Journal of Mathematical Inequalities
Volume 14, Number 1 (2020), 1–21
doi:10.7153/jmi-2020-14-01

INEQUALITIES FOR GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS WITH ONE PARAMETER

MIAO-KUN WANG, MIAO-YING HONG, YANG-FAN XU, ZHONG-HUA SHEN AND YU-MING CHU*

(*Communicated by J. Pečarić*)

Abstract. In the article, we establish several new inequalities for the generalized trigonometric and hyperbolic functions with one parameter, generalize the well known Mitrinović-Adamović, Lazarević, Huygens-type, Wilker-type and Cusa-Huygens-type inequalities to the cases of the generalized trigonometric and hyperbolic functions with one parameter.

1. Introduction

It is well known that the trigonometric and hyperbolic functions as well as their inverse functions are very common elementary functions in mathematics, they are widely used in all branches of mathematics [2, 3, 14, 17, 19, 20, 25, 30, 32, 35, 38, 41, 46, 50, 52, 61, 70, 74, 77, 81, 83, 85, 86, 87, 88], they are closely related to the bivariate means [15, 26, 54, 55, 56, 68, 73, 75, 78] and special functions [1, 7, 12, 13, 21, 22, 27, 28, 36, 51, 57, 58, 59, 60, 62, 65, 69, 72, 82]. Recently, many remarkable results [4, 5, 6, 16, 18, 23, 29, 33, 34, 39, 40, 42, 43, 45, 47, 53, 63, 64, 66, 67, 71, 76, 80, 89] in mathematics, physics, mechanics, game theory, control and optimization theory and so on have been established via these functions.

From the basic knowledge of calculus we clearly see that

$$\arcsin(x)=\int_0^x\frac{1}{(1-t^2)^{1/2}}dt,\quad 0\leqslant x\leqslant 1$$

and

$$\frac{\pi}{2}=\arcsin(1)=\int_0^1\frac{1}{(1-t^2)^{1/2}}dt.$$

Since the function $\arcsin(x)$ is differentiable on $(0,1)$, and strictly increasing from $[0,1]$ onto $[0,\pi/2]$, it has a differentiable inverse function $x\mapsto\sin(x)$ on $[0,\pi/2]$. By defining $\sin(x)=\sin(\pi-x)$ for $x\in[\pi/2,\pi]$ and extending $\sin(x)$ to $[-\pi,\pi]$ by

Mathematics subject classification (2010): 33B10, 26D05.
Keywords and phrases: Generalized trigonometric function, generalized hyperbolic function, Gamma function, Mitrinović-Adamović inequality, Lazarević inequality, Huygens-type inequality, Wilker-type inequality, Cusa-Huygens-type inequality.

* Corresponding author.

Paper JMI-14-01

oddness, and to $(-\infty,+\infty)$ by 2π-periodicity, we obtain the trigonometric sine function defined on the whole $\mathbb{R}$.

In 2012, Edmunds, Gurka and Lang [24] considered the Dirichlet problem for (p,q)-Laplacian and introduced generalized trigonometric sine function with two parameters as an eigenfunction. Indeed, for $p,q>1$, the authors defined

$$\operatorname{arcsin}_{p,q}(x)=\int_0^x (1-t^q)^{-1/p}dt, \quad x\in[0,1],$$

and the generalized circumference ratio

$$\pi_{p,q}=2\int_0^x (1-t^q)^{-1/p}dt.$$

Clearly $x\mapsto \operatorname{arcsin}_{p,q}(x)$ is strictly increasing from $[0,1]$ to $[0,\pi_{p,q}/2]$. Consequently, it also has the inverse function denoted by $\sin_{p,q}$. By the standard extension procedures as the classical sine function one has the generalized trigonometric sine function with two parameters on $(-\infty,+\infty)$.

In this paper, we focus on the generalized trigonometric function $\sin_{p,q}$ with $1/p+1/q=1$ (q is the conjugate of p), which was first studied by Lindqvist and Peetre in [48, 49]. Actually, for $p>1$, set

$$\operatorname{arcsin}_p^*(x)=\operatorname{arcsin}_{\frac{p}{p-1},p}(x)=\int_0^x (1-t^p)^{-\frac{p-1}{p}}dt, \quad x\in[0,1] \tag{1.1}$$

and

$$\frac{\pi_p^*}{2}=\operatorname{arcsin}_p^*(1)=\int_0^1 (1-t^p)^{-\frac{p-1}{p}}dt=\frac{1}{p}B\left(\frac{1}{p},\frac{1}{p}\right), \tag{1.2}$$

where $B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}dt=\Gamma(u)\Gamma(v)/\Gamma(u+v)$ $(u,v>0)$ is the classical Beta function [31, 37], and $\Gamma(u)=\int_0^{+\infty}t^{u-1}e^{-t}dt$ $(u>0)$ is the Gamma function [79, 84]. Then we also obtain the increasing function $\sin_p^*$ defined on the interval $[0,\pi_p^*/2]$ as the inverse function of $\operatorname{arcsin}_p^*$, and then extend it to $\mathbb{R}$ analogously. Since $\sin_p^*$ has one free parameter p, we called $\sin_p^*$ the generalized trigonometric sine function with one parameter,

Define $\cos_p^*:[0,\pi_p^*/2]\to[0,1]$ by

$$\cos_p^*(x)=\left[1-(\sin_p^* x)^p\right]^{1/p}. \tag{1.3}$$

It is clear to see that $\cos_p^*(0)=1$ and $\cos_p^*(\pi_p^*/2)=0$. We also can extend $\cos_p^*$ to $(-\infty,+\infty)$ by evenness about 0, oddness about $\pi_p^*/2$ and $2\pi_p^*$-periodicity. It follows the definitions of $\sin_p^*$ and $\cos_p^*$ that

$$|\cos_p^*(x)|^p+|\sin_p^*(x)|^p=1, \quad x\in\mathbb{R}.$$

Obviously, when $p=2$, the functions $\sin_p^*$ and $\cos_p^*$ agree with the classical trigonometric sine and cosine function $\sin x$ and $\cos x$, respectively.

In the recent past, $\sin_p^*$ and $\cos_p^*$ have been investigated by several authors from different points of view [9, 24, 48, 49]. For example, $\sin_p^*$ and $\cos_p^*$ are applied in the parametrization of so-called p-circle

$$|x|^p + |y|^p = R^p$$

by Lindqvist and Peetre in [49], in which the authors proved that the area closed by the p-circle is $\pi_p^* R^2$, and the q-length (l_q metric) of p-circle is $2\pi_p^* R$. In 2012, making use of elliptic functions, Edmunds, Gurka and Lang [24] established the duplication formula for $\sin_4^*(x)$:

$$\sin_4^*(2x) = \frac{2\sin_4^*(x)\cos_4^*(x)}{\left[1+4\sin_4^{*4}(x)\cos_4^{*4}(x)\right]^{1/2}},$$

which is similar to $\sin(2x) = 2\sin x\cos x$, furthermore, they showed that for all $p \in (1,+\infty)$, the functions $\sin_p^*(\pi_p^* nx)$ form a basis in Lebesgue space $L^r(0,1)$ for all $r \in (1,+\infty)$. In 2017, the basis properties of the generalized trigonometric functions with one parameter in $L^r(0,1)^n$ for any $r \in (1,+\infty)$ and $n \leqslant 3$ were investigated by Bakşi et al. in [9].

It is worthy mentioning that, another special case of $\sin_{p,q}$, $\sin_{p,p} = \sin_p$ $(p>1)$ and some other related functions have attracted the attention of many researchers. In particular, a lot of well known properties and inequalities for the classical trigonometric functions had been generalized to these cases [10, 11, 44].

The aim of this paper is to generalize some well known inequalities satisfied by classical trigonometric and hyperbolic functions, such as Mitrinović-Adamović inequality, Lazarević inequality, Huygens-type inequality, Wilker-type inequality, Cusa-Huygens-type inequality, to the cases of the generalized trigonometric and hyperbolic functions with one parameter (See Section 2).

2. Preliminaries and Basic Definitions

Throughout this section, we assume that $1 < p < +\infty$. At first, we introduce some definitions and formulas for the generalized trigonometric and hyperbolic functions with one parameter. By (1.1) and (1.3), we have, on $(0, \pi_p^*/2)$,

$$\frac{d\cos_p^*(x)}{dx} = -[\sin_p^*(x)]^{p-1}, \quad \frac{d\sin_p^*(x)}{dx} = [\cos_p^*(x)]^{p-1}.$$

Replacing x by $\arccos_p^*(x)$ in (1.3), one has

$$\arccos_p^*(x) = \int_x^1 \frac{dt}{(1-t^p)^{\frac{p-1}{p}}}, \quad x \in [0,1],$$

so that

$$\arccos_p^*(x) + \arcsin_p^*(x) = \frac{\pi_p^*}{2}.$$

The generalized tangent function with one parameter is defined as in the classical case:

$$\tan_p^*(x)=\frac{\sin_p^*(x)}{\cos_p^*(x)},\quad x\in\mathbb{R}\setminus\left\{k\pi_p^*+\frac{\pi_p^*}{2},k\in\mathbb{Z}\right\}.$$

It follows that

$$\frac{d\tan_p^*(x)}{dx}=\frac{1}{[\cos_p^*(x)]^2},\quad x\in\left(-\frac{\pi_p^*}{2},\frac{\pi_p^*}{2}\right).$$

Thus $x\mapsto\tan_p^*(x)$ is strictly increasing from $(-\pi_p^*/2,\pi_p^*/2)$ onto $(-\infty,+\infty)$.

