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A NOTE ON TWO RECENT RESULTS ABOUT POLYNOMIALS WITH RESTRICTED ZEROS

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Abstract. Let $P(z)$ be a polynomial of degree n and for any complex number α, let $D_\alpha P(z) := nP(z) + (\alpha - z)P'(z)$ denote the polar derivative of the polynomial $P(z)$ with respect to α. In this paper, we first extend a recently proved result contained in a paper published in this journal to the polar derivative of a polynomial. We shall also point out a fault in other result published in the same paper and discuss in detail the validity of that result.

1. Introduction

If $P(z)$ is a polynomial of degree n, then concerning the estimate of $|P'(z)|$ on the unit disk $|z| = 1$, we have

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The above inequality which is an immediate consequence of Bernstein's inequality on the derivative of Trigonometric polynomial is best possible with equality holding for the polynomial $P(z) = \lambda z^n, \lambda$ being a complex number.

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then the above inequality can be sharpened. In fact, Erdös conjectured and later Lax [5] proved that, if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

The above inequality is best possible and equality holds for all polynomials having their zeros on $|z| = 1$.

As an extension of (1.2), Malik [7] proved that, if $P(z) \neq 0$ in $|z| < k, k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.3}$$

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As a generalization of (1.3), Aziz and Shah [1] proved that, if $P(z)$ has no zeros in $|z|<k, k\geqslant 1$ except with $s-$fold zeros at the origin, then

$$\max_{|z|=1}|P'(z)| \leqslant \frac{n+ks}{1+k}\max_{|z|=1}|P(z)|. \tag{1.4}$$

Chan and Malik [2] generalized (1.3) in a different direction and proved that, if $P(z)=a_0+\sum\limits_{\nu=\mu}^{n}a_\nu z^\nu$, $\mu\geqslant 1$, is a polynomial of degree n which does not vanish in $|z|<k$ where $k\geqslant 1$, then

$$\max_{|z|=1}|P'(z)| \leqslant \frac{n}{1+k^\mu}\max_{|z|=1}|P(z)|. \tag{1.5}$$

As a refinement of (1.5), Pukhta [9] proved that, if $P(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leqslant\mu\leqslant n$, is a polynomial of degree n not vanishing in $|z|<k$, $k\geqslant 1$, then

$$\max_{|z|=1}|P'(z)| \leqslant \frac{n}{1+k^\mu}\left\{\max_{|z|=1}|P(z)|-\min_{|z|=k}|P(z)|\right\}. \tag{1.6}$$

Further, Kumar and Lal [6] generalized (1.6) by proving that, if $P(z)=z^s\left(a_0+\sum_{\nu=\mu}^{n-s}a_\nu z^\nu\right)$, $1\leqslant\mu\leqslant n-s$, $0\leqslant s\leqslant n-1$, is a polynomial of degree n having a zero of order s at the origin and remaining $n-s$ zeros in $|z|\geqslant k$, $k\geqslant 1$, then

$$\max_{|z|=1}|P'(z)| \leqslant \frac{n+sk^\mu}{1+k^\mu}\max_{|z|=1}|P(z)|-\frac{n-s}{k^s(1+k^\mu)}\min_{|z|=k}|P(z)|. \tag{1.7}$$

Very recently, K. M. Nakprasit and J. Somsuwan [8] proved the following generalization of (1.7).

THEOREM A. *If* $P(z)=(z-z_0)^s\left(a_0+\sum_{\nu=\mu}^{n-s}a_\nu z^\nu\right)$, $1\leqslant\mu\leqslant n-s$, $0\leqslant s\leqslant n-1$, *is a polynomial of degree* n *having a zero of order* s *at* z_0 *with* $|z_0|<1$ *and the remaining* $n-s$ *zeros in* $|z|\geqslant k$, $k\geqslant 1$, *then*

$$\max_{|z|=1}|P'(z)| \leqslant \left[\frac{s}{(1-|z_0|)}+\frac{A}{(1-|z_0|)^s}\right]\max_{|z|=1}|P(z)|-\frac{A}{(k+|z_0|)^s}\min_{|z|=k}|P(z)|, \tag{1.8}$$

where

$$A=\frac{(1+|z_0|)^{s+1}(n-s)}{(1+k^\mu)(1-|z_0|)}. \tag{1.9}$$

In the same paper K. M. Nakprasit and J. Somsuwan also claim to have proved the following result.

