n(x)) e^{-\mu k/n} f\left(\frac{k}{n}\right), \; x \in [0,1], \; f \in C[0,1],$$

where

$$a_n(x) = \frac{e^{x/n} - 1}{e^{1/n} - 1}.$$

The operator $\mathcal{G}_n$ preserves the exponential functions exp_μ and exp_μ^2, where $exp_\mu : [0,1] \to \mathbb{R}, \; exp_\mu(x) = e^{\mu x}$. It is a special case of an operator introduced by Morigi and Neamtu in [2].

A function $f : [0,1] \to \mathbb{R}$ is said to be

1. convex with respect to the $\{exp_\mu\}$ if

$$\begin{vmatrix} e^{\mu x_0} & e^{\mu x_1} \\ f(x_0) & f(x_1) \end{vmatrix} \geqslant 0, \; 0 < x_0 < x_1 < 1, \tag{1}$$

Mathematics subject classification (2010): 41A36, 41A25, 41A40.

Keywords and phrases: conjecture; better approximation; exponential functions; generalized convex functions.

2. convex with respect to the $\{exp_\mu, exp_\mu^2\}$ if

$$\begin{vmatrix} e^{\mu x_0} & e^{\mu x_1} & e^{\mu x_2} \\ e^{2\mu x_0} & e^{2\mu x_1} & e^{2\mu x_2} \\ f(x_0) & f(x_1) & f(x_2) \end{vmatrix} \geqslant 0,\ 0 < x_0 < x_1 < x_2 < 1. \tag{2}$$

If $f \in C[0,1]$, then the inequalities (1) and (2) hold by continuity for $0 \leqslant x_0 < x_1 \leqslant 1$ and respectively for $0 \leqslant x_0 < x_1 < x_2 \leqslant 1$.

The spaces of convex functions with respect to the $\{exp_\mu\}$ and $\{exp_\mu, exp_\mu^2\}$ are denoted by $\mathscr{C}(exp_\mu)$ and $\mathscr{C}(exp_\mu, exp_\mu^2)$ respectively.

From [1, Proposition 1 and Proposition 2] we have

1. if $f \in C^1[0,1]$, then

$$f \in \mathscr{C}(exp_\mu) \Leftrightarrow f/\exp_\mu \text{ is increasing,} \tag{3}$$

2. if $f \in C^2[0,1]$, then

$$f \in \mathscr{C}(exp_\mu, exp_\mu^2) \Leftrightarrow f'' - 3\mu f' + 2\mu^2 f \geqslant 0. \tag{4}$$

In [1] the following conjecture about a better approximation of the operator $\mathscr{G}_n$ than the classical Bernstein operator B_n for some generalized convex functions was given:

Conjecture. If $f \in C[0,1]$ is such that $f \in \mathscr{C}(exp_\mu)$ and $f \in \mathscr{C}(exp_\mu, exp_\mu^2)$, then for all $n \in \mathbb{N}$ and for all $x \in [0,1]$ one has that

$$f(t) \leqslant \mathscr{G}_n(f)(t) \leqslant B_n(f)(t).$$

The first inequality holds by using [3, Theorem 1].

We give examples which show that the second inequality is not true for some $n \in \mathbb{N}$, $x \in [0,1]$ and $\mu > 0$.

Let the function $f : [0,1] \to \mathbb{R},\ f(x) = e^x(x^2 + c)$ with $c \in \mathbb{R}$. We observe that the function f satisfy both conditions (3) and (4) for $\mu = 1$. Thus $f \in C(exp_1)$ and $f \in C(exp_1, exp_1^2)$.

For $c = -7$ the function f is concave over the all interval $[0,1]$. It follows that, for all $n \in \mathbb{N}$,

$$B_n(f)(t) \leqslant f(t),\ t \in [0,1].$$

For $c = -2$ the function f is convex over the all interval $[0,1]$ and we have

$$f(t) \leqslant B_2(f)(t) \leqslant \mathscr{G}_2(f)(t),\ t \in [0, x^*],$$

where $x^* = 0.128521...$

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(Received February 6, 2019)

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