Analogously, we also define the generalized hyperbolic sine, cosine and tangent functions as follows (see [11]):

$$\operatorname{arcsinh}_p^*(x)=\int_0^x(1+t^p)^{-\frac{p-1}{p}}\,dt,\quad x\in[0,+\infty),$$

$$m_p^*=\int_0^{+\infty}(1+t^p)^{-\frac{p-1}{p}}\,dt$$

and

$$\sinh_p^*(x):(0,m_p^*)\to(0,+\infty).$$

Noting that $m_p^*=+\infty$ for $1<p\leqslant 2$, and for $p>2$,

$$m_p^*=\int_0^{+\infty}(1+t^p)^{-\frac{p-1}{p}}\,dt=\frac{1}{p}\int_0^1 x^{-2/p}(1-x)^{1/p-1}dx=\frac{1}{p}B\left(1-\frac{2}{p},\frac{1}{p}\right)$$

by substituting $1+t^p=1/x$, then using the formula $\Gamma(2z)=2^{2z-1}\Gamma(z)\Gamma(z+1/2)/\sqrt{\pi}$ gives rise to

$$m_p^*=\frac{1}{p}B\left(1-\frac{2}{p},\frac{1}{p}\right)=\frac{1}{4^{1/p}p}B\left(\frac{1}{p},\frac{1}{2}-\frac{1}{p}\right).$$

For $x\in(-\infty,0)$, let $\arcsin_p^*(x)=-\arcsin_p^*(-x)$, then $\sinh(x)=-\sinh(-x)$ for $x\in(-m_p^*,0)$. Similarly, define

$$\cosh_p^*(x)=[1+|\sinh_p^*(x)|^p]^{1/p},\quad x\in(-m_p^*,m_p^*),$$

$$\tanh_p^*(x)=\frac{\sinh_p^*(x)}{\cosh_p^*(x)},\quad x\in(-m_p^*,m_p^*).$$

Then it follows that

$$[\cosh_p^*(x)]^p-|\sinh_p^*(x)|^p=1,\quad x\in(-m_p^*,m_p^*),$$

and for $x\in(0,m_p^*)$, one has

$$\frac{d\sinh_p^*(x)}{dx}=[\cosh_p^*(x)]^{p-1},\quad \frac{d\cosh_p^*(x)}{dx}=[\sinh_p^*(x)]^{p-1},$$

$$\frac{d\tanh_p^*(x)}{dx}=\frac{1}{[\cosh_p^*(x)]^2}.$$

Obviously, the functions $\sinh_p^*$, $\cosh_p^*$ and $\tanh_p^*$ are strictly increasing on $(0,m_p^*)$, and $\sinh_p^*(0)=\tanh_p^*(0)=0$, $\cosh_p^*(0)=\tanh_p^*(m_p^*)=1$, $\sinh_p^*(m_p^*)=\cosh_p^*(m_p^*)=+\infty$.

LEMMA 2.1. (See [8, Theorem 1.25, l'Hôptial Monotone Rule]) *Let* $-\infty < a < b < +\infty$ *and* $f,g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b)*. Assume that* $g'(x) \neq 0$ *for each* $x \in (a,b)$*. If* f'/g' *is (strictly) increasing (decreasing) on* (a,b)*, then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \textit{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

LEMMA 2.2. *Let* $p \in (1,+\infty)$*. Then the following statements hold:*

(1) *The function* $f(x) = x/(\sin_p^*(x)[\cos_p^*(x)]^{1-p})$ *is strictly decreasing from* $(0,\pi_p^*/2)$ *onto* $(0,1)$*;*

(2) *The function* $g(x) = \sinh_p^*(x)[\cosh_p^*(x)]^{1-p}/x$ *is strictly decreasing on* $(0,m_p^*)$*, and the range of* g *is* $(2-p,1)$ *if* $p \in (1,2]$*, while the range is* $(0,1)$ *if* $p \in (2,+\infty)$*;*

(3) *The inequality*

$$\frac{\sinh_p^*(x)}{x} - 1 > (p-1)\left[\frac{1-\tanh_p^*(x)/x}{2}\right] \tag{2.1}$$

holds for all $x \in (0,m_p^*)$*,*

Proof. For part (1), let $f_1(x) = x$, $f_2(x) = \sin_p^*(x)[\cos_p^*(x)]^{1-p}$. Then $f_1(0) = f_2(0) = 0$, and simple computations one has

$$\frac{f_1'(x)}{f_2'(x)} = \frac{1}{1-[\sin_p^*(x)]^p(1-p)[\cos_p^*(x)]^{-p}} = \frac{1}{1+(p-1)[\tan_p^*(x)]^p},$$

which is strictly decreasing on $(0,\pi_p^*/2)$ due to $\tan_p^*(x)$ is strictly increasing from $(0,\pi_p^*/2)$ onto $(0,+\infty)$. By application of Lemma 2.1, $f(x)$ is also strictly decreasing on $(0,\pi_p^*/2)$. Moreover,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{f_1'(x)}{f_2'(x)} = 1, \qquad \lim_{x\to \frac{\pi_p^*}{2}^-} f(x) = 0.$$

For part (2), let $g_1(x) = \sinh_p^*(x)[\cosh_p^*(x)]^{1-p}$, $g_2(x) = x$. Then $g_1(0) = g_2(0) = 0$ and

$$\frac{g_1'(x)}{g_2'(x)} = 1+[\sinh_p^*(x)]^p(1-p)[\cosh_p^*(x)]^{-p} = 1+(1-p)[\tanh_p^*(x)]^p,$$

which is strictly decreasing on $(0,m_p^*)$ since $\tanh_p^*(x)$ is strictly increasing from $(0,m_p^*)$ onto $(0,1)$. By Lemma 2.1, $g(x)$ is also strictly decreasing on $(0,m_p^*)$. Moreover, by l'Hôptial's Rule we get $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} g_1'(x)/g_2'(x) = 1$, and

$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty}\frac{g_1'(x)}{g_2'(x)} = 1+(1-p) = 2-p \ \text{ if } p \in (1,2],$$

while if $p \in (2, +\infty)$, then

$$\lim_{x \to m_p^*} g(x) = \lim_{x \to m_p^*} \frac{\sinh_p^*(x)}{\cosh_p^*(x)} \cdot \frac{1}{x[\cosh_p^*(x)]^{p-2}} = 0.$$

For part (3), define h on $(0, m_p^*)$ by

$$h(x) = 2\sinh_p^*(x) + (p-1)\tanh_p^*(x) - (1+p)x.$$

Then taking derivative twice in succession yields

$$h'(x) = 2[\cosh_p^*(x)]^{p-1} + (p-1)[\cosh_p^*(x)]^{-2} - (1+p)$$

and

$$\begin{aligned} h''(x) &= 2(p-1)[\cosh_p^*(x)]^{p-2}[\sinh_p^*(x)]^{p-1} + (p-1)(-2)[\cosh_p^*(x)]^{-3}[\sinh_p^*(x)]^{p-1} \\ &= 2(p-1)[\sinh_p^*(x)]^{p-1}[\cosh_p^*(x)]^{-3}([\cosh_p^*(x)]^{p+1} - 1). \end{aligned}$$

It is clear to see that $h''(x) > 0$ for all $x \in (0, m_p^*)$, so that $h'(x)$ is strictly increasing on $(0, m_p^*)$. Consequently, $h'(x) > h'(0) = 0$ for all $x \in (0, m_p^*)$, and therefore $h(x)$ is strictly increasing $(0, m_p^*)$ and $h(x) > h(0) = 0$ for all $x \in (0, m_p^*)$. This leads to inequality (2.1) immediately. $\square$

LEMMA 2.3. *Let*

$$F(x) = \log\left(\frac{2x^2}{x+1}\right) + 2\log\Gamma(x) - \log\Gamma(2x), \quad x \in (0,1).$$

Then there exists unique $x_0 = 0.435\cdots \in (0,1)$ *such that* $F(x) < 0$ *for* $x \in (0, x_0)$, $F(x) > 0$ *for* $x \in (x_0, 1)$ *and* $F(x_0) = 0$.

Proof. Employing digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$ and its duplication formula $2\psi(2x) = \psi(x) + \psi(x+1/2) + \log 4$, we obtain

$$\begin{aligned} F'(x) &= \frac{x+2}{x(x+1)} + 2\psi(x) - 2\psi(2x) = \frac{2}{x} - \frac{1}{x+1} + \psi(x) - \psi\left(x + \frac{1}{2}\right) - \log 4 \\ &= \frac{2}{x} - \frac{1}{x+1} + \psi(x+1) - \frac{1}{x} - \psi\left(x + \frac{1}{2}\right) - \log 4 \\ &= \frac{1}{x} - \frac{1}{x+1} + \psi(x+1) - \psi\left(x + \frac{1}{2}\right) - \log 4. \end{aligned}$$

Continuing to differentiate F' yields

$$F''(x) = \left[-\frac{1}{x^2} + \frac{1}{(x+1)^2}\right] + \left[\psi'(x+1) - \psi'\left(x + \frac{1}{2}\right)\right] < 0$$

for all $x \in (0,1)$ since $\psi'(x) = \sum_{k=0}^{\infty}[1/(x+k)^2]$. Hence $F'(x)$ is strictly decreasing on $(0,1)$. Noting that

$$F'(0^+) = +\infty, \quad F'(1^-) = -\frac{1}{2},$$

we conclude that there exists unique $\xi \in (0,1)$ such that $F'(x) > 0$ for $x \in (0,\xi)$ and $F'(x) < 0$ for $x \in (\xi,1)$. This together with the limiting values $F(0^+) = 0$ and

$$\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} [\log 2 - \log(x+1) + 2\log\Gamma(x+1) - \log\Gamma(2x+1) + \log(2x)] = -\infty,$$

leads to the conclusion that there exists $x_0 \in (0,1)$ such that $F(x) < 0$ for $x \in (0,x_0)$ and $F(x) > 0$ for $x \in (x_0,1)$. Furthermore, numerical computations show that

$$F(0.4352) = -0.0000264\cdots, \quad F(0.4353) = 0.000612\cdots,$$

Thus $x_0 = 0.435\cdots \in (0.4352, 0.4353)$. This completes the proof of Lemma 2.3. □

COROLLARY 2.4. *Let $p \in (1,+\infty)$. Then there exists unique $p_0 = 2.297\cdots \in (1,+\infty)$ such that $\pi^*_{p_0} = p_0 + 1$, $\pi^*_p > p+1$ for $p \in (1,p_0)$ and $\pi^*_p < p+1$ for $p \in (p_0,+\infty)$.*

Proof. Let $x = 1/p \in (0,1)$. Then

$$\log\left(\frac{\pi^*_p}{p+1}\right) = \log\left[\frac{(2/p)B(1/p,1/p)}{p+1}\right] = \log\left[\frac{2xB(x,x)}{1/x+1}\right]$$
$$= \log\left(\frac{2x^2}{x+1}\right) + 2\log\Gamma(x) - \log\Gamma(2x).$$

By Lemma 2.3, Corollary 2.4 follows. □

LEMMA 2.5. *Let*

$$G(x) = \log\Gamma(x+1) + \log\Gamma\left(\frac{1}{2} - x\right) + \log(1-x)$$
$$- x\log 4 - \frac{1}{2}\log\pi - \log(1+x), x \in \left(0, \frac{1}{2}\right).$$

Then there exists unique $x_1 = 0.37027\cdots \in (0,1)$ such that $G(x) < 0$ for $x \in (0,x_1)$, $G(x) > 0$ for $x \in (x_1, 1/2)$ and $G(x_1) = 0$.

Proof. Simple computations yield

$$\lim_{r\to 0^+} G(x) = 0, \quad \lim_{x\to \frac{1}{2}^-} G(x) = +\infty, \tag{2.2}$$

$$G'(x) = \psi(x+1) - \psi\left(\frac{1}{2} - x\right) - \frac{1}{1-x} - \log 4 - \frac{1}{1+x}$$

$$= \psi(x+1) - \psi\left(\frac{3}{2} - x\right) + \frac{2}{1-2x} - \frac{1}{1-x} - \log 4 - \frac{1}{1+x},$$

$$\lim_{r\to 0^+} G'(x) = \psi(1) - \psi\left(\frac{1}{2}\right) - \log 4 - 2 = -2, \quad \lim_{x\to \frac{1}{2}^-} G'(x) = +\infty \tag{2.3}$$

and

$$G''(x) = \psi'(x+1) + \psi'\left(\frac{1}{2} - x\right) - \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(x+1+k)^2} + \sum_{k=0}^{\infty} \frac{1}{(1/2-x+k)^2} - \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2}$$

$$> \sum_{k=1}^{\infty} \frac{1}{(x+1+k)^2} + \sum_{k=1}^{\infty} \frac{1}{(1/2-x+k)^2} > 0. \tag{2.4}$$

It follows from (2.2)-(2.4) and similar argument in Lemma 2.4 that there exists $x_1 \in (0,1)$ such that $G(x) < 0$ for $x \in (0, x_1)$, $G(x) > 0$ for $x \in (x_1, 1/2)$ and $G(x_1) = 0$. Furthermore, numerical computations show that

$$g(0.37027) = -0.0000158\cdots, \quad g(0.37028) = 0.0000270\cdots.$$

Thus $x_0^* \in (0.37027, 0.37028)$. This completes the proof of Lemma 2.5. □

COROLLARY 2.6. *Let $p \in (2, +\infty)$. Then there exists unique $p_1 = 2.7007\cdots \in (2, +\infty)$ such that $m_{p_1}^* = (p_1+1)/(p_1-1)$, $m_p^* > (p+1)/(p-1)$ for $p \in (2, p_1)$ and $m_p^* < (p+1)/(p-1)$ for $p \in (p_1, +\infty)$.*

Proof. Let $x = 1/p \in (0, 1/2)$. Then

$$\log\left[\frac{m_p^*(p-1)}{p+1}\right] = \log\left[\frac{B(1/p, 1/2-1/p)(p-1)}{4^{-1/p}\cdot p(p+1)}\right]$$

$$= \log\left[\frac{\Gamma(1/p)\Gamma(1/2-1/p)(p-1)}{\sqrt{\pi}4^{-1/p}\cdot p(p+1)}\right]$$

$$= \log\Gamma(x) + \log\Gamma\left(\frac{1}{2} - x\right) + \log\left(\frac{1}{x} - 1\right) - x\log 4 - \frac{1}{2}\log\pi - \log\frac{1}{x} - \log\left(\frac{1}{x} + 1\right)$$

$$= \log\Gamma(x+1) + \log\Gamma\left(\frac{1}{2} - x\right) + \log(1-x) - x\log 4 - \frac{1}{2}\log\pi - \log(1+x).$$

By Lemma 2.5, Corollary 2.6 follows. □

LEMMA 2.7. *Let $p \in (1, +\infty)$ and*

$$J(t) = \frac{2(p-1)^2}{p+1} t^{p/2} - \frac{(p-1)(p-3)}{p+1} - (p-1)t, \quad t \in (0,1).$$

Then the following statements holds

(1) *If* $p \in (1,2]$, *then* $J(t) > 0$ *for* $t \in (0,1)$, $J(t) < 0$ *for* $t \in (1,+\infty)$ *and* $J(1) = 0$*;*

(2) *If* $p \in (2, 1+\sqrt{2})$, *then* $J(t) > 0$ *for all* $t \in (0,1)$, *and there exists* $t_0 \in (1,+\infty)$ *such that* $J(t) < 0$ *for* $t \in (1,t_0)$ *and* $J(t) > 0$ *for* $t \in (t_0,+\infty)$*;*

(3) *If* $p = 1+\sqrt{2}$, *then* $J(t) > 0$ *for all* $t \in (0,+\infty)$*;*

(4) *If* $p \in (1+\sqrt{2}, 3)$, *then* $J(t) > 0$ *for* $t \in (1,+\infty)$, *and there exists* $t_0^* \in (0,1)$ *such that* $J(t) > 0$ *for* $t \in (0,t_0^*)$ *and* $J(t) < 0$ *for* $t \in (t_0^*,1)$*;*

(5) *If* $p \in [3,+\infty)$, *then* $J(t) < 0$ *for* $t \in (0,1)$, $J(t) > 0$ *for* $t \in (1,+\infty)$ *and* $J(1) = 0$.

Proof. Clearly when $p = 2$, $J(t) = -t/3 + 1/3$, the assertion of $J(t)$ for $p = 2$ in part (1) holds true. In the remaining proof, we assume that $p \neq 2$. Note that

$$J(0) = -\frac{(p-1)(p-3)}{p+1}, \quad J(1) = 0.$$

We clearly see that $J(0) > 0$ for $p \in (1,3)$, $J(0) < 0$ for $p \in (3,+\infty)$ and $J(0) = 0$ for $p = 3$.

Differentiating J gives

$$J'(t) = \frac{p(p-1)^2}{p+1} t^{p/2-1} - (p-1) = (p-1)t^{p/2-1} J_1(t),$$

where $J_1(t) = p(p-1)/(p+1) - t^{1-p/2}$ is monotone on $(0,+\infty)$ and

$$J_1(1) = \frac{p(p-1) - p - 1}{p+1} = \frac{p^2 - 2p - 1}{p+1}.$$

We divide the proof into five cases.

Case 1 $p \in (1,2)$. Then $J_1(0^+) = p(p-1)/(p+1) > 0$ and $J_1(1^-) < 0$, so that there exists $\lambda \in (0,1)$ such that $J_1'(t) > 0$ for $t \in (0,\lambda)$ and $J_1'(t) < 0$ for $t \in (\lambda,+\infty)$. Hence $J(t)$ is strictly increasing on $(0,\lambda)$ and strictly decreasing on $(\lambda,+\infty)$. It follows from $J(0) > 0$ and $J(1) = 0$ that $J(t) > 0$ for $t \in (0,1)$ and $J(t) < 0$ for $t \in (1,+\infty)$.

Case 2 $p \in (2, 1+\sqrt{2})$. Then $J_1(0^+) = -\infty$, $J_1(1^-) < 0$ and $J_1(+\infty) = p(p-1)/(p+1) > 0$, so that there exists $\delta \in (1,+\infty)$ such that $J_1'(t) < 0$ for $t \in (0,\delta)$ and $J_1'(t) > 0$ for $t \in (\delta,+\infty)$. Hence $J(t)$ is strictly decreasing on $(0,\delta)$ and strictly increasing on $(\delta,+\infty)$. Since $J(0) > 0$ and $J(1) = 0$, we conclude that $J(t) > 0$ for $t \in (0,1)$, and there exists $t_0 \in (\delta,+\infty)$ such that $J(t) < 0$ for $t \in (1,t_0)$ and $J(t) > 0$ for $t \in (t_0,+\infty)$.

Case 3 $p = 1+\sqrt{2}$. Then $J_1(0^+) = -\infty$, $J_1(1^-) = 0$ and $J_1(+\infty) = p(p-1)/(p+1) > 0$, so that $J_1'(t) < 0$ for $t \in (0,1)$ and $J_1'(t) > 0$ for $t \in (1,+\infty)$. Hence $J(t)$ is strictly decreasing on $(0,1)$ and strictly increasing on $(1,+\infty)$. This together with $J(0) > 0$ and $J(1) = 0$ leads to the conclusion that $J(t) \geqslant 0$ for all $t \in (0,+\infty)$ and $J(t) = 0$ if and only if $t = 1$.

Case 4 $p \in (1+\sqrt{2}, 3)$. Then $J_1(0^+) = -\infty$, $J_1(1^-) > 0$ and $J_1(+\infty) = p(p-1)/(p+1) > 0$, so that there exists $\lambda^* \in (0,1)$ such that $J_1'(t) < 0$ for $t \in (0, \lambda^*)$ and $J_1'(t) > 0$ for $t \in (\lambda^*, +\infty)$. Hence $J(t)$ is strictly decreasing on $(0, \lambda^*)$ and strictly increasing on $(\lambda^*, +\infty)$. Therefore, part (4) directly follows from the piecewise monotonicity of $J(t)$ on $(0, +\infty)$ together with $J(0) > 0$ and $J(1) = 0$.

Case 5 $p \in [3, +\infty)$. Then making use of the similar argument in Case 4, we conclude that there exists $\delta^* \in (0,1)$ such that $J(t)$ is strictly decreasing on $(0, \delta^*)$ and strictly increasing on $(\delta^*, +\infty)$. Noting that $J(0) \leqslant 0$ in this case, the assertion in part (5) takes place. □

REMARK 2.8. It is apparent from Lemma 2.2(1) and (2) that Lemma 2.1 is a key tool to prove some monotonicity theorems involving generalized trigonometric and hyperbolic functions. In fact, by application of Lemma 2.1, we can easily show that

(1) The function $x \mapsto \sin_p^*(x)/x$ is strictly decreasing from $(0, \pi_p^*/2)$ onto $(2/\pi_p^*, 1)$;

(2) The function $x \mapsto \tan_p^*(x)/x$ is strictly increasing from $(0, \pi_p^*/2)$ onto $(1, +\infty)$;

(3) The function $x \mapsto \sinh_p^*(x)/x$ is strictly increasing from $(0, m_p^*)$ onto $(1, +\infty)$;

(4) The function $x \mapsto \tanh_p^*(x)/x$ is strictly decreasing on $(0, m_p^*)$, and if $p \in (1, 2]$, then the range is $(0, 1)$, while if $p \in (2, +\infty)$, then the range is $(1/m_p^*, 1)$.

3. Main Results

THEOREM 3.1. (Generalized Mitrinović-Adamović inequality) *Let $p \in (1, +\infty)$. Then the function*

$$f(x) = \frac{\log\left[\sin_p^*(x)/x\right]}{\log[\cos_p^*(x)]}$$

is strictly increasing from $(0, \pi_p^*/2)$ onto $(0, (p-1)/(p+1))$. In particular, the inequality

$$[\cos_p^*(x)]^\alpha < \frac{\sin_p^*(x)}{x} < [\cos_p^*(x)]^\beta \tag{3.1}$$

holds for all $p \in (1, +\infty)$ and $x \in (0, \pi_p^*/2)$ with the best constants $\alpha = (p-1)/(p+1)$ and $\beta = 0$.

Proof. Let $f_1(x) = \log[\sin_p^*(x)/x]$, $f_2(x) = \log[\cos_p^*(x)]$. Then $f_1(x) = f_2(x) = 0$, and

$$f_1'(x) = \frac{x[\cos_p^*(x)]^{p-1} - \sin_p^*(x)}{x \sin_p^*(x)}, \quad f_2'(x) = -\frac{[\sin_p^*(x)]^{p-1}}{\cos_p^*(x)},$$

and thereby

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\sin_p^*(x)\cos_p^*(x) - x[\cos_p^*(x)]^p}{x[\sin_p^*(x)]^p}.$$

Write $f_{11}(x) = \sin_p^*(x)\cos_p^*(x) - x[\cos_p^*(x)]^p$, $f_{22}(x) = x[\sin_p^*(x)]^p$, then $f_{11}(0) = f_{22}(0) = 0$, and

$$f_{11}'(x) = [\cos_p^*(x)]^p - [\sin_p^*(x)]^p - [\cos_p^*(x)]^p + xp[\cos_p^*(x)]^{p-1}[\sin_p^*(x)]^{p-1}$$

$$= px[\sin_p^*(x)\cos_p^*(x)]^{p-1} - [\sin_p^*(x)]^p,$$

$$f_{22}'(x) = px[\sin_p^*(x)\cos_p^*(x)]^{p-1} + [\sin_p^*(x)]^p$$

and

$$\frac{f_{11}'(x)}{f_{22}'(x)} = \frac{px[\sin_p^*(x)\cos_p^*(x)]^{p-1} - [\sin_p^*(x)]^p}{px[\sin_p^*(x)\cos_p^*(x)]^{p-1} + [\sin_p^*(x)]^p}$$
$$= 1 - \frac{2[\sin_p^*(x)]^p}{px[\sin_p^*(x)\cos_p^*(x)]^{p-1} + [\sin_p^*(x)]^p} = 1 - \frac{2}{1 + \dfrac{px}{\sin_p^*(x)[\cos_p^*(x)]^{1-p}}}.$$

Lemma 2.2(1) shows that $f_{11}'(x)/f_{22}'(x)$ is strictly decreasing on $(0, \pi_p^*/2)$. Applying Lemma 2.1 twice, we derive that $f(x)$ is also strictly decreasing on $(0, \pi_p^*/2)$. Moreover,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} \frac{f_{11}'(x)}{f_{22}'(x)} = \frac{p-1}{p+1}, \quad \lim_{x\to \frac{\pi_p^*}{2}^-} f(x) = \lim_{x\to \frac{\pi_p^*}{2}^-} \frac{f_1'(x)}{f_2'(x)} = 0.$$

Therefore, the proof of inequality (3.1) is completed. □

THEOREM 3.2. (Generalized Lazarević-type inequality) *Let* $p \in (1, +\infty)$. *Then the function*

$$f(x) = \frac{\log[\sinh_p^*(x)/x]}{\log[\cosh_p^*(x)]}$$

is strictly increasing on $(0, m_p^*)$. *And if* $p \in (1,2]$, *then the range of* $f(x)$ *is* $((p-1)/(p+1), p-1)$, *while if* $p \in (2, +\infty)$, *then the range of* $f(x)$ *is* $((p-1)/(p+1), 1)$. *In particular, for all* $p \in (1,2]$ $(p \in (2, +\infty))$, *the inequality*

$$[\cosh_p^*(x)]^\alpha < \frac{\sinh_p^*(x)}{x} < [\cosh_p^*(x)]^\beta \tag{3.2}$$

holds for all $x \in (0, m_p^*)$, *with the best constant* $\alpha = (p-1)/(p+1)$, $\beta = p-1$ $(\beta = 1)$.

Proof. Let $f_1(x) = \log[\sinh_p^*(x)/x]$ and $f_2(x) = \log[\cosh_p^*(x)]$. Then $f_1(0) = f_2(0) = 0$ and

$$f_1'(x) = \frac{x}{\sinh_p^*(x)} \frac{-\sinh_p^*(x) + [\cosh_p^*(x)]^{p-1}x}{x^2} = \frac{[\cosh_p^*(x)]^{p-1}x - \sinh_p^*(x)}{x\sinh_p^*(x)},$$

$$f_2'(x) = \frac{[\sinh_p^*(x)]^{p-1}}{\cosh_p^*(x)}.$$

So that

$$\frac{f_1'(x)}{f_2'(x)} = \frac{[\cosh_p^*(x)]^p x - \sinh_p^*(x)\cosh_p^*(x)}{x[\sinh_p^*(x)]^p}.$$

If we also let $f_{11}(x)=[\cosh_p^*(x)]^p x-\sinh_p^*(x)\cosh_p^*(x)$ and $f_{22}(x)=x[\sinh_p^*(x)]^p$, then $f_{11}(0)=f_{22}(0)=0$ and

$$f_{11}'(x)=[\cosh_p^*(x)]^p+xp[\cosh_p^*(x)]^{p-1}[\sinh_p^*(x)]^{p-1}-[\cosh_p^*(x)]^p-[\sinh_p^*(x)]^p,$$

$$f_{22}'(x)=[\sinh_p^*(x)]^p+px[\cosh_p^*(x)\sinh_p^*(x)]^{p-1},$$

and thereby

$$\begin{aligned}\frac{f_{11}'(x)}{f_{22}'(x)}&=\frac{xp[\cosh_p^*(x)\sinh_p^*(x)]^{p-1}-[\sinh_p^*(x)]^p}{xp[\cosh_p^*(x)\sinh_p^*(x)]^{p-1}+[\sinh_p^*(x)]^p}\\&=-1+\frac{2px[\cosh_p^*(x)\sinh_p^*(x)]^{p-1}}{[\sinh_p^*(x)]^p+px[\cosh_p^*(x)\sinh_p^*(x)]^{p-1}}\\&=-1+\frac{2p}{p+\dfrac{\sinh_p^*(x)[\cosh_p^*(x)]^{1-p}}{x}}.\end{aligned}$$

We divide the proof into two cases.

Case 1 $p\in(1,2]$. Then the function $f_{11}'(x)/f_{22}'(x)$ is strictly increasing on $(0,+\infty)$ since $x\mapsto\sinh_p^*(x)[\cosh_p^*(x)]^{1-p}/x$ is strictly decreasing from $(0,+\infty)$ onto $(2-p,1)$ by Lemma 2.2(2). Applying Lemma 2.1, $f(x)$ is also strictly increasing on $(0,+\infty)$. Moreover, by the l'Hôptial rule, we obtain

$$\lim_{x\to0^+}f(x)=\lim_{x\to0}\frac{f_{11}'(x)}{f_{22}'(x)}=\frac{p-1}{p+1},$$

$$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}\frac{f_{11}'(x)}{f_{22}'(x)}=p-1.$$

Case 2 $p\in(2,+\infty)$. Then the function $f_{11}'(x)/f_{22}'(x)$ is strictly increasing on $(0,m_p^*)$ since $x\mapsto\sinh_p^*(x)[\cosh_p^*(x)]^{1-p}/x$ is strictly decreasing from $(0,m_p^*)$ onto $(0,1)$ by Lemma 2.2(2), so is $f(x)$ by Lemma 2.1. Moreover, $\lim\limits_{x\to0^+}f(x)=(p-1)/(p+1)$, and by Lemma 2.2(2) and the l'Hôptial rule we get

$$\lim_{x\to m_p^*}f(x)=-1+\frac{2p}{p+0}=1.\quad\square$$

THEOREM 3.3. (Generalized Huygens-type inequality) *Let* $p\in(1,+\infty)$. *Then the inequality*

$$(p+1)\frac{\sin_p^*(x)}{x}+(p-1)\frac{1}{\cos_p^*(x)}>2p\tag{3.3}$$

holds for all $x\in(0,\pi_p^*/2)$, *and the inequality*

$$(p+1)\frac{\sinh_p^*(x)}{x}+(p-1)\frac{1}{\cosh_p^*(x)}>2p\tag{3.4}$$

holds for all $x\in(0,m_p^*)$.

Proof. By substituting $t=1/2+1/(2p)$, $a=\sin_p^*(x)/x$ and $b=1/\cos_p^*(x)$ into the Bernoulli inequality $ta+(1-t)b>a^t b^{1-t}$ $(a,b>0,\ t\in(0,+\infty))$, we obtain

$$\left(\frac{1}{2}+\frac{1}{2p}\right)\frac{\sin_p^*(x)}{x}+\left(\frac{1}{2}-\frac{1}{2p}\right)\frac{1}{\cos_p^*(x)}>\left(\frac{\sin_p^*(x)}{x}\right)^{1/2+1/(2p)}\left(\frac{1}{\cos_p^*(x)}\right)^{1/2-1/(2p)}.$$

According to the left hand of inequality (3.1), one has

$$\left(\frac{\sin_p^*(x)}{x}\right)^{1/2+1/(2p)}\left(\frac{1}{\cos_p^*(x)}\right)^{1/2-1/(2p)}$$

$$>\left(\left[\cos_p^*(x)\right]^{(p-1)/(p+1)}\right)^{1/2+1/(2p)}\left(\frac{1}{\cos_p^*(x)}\right)^{1/2-1/(2p)}=1,$$

so that inequality (3.3) follows. Similarly, it follows from the Bernoulli inequality and inequality (3.2) that

$$\begin{aligned}&\left(\frac{1}{2}+\frac{1}{2p}\right)\frac{\sinh_p^*(x)}{x}+\left(\frac{1}{2}-\frac{1}{2p}\right)\frac{1}{\cosh_p^*(x)}\\ &\quad>\left(\frac{\sinh_p^*(x)}{x}\right)^{1/2+1/(2p)}\left(\frac{1}{\cosh_p^*(x)}\right)^{1/2-1/(2p)}\\ &\quad>\left(\left[\cosh_p^*(x)\right]^{(p-1)/(p+1)}\right)^{1/2+1/(2p)}\left(\frac{1}{\cosh_p^*(x)}\right)^{1/2-1/(2p)}=1.\end{aligned}$$

That is,

$$(p+1)\frac{\sinh_p^*(x)}{x}+(p-1)\frac{1}{\cosh_p^*(x)}>2p$$

for all $x\in(0,m_p^*)$. □

THEOREM 3.4. (Generalized Wilker-type inequality) *Let* $p\in(1,+\infty)$. *Then the inequality*

$$\left[\frac{\sinh_p^*(x)}{x}\right]^p+\frac{p(p-1)}{2}\frac{\tanh_p^*(x)}{x}>\frac{p^2-p+2}{2}\tag{3.5}$$

holds for all $x\in(0,m_p^*)$.

Proof. Substituting $t=\sinh_p^*(x)/x-1$ and $\alpha=p$ into another Bernoulli inequality $(1+t)^\alpha>1+\alpha t$ $(\alpha>0,\ t>0)$, one has

$$\left[\frac{\sinh_p^*(x)}{x}\right]^p>1+p\left[\frac{\sinh_p^*(x)}{x}-1\right].$$

This together with Lemma 2.2(3) yields

$$\left[\frac{\sinh_p^*(x)}{x}\right]^p > 1+p\left[\frac{\sinh_p^*(x)}{x}-1\right] > 1+\frac{p(p-1)}{2}\left[1-\frac{\tanh_p^*(x)}{x}\right].$$

By transposition of terms in the above inequality we immediately obtain (3.5). □

THEOREM 3.5. (Generalized Cusa-Hugens-type inequality for trigonometric functions) *Let* $p\in(1,+\infty)$ *and* p_0 *be defined as in Corollary 2.4. Then the inequality*

$$\frac{\sin_p^*(x)}{x} < \frac{2}{p+1}+\frac{p-1}{p+1}\cos_p^*(x) \tag{3.6}$$

holds for all $x\in(0,\pi_p^*/2)$ *if and only if* $p\in(1,p_0]$, *and the inverse inequality*

$$\frac{\sin_p^*(x)}{x} > \frac{2}{p+1}+\frac{p-1}{p+1}\cos_p^*(x) \tag{3.7}$$

holds for all $x\in(0,\pi_p^*/2)$ *if and only if* $p\in[1+\sqrt{2},\infty)$.

Proof. Let $\alpha=2/(p+1)\in(0,1)$ and

$$f(x)=\alpha x+(1-\alpha)x\cos_p^*(x)-\sin_p^*(x),\quad x\in\left(0,\pi_p^*/2\right).$$

Then simple computations lead to

$$f(0)=0, \tag{3.8}$$

$$f\left(\frac{\pi_p^*}{2}\right)=\frac{\pi_p^*}{2}\alpha-1, \tag{3.9}$$

$$f'(x)=\alpha+(1-\alpha)\cos_p^*(x)-(1-\alpha)x[\sin_p^*(x)]^{p-1}-[\cos_p^*(x)]^{p-1},$$

$$f'(0)=0, \tag{3.10}$$

$$f'\left(\frac{\pi_p^*}{2}\right)=\alpha-(1-\alpha)\frac{\pi_p^*}{2}, \tag{3.11}$$

$$\begin{aligned} f''(x) &= -(1-\alpha)[\sin_p^*(x)]^{p-1}-(1-\alpha)[\sin_p^*(x)]^{p-1} \\ &\quad -(1-\alpha)x(p-1)[\sin_p^*(x)]^{p-2}[\cos_p^*(x)]^{p-1} \\ &\quad +(p-1)[\cos_{p^*}(x)]^{p-2}[\sin_p^*(x)]^{p-1} \\ &= [\sin_p^*(x)]^{p-2}[\cos_p^*(x)]^{p-1}g(x), \end{aligned} \tag{3.12}$$

where

$$g(x)=-2(1-\alpha)\sin_p^*(x)\left[\cos_p^*(x)\right]^{1-p}-(1-\alpha)(p-1)x+(p-1)\tan_p^*(x).$$

It is easy to check that

$$g(0)=0, \tag{3.13}$$

$$g\left(\frac{\pi_p^*}{2}\right)=\begin{cases}+\infty, & p\in(1,2],\\ -\infty, & p\in(2,+\infty)\end{cases} \tag{3.14}$$

and

$$g'(x)=-2(1-\alpha)+2(1-\alpha)(1-p)[\cos_p^*(x)]^{-p}-1] \\ -(1-\alpha)(p-1)+(p-1)[\cos_p^*(x)]^{-2}.$$

If we denote $t=\left[\cos_p^*(x)\right]^{-2}$, then $t\in(1,+\infty)$, and $g'(x)=-J(t)$, where $J(t)$ is defined as in Lemma 2.7. The remaining proof can be divided into three cases: $p\in(1,2]$, $p\in(2,1+\sqrt{2})$ and $p\in[1+\sqrt{2},+\infty)$.

Case 1 $p\in(1,2]$. Then it follows from Lemma 2.7(1) that $g'(x)>0$ for all $x\in(0,\pi_p^*/2)$. Thus from equations (3.8), (3.10), (3.12) and (3.13) we conclude that $f(x)>0$ for all $x\in(0,\pi_p^*/2)$, namely, inequality (3.6) holds for all $x\in(0,\pi_p^*/2)$ and $p\in(1,2]$.

Case 2 $p\in(2,1+\sqrt{2})$. Then it follows from Lemma 2.7(2) that there exists $\zeta\in(0,\pi_p^*/2)$ such that $g'(x)>0$ for $x\in(0,\zeta)$ and $g'(x)<0$ for $x\in(\zeta,\pi_p^*/2)$, so that $g(x)$ is strictly increasing on $(0,\zeta)$ and strictly decreasing on $(\zeta,\pi_p^*/2)$. This together with (3.13) and (3.14) leads to the conclusion that there exists $\zeta^*\in(0,\pi_p^*/2)$ such that $g(x)>0$ for $x\in(0,\zeta^*)$ and $g(x)<0$ for $x\in(\zeta^*,\pi_p^*/2)$, so that $f'(x)$ is strictly increasing on $(0,\zeta^*)$ and strictly decreasing on $(\zeta^*,\pi_p^*/2)$ by (3.12). Since $f'(0)=0$, the sign characterize of $f'(x)$ on $(0,\pi_p^*/2)$ is either positive or positive then negative. Hence $f(x)$ is either strictly increasing or increasing then decreasing on $(0,\pi_p^*/2)$. Note that $f(0)=0$, and $f(\pi_p^*/2)=\pi_p^*/2\alpha-1\geqslant\ (<)0$ for $p\in(1,p_0]$ $(p_0,1+\sqrt{2})$ by Corollary 2.4, we obtain that $f(x)>0$ for all $x\in(0,\pi_p^*/2)$ if and only if $p\leqslant p_0$, namely, inequality (3.6) holds for all $x\in(0,\pi_p^*/2)$ and $p\in(2,p_0]$, and there exists no $p\in(p_0,1+\sqrt{2})$ such that $f(x)<0$ or $f(x)>0$ holds for all $x\in(0,\pi_p^*/2)$.

Case 3 $p\in[1+\sqrt{2},+\infty)$. Then it follows from Lemma 2.3(3)-(5) that $g'(x)<0$ for all $x\in(0,\pi_p^*/2)$. Thus from equations (3.8), (3.10), (3.12) and (3.13) we know that $f(x)<0$ for all $x\in(0,\pi_p^*/2)$, namely, inequality

$$\frac{\sin_p^*(x)}{x}>\frac{2}{p+1}+\frac{p-1}{p+1}\cos_p^*(x)$$

holds for all $x\in(0,\pi_p^*/2)$ and $p\in[1+\sqrt{2},+\infty)$. □

THEOREM 3.6. (Generalized Cusa-Hugens-type inequality for hyperbolic functions) *Let $p\in(1,+\infty)$ and p_1 be defined as in Corollary 2.6. Then the inequality*

$$\frac{\sinh_p^*(x)}{x}<\frac{2}{p+1}+\frac{p-1}{p+1}\cosh_p^*(x) \tag{3.15}$$

holds for all $x\in(0,m_p^*)$ *if and only if* $p\in(1,1+\sqrt{2}]$, *and the inverse inequality*

$$\frac{\sinh_p^*(x)}{x}>\frac{2}{p+1}+\frac{p-1}{p+1}\cosh_p^*(x) \tag{3.16}$$

holds for all $x\in(0,m_p^*)$ *if and only if* $p\in[p_1,\infty)$.

Proof. Let $\alpha=2/(p+1)$, and

$$f(x)=\alpha x+(1-\alpha)x\cosh_p^*(x)-\sinh_p^*(x),\quad x\in(0,m_p^*). \tag{3.17}$$

Then by simple computations we have

$$f(0)=0, \tag{3.18}$$

$$f'(x)=\alpha+(1-\alpha)\cosh_p^*(x)+(1-\alpha)x[\sinh_p^*(x)]^{p-1}-[\cosh_p^*(x)]^{p-1},$$

$$f'(0)=0, \tag{3.19}$$

$$\begin{aligned} f''(x)&=2(1-\alpha)[\sinh_p^*(x)]^{p-1}+(1-\alpha)(p-1)x[\sinh_p^*(x)]^{p-2}[\cosh_p^*(x)]^{p-1}\\ &\quad-(p-1)[\cosh_p^*(x)]^{p-2}[\sinh_p^*(x)]^{p-1}\\ &=[\sinh_p^*(x)]^{p-2}[\cosh_p^*(x)]^{p-1}g(x), \end{aligned} \tag{3.20}$$

where

$$g(x)=2(1-\alpha)\sinh_p^*(x)[\cosh_p^*(x)]^{1-p}+(1-\alpha)(p-1)x-(p-1)\tanh_p^*(x). \tag{3.21}$$

Moreover,

$$g(0)=0, \tag{3.22}$$

$$\begin{aligned} g'(x)&=2(1-\alpha)+2(1-\alpha)(1-p)[\tanh_p^*(x)]^{p}+(1-\alpha)(p-1)-(p-1)[\cosh_p^*(x)]^{-2}\\ &=2(1-\alpha)+2(1-\alpha)(p-1)[\cosh_p^*(x)]^{-p}-(1-\alpha)(p-1)-(p-1)[\cosh_p^*(x)]^{-2}. \end{aligned}$$

If we denote $t=[\cosh_p^*(x)]^{-2}$, then $t\in(0,1)$, and $g'(x)=J(t)$, which is defined as in Lemma 2.7. The remaining proof can be divided into three cases: $p\in(1,1+\sqrt{2}]$, $p\in(1+\sqrt{2},3)$ and $p\in[3,+\infty)$.

Case 1 $p\in(1,1+\sqrt{2}]$. Then it follows from Lemma 2.7(1)-(3) that $g'(x)>0$ for all $x\in(0,m_p^*)$. Thus from equations (3.18)-(3.22) we conclude that $f(x)>0$ for all $x\in(0,m_p^*)$, namely, inequality (3.15) holds for all $x\in(0,m_p^*)$ and $p\in(1,1+\sqrt{2}]$.

Case 2 $p\in(1+\sqrt{2},3)$. Then Lemma 2.7(4) implies that there exists $\eta\in(0,m_p^*)$ such that $g'(x)<0$ for $x\in(0,\eta)$ and $g'(x)>0$ for $x\in(\eta,m_p^*)$, so that $g(x)$ is strictly decreasing on $(0,\eta)$ and strictly increasing on (η,m_p^*).

Subcase 2.1 $p\in(1+\sqrt{2},p_1)$. Then Corollary 2.6 shows that $g(m_p^*)=(p-1)[(1-\alpha)m_p^*-1]>0$, so that there exists $\eta^*\in(0,m_p^*)$ such that $g(x)<0$ for $x\in$

$(0,\eta^*)$ and $g(x)>0$ for $x\in(\eta^*,m_p^*)$. Hence by (3.20), $f'(x)$ is strictly decreasing on $(0,\eta^*)$ and strictly increasing on (η^*,m_p^*). Furthermore, the limiting values

$$\lim_{r\to m_p^*} f(x) = \lim_{x\to m_p^*} \cosh_p^*(x)\left(\alpha x[\cosh_p^*(x)]^{-1}+(1-\alpha)x-\tanh_p^*(x)\right)=+\infty$$

and

$$\begin{aligned}\lim_{r\to m_p^*} f'(x) &= \lim_{x\to m_p^*}[\cosh_p^*(x)]^{p-1}\Big(\alpha[\cosh_p^*(x)]^{1-p}+(1-\alpha)[\cosh_p^*(x)]^{2-p}\\ &\qquad +(1-\alpha)x[\tanh_p^*(x)]^{p-1}-1\Big)\\ &=+\infty\end{aligned}$$

imply that there exists $\eta^{**}\in(0,m_p^*)$ such that $f(x)<0$ for $x\in(0,\eta^{**})$ and $f(x)>0$ for $x\in(\eta^{**},m_p^*)$. Consequently, inequality (3.16) holds for $x\in(0,\eta^{**})$ and inequality (3.15) holds for $x\in(\eta^{**},m_p^*)$.

Subcase 2.2 $p\in[p_1,3)$. Then Corollary 2.6 leads to the conclusion that $g(m_p^*)=(p-1)[(1-\alpha)m_p^*-1]\leqslant 0$, so that $g(x)<0$ for all $x\in(0,m_p^*)$. Therefore, from (3.18)-(3.22) we know that $f(x)<0$ for all $x\in(0,m_p^*)$, namely, inequality (3.16) holds for all $x\in(0,m_p^*)$ and $p\in[p_1,3)$.

Case 3 $p\in[3,+\infty)$. Then Lemma 2.7(5) shows that $g'(x)<0$ for all $x\in(0,m_p^*)$. This together with equations (3.18)-(3.22) leads to the conclusion that $f(x)<0$ for all $x\in(0,m_p^*)$, namely, inequality (3.16) holds for all $x\in(0,m_p^*)$ and $p\in[3,+\infty)$. □

Acknowledgements. The research was supported by the Natural Science Foundation of China (Grant Nos. 11701176, 61673169, 11301127, 11626101, 11601485), the Natural Science Foundation of Zhejiang Province (Grant No. LQY18G030001) and the research project for college students of Huzhou University (Grant No. 2019-111).

REFERENCES

[1] M. ADIL KHAN, Y.-M. CHU, A. KASHURI, R. LIKO AND G. ALI, *Conformable fractional integrals versions of Hermite-Hadamard inequalities and their generalizations*, J. Funct. Spaces, **2018** (2018), Article ID 6928130, 9 pages.

[2] M. ADIL KHAN, Y.-M. CHU, T. U. KHAN AND J. KHAN, *Some new inequalities of Hermite-Hadamard type for s-convex functions with applications*, Open Math., **15**, 1 (2017), 1414–1430.

[3] M. ADIL KHAN, M. HANIF, Z. A. KHAN, K. AHMAD AND Y.-M. CHU, *Association of Jensen's inequality for s-convex function with Csiszár divergence*, J. Inequal. Appl., **2019** (2019), Article 162, 14 pages.

[4] M. ADIL KHAN, A. IQBAL, M. SULEMAN AND Y.-M. CHU, *Hermite-Hadamard type inequalities for fractional integrals via Green's function*, J. Inequal. Appl., **2018** (2018), Article 161, 15 pages.

[5] M. ADIL KHAN, Y. KHURSHID, T.-S. DU AND Y.-M. CHU, *Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals*, J. Funct. Spaces, **2018** (2018), Article ID 5357463, 12 pages.

[6] M. ADIL KHAN, S.-H. WU, H. ULLAH AND Y.-M. CHU, *Discrete majorization type inequalities for convex functions on rectangles*, J. Inequal. Appl., **2019** (2019), Article 16, 18 pages.

[7] M. ADIL KHAN, S. ZAHEER ULLAH AND Y.-M. CHU, *The concept of coordinate strongly convex functions and related inequalities*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **113**, 3 (2019), 2235–2251.

[8] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.

[9] Ö. BAKŞI, P. GURKA, J. LANG AND O. MÉNDEZ, *Basis properties of Lindqvist-Peetre functions on $L^r(0,1)^n$*, Rev. Mat. Complut., **30**, 1 (2017), 1–12.

[10] Á. BARICZ, B. A. BHAYO AND M. VUORINEN, *Turán inequalities for generalized inverse trigonometric functions*, Filomat, **29**, 2 (2015), 303-313.

[11] B. A. BHAYO AND M. VUORINEN, *On generalized trigonometric functions with two parameters*, J. Approx. Theory, **164**, 10 (2012), 1415–1426.

[12] Z.-W. CAI, J.-H. HUANG AND L.-H. HUANG, *Generalized Lyapunov-Razumikhin method for retarded differential inclusions: applications to discontinuous neural networks*, Discrete Contin. Dyn. Syst., **22B**, 9 (2017), 3591–3614.

[13] Z.-W. CAI, J.-H. HUANG AND L.-H. HUANG, *Periodic orbit analysis for the delayed Filippov system*, Proc. Amer. Math. Soc., **146**, 11 (2018), 4667–4682.

[14] Y.-M. CHU, M. ADIL KHAN, T. ALI AND S. S. DRAGOMIR, *Inequalities for α-fractional differentiable functions*, J. Inequal. Appl., **2017** (2017), Article 93, 12 pages.

[15] Y.-M. CHU, M.-K. WANG AND S.-L. QIU, *Optimal combinations bounds of root-square and arithmetic means for Toader mean*, Proc. Indian Acad. Sci. Math. Sci., **122**, 1 (2012), 41–51.

[16] Z.-F. DAI, *Comments on a new class of nonlinear conjugate gradient coefficients with global convergence properties*, Appl. Math. Comput., **276** (2016), 297–300.

[17] Z.-F. DAI, X.-H. CHEN AND F.-H. WEN, *A modified Perry's conjugate gradient method-based derivative-free method for solving large-scale nonlinear monotone equations*, Appl. Math. Comput., **270** (2015), 378–386.

[18] Z.-F. DAI, D.-H. LI AND F.-H. WEN, *Worse-case conditional value-at-risk for asymmetrically distributed asset scenarios returns*, J. Comput. Anal. Appl., **20**, 2 (2016), 237–251.

[19] Z.-F. DAI AND F.-H. WEN, *Another improved Wei-Yao-Liu nonlinear conjugate gradient method with sufficient descent property*, Appl. Math. Comput., **218**, 14 (2012), 7421–7430.

[20] Z.-F. DAI AND F.-H. WEN, *Robust CVaR-based portfolio optimization under a genal affine data perturbation uncertainty set*, J. Comput. Anal. Appl., **16**, 1 (2014), 93–103.

[21] L. DUAN, X.-W. FANG AND C.-X. HUANG, *Global exponential convergence in a delayed almost periodic Nicholson's blowflies model with discontinuous harvesting*, Math. Methods Appl. Sci., **41**, 5 (2018), 1954–1965.

[22] L. DUAN AND C.-X. HUANG, *Existence and global attractivity of almost periodic solutions for a delayed differential neoclassical growth model*, Math. Methods Appl. Sci., **40**, 3 (2017), 814–822.

[23] L. DUAN, L.-H. HUANG, Z.-Y. GUO AND X.-W. FANG, *Periodic attractor for reaction-diffusion high-order Hopfield neural networks with time-varying delays*, Comput. Math. Appl., **73**, 2 (2017), 233–245.

[24] D. E. EDMUNDS, P. GURKA AND J. LANG, *Properties of generalized trigonometric functions*, J. Approx. Theory, **164**, 1 (2012), 47–56.

[25] X.-P. FANG, Y.-J. DENG AND J. LI, *Plasmon resonance and heat generation in nanostructures*, Math. Methods Appl. Sci., **38**, 18 (2015), 4663–4672.

[26] X-H. HE, W.-M. QIAN, H.-Z. XU AND Y.-M. CHU, *Sharp power mean bounds for two Sándor-Yang means*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **113**, 3 (2019), 2627–2638.

[27] H.-J. HU AND L.-Z. LIU, *Weighted inequalities for a general commutator associated to a singular integral operator satisfying a variant of Hörmander's condition*, Math. Notes, **101**, 5-6 (2017), 830–840.

[28] H.-J. HU AND X.-F. ZOU, *Existence of an extinction wave in the Fisher equation with a shifting habitat*, Proc. Amer. Math. Soc., **145**, 11 (2017), 4763–4771.

[29] C.-X. HUANG AND J.-D. CAO, *Stochastic dynamics of nonautonomous Cohen-Grossberg neural networks*, Abstr. Appl. Anal., **2011** (2011), Article ID 297147, 17 pages.

[30] C.-X. HUANG, S. GUO AND L.-Z. LIU, *Boundedness on Morrey space for Toeplitz type operator associated to singular integral operator with variable Calderón-Zygmund kernel*, J. Math. Inequal., **8**, 3 (2014), 453–464.

[31] T.-R. HUANG, B.-W. HAN, X.-Y. MA AND Y.-M. CHU, *Optimal bounds for the generalized Euler-Mascheroni constant*, J. Inequal. Appl., **2018** (2018), Article 118, 9 pages.

[32] C.-X. HUANG, H.-F. KUANG, X.-H., CHEN AND F.-H. WEN, *An LMI approach for dynamics of switched cellular neural networks with mixed delays*, Abstr. Appl. Anal., **2013** (2013), Article ID 870486, 8 pages.

[33] C.-X. HUANG AND B.-W. LIU, *New studies on dynamic analysis of inertial neural networks involving non-reduced order method*, Neurocomputing, **325**, 24 (2019), 283–287.

[34] C.-X. HUANG, B.-W. LIU, X.-M. TIAN, L.-S. YANG AND X.-X. ZHANG, *Global convergence on asymptotically almost periodic SICNNs with nonlinear decay functions*, Neural Process. Lett., **49**, 2 (2019), 625–641.

[35] C.-X. HUANG, C.-L. PENG, X.-H. CHEN AND F.-H. WEN, *Dynamics analysis of a class of delayed economic model*, Abstr. Appl. Anal., **2013** (2013), Article ID 962738, 12 pages.

[36] C.-X. HUANG, Y.-C. QIAN, L.-H. HUANG AND R. P. AGARWAL, *Dynamical behaviors of a food-chain model with stage structure and time delays*, Adv. Difference Equ., **2018** (2018), Article 186, 26 pages.

[37] T.-R. HUANG, S.-Y. TAN, X.-Y. MA AND Y.-M. CHU, *Monotonicity properties and bounds for the complete p-elliptic integrals*, J. Inequal. Appl., **2018** (2018), Article 239, 11 pages.

[38] C.-X. HUANG, Z.-C. YANG, T.-S. YI AND X.-F. ZHOU, *On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities*, J. Differential Equations, **256**, 7 (2014), 2101–2114.

[39] C.-X. HUANG AND H. ZHANG, *On Periodicity of non-autonomous inertial neural networks involving proportional delays and non-reduced order method*, Int. J. Biomath., **12**, 2 (2019), Article ID 1950016, 13 pages.

[40] C.-X. HUANG, H. ZHANG AND L.-H. HUANG, *Almost periodicity analysis for a delayed Nicholson's blowflies model with nonlinear density-dependent mortality term*, Commun. Pure Appl. Anal., **18** (2019), 3337–3349.

[41] Y.-J. JIANG AND J.-T. MA, *Spectral collocation methods for Volterra-integro differential equations with noncompact kernels*, J. Comput. Appl. Math., **244** (2013), 115–124.

[42] Y.-J. JIANG AND X.-J. XU, *A monotone finite volume method for time fractional Fokker-Planck equations*, Sci. China Math., **62**, 4 (2019), 783–794.

[43] Y. KHURSHID, M. ADIL KHAN, Y.-M. CHU AND Z. A. KHAN, *Hermite-Hadamard-Fejér inequalities for conformable fractional integrals via preinvex functions*, J. Funct. Spaces, **2019** (2019), Article ID 3146210, 9 pages.

[44] R. KLÉN, M. VUORUNEN AND X.-H. ZHANG, *Inequalities for the generalized trigonometric and hyperbolic functions*, J. Math. Anal. Appl., **409**, 1 (2014), 521–529.

[45] J.-L. LI, G.-Y. SUN AND R.-M. ZHANG, *The numerical solution of scattering by infinite rough interfaces based on the integral equation method*, Comput. Math. Appl., **71**, 7 (2016), 1491–1502.

[46] X.-F. LI, G.-J. TANG AND B.-Q. TANG, *Stress field around a strike-slip fault in orthotropic elastic layers via a hypersingular integral equation*, Comput. Math. Appl., **66**, 11 (2013), 2317–2326.

[47] J. LI, Y.-Y. YANG AND D.-X. XIE, *On the analysis and application of an ion size-modified Poisson-Boltzmann equation*, Nonlinear Anal. Real World Appl., **47** (2019), 188–203.

[48] P. LINDQVIST, *Some remarkable sine and cosine functions*, Ricerche Mat., **44**, 2 (1995), 269–290.

[49] P. LINDQVIST AND J. PEETRE, *p-arclength of the q-circle*, Math. Student, **72**, 1-4 (2003), 139–145.

[50] Y.-C. LIU AND J. WU, *Fixed point theorems in piecewise continuous function spaces and applications to some nonlinear problems*, Math. Methods Appl. Sci., **37**, 4 (2014), 508–517.

[51] Z.-Y. LIU, N.-C. WU, X.-R. QIN AND Y.-L. ZHANG, *Trigonometric transform splitting methods for real symmetric Toeplitz systems*, Comput. Math. Appl., **75**, 8 (2018), 2782–2794.

[52] Z.-Y. LIU, Y.-L. ZHANG, J. SANTOS ADN R. RALHA, *On computing complex square roots of real matrices*, Appl. Math. Lett., **25**, 10 (2012), 1565–1568.

[53] J.-Y. PENG AND Y. ZHANG, *Heron triangles with figurate number sides*, Acta Math. Hungar., **157**, 2 (2019), 478–488.

[54] W.-M. QIAN, Z.-Y. HE, H.-W. ZHANG AND Y.-M. CHU, *Sharp bounds for Neuman means in terms of two-parameter contraharmonic and arithmetic mean*, J. Inequal. Appl., **2019** (2019), Article 168, 13 pages.

[55] W.-M. QIAN, H.-Z. XU AND Y.-M. CHU, *Improvements of bounds for the Sándor-Yang means*, J. Inequal. Appl., **2019** (2019), Article 73, 8 pages.

[56] W.-M. QIAN, Y.-Y. YANG, H.-W. ZHANG AND Y.-M. CHU, *Optimal two-parameter geometric and arithmetic mean bounds for the Sándor-Yang mean*, J. Inequal. Appl., **2019** (2019), Article 287, 12 pages.

[57] S. RASHID, M. A. NOOR, K. I. NOOR, F. SAFDAR AND Y.-M. CHU, *Hermite-Hadamard type inequalities for the class of convex functions on time scale*, Math., **2009**, 7 (2019), Article 956, 20 pages, DOI:10.3390/math7100956.

[58] Y.-Q. SONG, M. ADIL KHAN, S. ZAHEER ULLAH AND Y.-M. CHU, *Integral inequalities involving strongly convex functions*, J. Funct. Spaces, **2018** (2018), Article ID 6595921, 8 pages.

[59] Y.-X. TAN, C.-X. HUANG, B.-SUN AND T. WANG, *Dynamics of a class of delayed reaction-diffusion systems with Neumann boundary condition*, J. Math. Anal. Appl., **458**, 2 (2018), 1115–1130.

[60] Y.-X. TAN, K. JIANG, *Existence and global exponential stability of almost periodic solution for delayed competitive neural networks with discontinuous activations*, Math. Methods Appl. Sci., **39**, 11 (2016), 2821–2839.

[61] W.-S. TANG AND Y.-J. SUN, *Construction of Runge-Kutta type methods for solving ordinary differential equations*, Appl. Math. Comput., **234** (2014), 179–191.

[62] W.-S. TANG AND J.-J. ZHANG, *Symplecticity-preserving continuous-stage Runge-Kutta-Nyström methods*, Appl. Math. Comput., **323** (2018), 204–219.

[63] Z.-L. TIAN, Y. LIU, Y. ZHANG, Z.-Y. LIU AND M.-Y. TIAN, *The general inner-outer iteration method based on regular splittings for the PageRank problem*, Appl. Math. Comput., **356** (2019), 479–501.

[64] W.-S. WANG, *On A-stable one-leg methods for solving nonlinear Volterra functional differential equations*, Appl. Math. Comput., **314** (2017), 380–390.

[65] W.-S. WANG AND Y.-Z. CHEN, *Fast numerical valuation of options with jump under Merton's model*, J. Comput. Appl. Math., **318** (2017), 79–92.

[66] W.-S. WANG, Y.-Z. CHEN AND H. FENG, *On the variable two-step IMEX BDF method for parabolic integro-differential equations with nonsmooth initial data arising in finance*, SIAM J. Numer. Anal., **57**, 3 (2019), 1289–1317.

[67] J.-F. WANG, X.-H. CHEN AND L.-H. HUANG, *The number and stability of limit cycles for planar piecewise linear systems of node-saddle type*, J. Math. Anal. Appl., **469**, 1 (2019), 405–427.

[68] M.-K. WANG, H.-H. CHU AND Y.-M. CHU, *Precise bounds for the weighted Hölder mean of the complete p-elliptic integrals*, J. Math. Anal. Appl., **480**, 2 (2019), Article ID 123388, 9 pages, DOI: 10.1016/j.jmaa.2019.123388.

[69] M.-K. WANG, Y.-M. CHU AND W. ZHANG, *Monotonicity and inequalities involving zero-balanced hypergeometric function*, Math. Inequal. Appl., **22**, 2 (2019), 601–617.

[70] M.-K. WANG, Y.-M. CHU AND W. ZHANG, *Precise estimates for the solution of Ramanujan's generalized modular equation*, Ramanujan J., **49**, 3 (2019), 653–668.

[71] T. WANG AND H. GUO, *Existence and nonexistence of nodal solutions for Choquard type equations with perturbation*, J. Math. Anal. Appl., **480**, 2 (2019), Article ID 123438, 20 pages, DOI: 10.1016/j.jmaa.2019.123438.

[72] J.-F. WANG, C.-X. HUANG AND L.-H. HUANG, *Discontinuity-induced limit cycles in a general planar piecewise linear system of saddle-focus type*, Nonlinear Anal. Hybrid Syst., **33** (2019), 162–178.

[73] J.-L. WANG, W.-M. QIAN, Z.-Y. HE AND Y.-M. CHU, *On approximating the Toader mean by other bivariate means*, J. Funct. Spaces, **2019** (2019), Article ID 6082413, 7 pages.

[74] M.-K. WANG, W. ZHANG AND Y.-M. CHU, *Monotonicity, convexity and inequalities involving the generalized elliptic integrals*, Acta Math. Sci., **39B**, 5 (2019), 1440–1450.

[75] S.-H. WU AND Y.-M. CHU, *Schur m-power convexity of generalized geometric Bonferroni mean involving three parameters*, J. Inequal. Appl., **2019** (2019), Article 57, 11 pages.

[76] C.-E. XIAO, J.-B. LIU AND Y.-L. LIU, *An inverse pollution problem in porous media*, Appl. Math. Comput., **218**, 7 (2011), 3649–3653.

[77] D.-X. XIE AND J. LI, *A new analysis of electrostatic free energy minimization and Poisson-Boltzmann equation for protein in ionic solvent*, Nonlinear Anal. Real World Appl., **21** (2015), 185–196.

[78] H.-Z. XU, Y.-M. CHU AND W.-M. QIAN, *Sharp bounds for the Sándor-Yang means in terms of arithmetic and contra-harmonic means*, J. Inequal. Appl., **2018** (2018), Article 127, 13 pages.

[79] Z.-H. YANG, W.-M. QIAN, Y.-M. CHU AND W. ZHANG, *On rational bounds for the gamma function*, J. Inequal. Appl., **2017** (2017), Article 210, 17 pages.

[80] S. ZAHEER ULLAH, M. ADIL KHAN AND Y.-M. CHU, *Majorization theorems for strongly convex functions*, J. Inequal. Appl., **2019** (2019), Article 58, 13 pages.

[81] S. ZAHEER ULLAH, M. ADIL KHAN AND Y.-M. CHU, *A note on generalized convex functions*, J. Inequal. Appl., **2019** (2019), Article 291, 10 pages.

[82] S. ZAHEER ULLAH, M. ADIL KHAN, Z. A. KHAN AND Y.-M. CHU, *Integral majorization type inequalities for the functions in the sense of strong convexity*, J. Funct. Spaces, **2019** (2019), Article ID 9487823, 11 pages.

[83] L. ZHANG AND S.-Y. JIAN, *Further studies on the Wei-Yao-Liu nonlinear conjugate gradient method*, Appl. Math. Comput., **219**, 14 (2013), 7616–7621.

[84] T.-H. ZHAO, Y.-M. CHU AND H. WANG, *Logarithmically complete monotonicity properties relating to the gamma function*, Abstr. Appl. Anal., **2011** (2011), Article ID 896483, 13 pages.

[85] W.-J. ZHOU, *On the convergence of the modified Levenberg-Marquardt method with a nonmonotone second order Armijo type line search*, J. Comput. Appl. Math., **239** (2013), 152–161.

[86] X.-S. ZHOU, *Weighted sharp function estimate and boundedness for commutator associated with singular integral operator satisfying a variant of Hörmander's condition*, J. Math. Inequal., **9**, 2 (2015), 587–596.

[87] W.-J. ZHOU, AND X.-L. CHEN, *On the convergence of a modified regularized Newton method for convex optimization with singular solutions*, J. Comput. Appl. Math., **239** (2013), 179–188.

[88] W.-J. ZHOU AND F. WANG, *A PRP-based residual method for large-scale monotone nonlinear equations*, Appl. Math. Comput., **261** (2015), 1–7.

[89] Q.-X. ZHU, C.-X. HUANG AND X.-S. YANG, *Exponential stability for stochastic jumping BAM neural networks with time-varying and distributed delays*, Nonlinear Anal. Hybrid Syst., **5**, 1 (2011), 52–77.

(Received June 24, 2019)

Miao-Kun Wang
Department of Mathematics
Huzhou University
Huzhou 313000, P. R. China
e-mail: wmk000@126.com, wangmiaokun@zjhu.edu.cn

Miao-Ying Hong
Department of Mathematics
Huzhou University
Huzhou 313000, P. R. China
e-mail: 1146519729@qq.com

Yang-Fan Xu
Department of Mathematics
Huzhou University
Huzhou 313000, P. R. China
e-mail: Yangfanxu@126.com

Zhong-Hua Shen
Department of Mathematics
Hangzhou Normal University
Hangzhou 311121, P. R. China
e-mail: ahtshen@126.com

Yu-Ming Chu
College of Science
Hunan City University
Yiyang 413000, P. R. China
and
School of Mathematics and Statistics
Changsha University of Science & Technology
Changsha 410114, P. R. China
e-mail: chuyuming2005@126.com