THEOREM B. *If* $P(z)=(z-z_0)^s\left(a_0+\sum_{\nu=\mu}^{n-s}a_\nu z^\nu\right)$, $1\leqslant\mu\leqslant n-s$, $0\leqslant s\leqslant n-1$, *is a polynomial of degree* n *having a zero of order* s *at* z_0 *with* $|z_0|<1$ *and the remaining* $n-s$ *zeros on* $|z|=k$, $k\leqslant 1$, *then*

$$\max_{|z|=1}|P'(z)| \leqslant \left[\frac{s}{(1-|z_0|)}+\frac{(1+|z_0|)^{s+1}(n-s)}{(k^{n-s-2\mu+1}+k^{n-s-\mu+1})(1-|z_0|)^{s+1}}\right]\max_{|z|=1}|P(z)|. \tag{1.10}$$

Let $D_\alpha P(z)$ denote the polar derivative of the polynomial $P(z)$ of degree n with respect to the point α. Then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left\{\frac{D_\alpha P(z)}{\alpha}\right\} = P'(z),$$

uniformly with respect to z for $|z| \leqslant R$, $R > 0$. As a generalization of (1.6) to the polar derivative of $P(z)$, Dewan, Singh and Mir [3] proved that, if $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geqslant 1$, then for every α with $|\alpha| \geqslant 1$,

$$\max_{|z|=1}|D_\alpha P(z)| \leqslant \frac{n}{1+k^\mu}\left\{(|\alpha|+k^\mu)\max_{|z|=1}|P(z)| - (|\alpha|-1)\min_{|z|=k}|P(z)|\right\}. \tag{1.11}$$

Here, we shall extend Theorem A to the polar derivative of a polynomial and thereby obtain generalizations of (1.7) and (1.8). Besides, we point out a fault in Theorem B and discuss in detail the validity of this result.

THEOREM 1. *If $P(z) = (z-z_0)^s\left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu\right)$, $1 \leqslant \mu \leqslant n-s$, $0 \leqslant s \leqslant n-1$, is a polynomial of degree n having a zero of order s at z_0 with $|z_0| < 1$ and the remaining $n-s$ zeros in $|z| \geqslant k$, $k \geqslant 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant 1$, we have*

$$\max_{|z|=1}|D_\alpha P(z)| \leqslant \left[\frac{s(|\alpha|+|z_0|)}{(1-|z_0|)} + \frac{A(|\alpha|+k^\mu)}{(1-|z_0|)^s}\right]\max_{|z|=1}|P(z)| - \frac{A(|\alpha|-1)}{(k+|z_0|)^s}\min_{|z|=k}|P(z)|, \tag{1.12}$$

where A is defined by (1.9).

Clearly Theorem 1 generalizes inequality (1.8) and to obtain (1.8) from the above theorem, simply divide both sides of (1.12) by $|\alpha|$ and let $|\alpha| \to \infty$.

REMARK 1. By letting $z_0 = 0$ in Theorem 1, we get $A = \frac{n-s}{1+k^\mu}$ and

$$\max_{|z|=1}|D_\alpha P(z)| \leqslant \left[\frac{s(|\alpha|-1)k^\mu + n(|\alpha|+k^\mu)}{(1+k^\mu)}\right]\max_{|z|=1}|P(z)| - \frac{(n-s)(|\alpha|-1)}{k^s(1+k^\mu)}\min_{|z|=k}|P(z)|,$$

which is a generalization of (1.7).

2. Proof of Theorem 1 and some comments on Theorem B

2.1. Proof of Theorem 1. Let $P(z) = (z-z_0)^s\phi(z)$ where $\phi(z) = a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu$, $1 \leqslant \mu \leqslant n-s$, is a polynomial of degree $n-s$ having no zeros in $|z| < k$, $k \geqslant 1$. Applying the inequality (1.11) to the polynomial $\phi(z)$, we get for $|\alpha| \geqslant 1$,

$$|D_\alpha \phi(z)| \leqslant \frac{n-s}{1+k^\mu}\left\{(|\alpha|+k^\mu)\max_{|z|=1}|\phi(z)| - (|\alpha|-1)m'\right\}, \tag{2.1}$$

where $m' = \min_{|z|=k} |\phi(z)|$.
Now

$$
\begin{aligned}
D_\alpha P(z) &= nP(z) + (\alpha - z)P'(z) \\
&= n(z-z_0)^s \phi(z) + (\alpha - z)\left\{s(z-z_0)^{s-1}\phi(z) + (z-z_0)^s \phi'(z)\right\} \\
&= n(z-z_0)^s \phi(z) + s(\alpha - z_0)(z-z_0)^{s-1}\phi(z) \\
&\qquad - s(z-z_0)^s\phi(z) + (\alpha - z)(z-z_0)^s\phi'(z) \\
&= (z-z_0)^s\left\{(n-s)\phi(z) + (\alpha - z)\phi'(z)\right\} + s(\alpha - z_0)(z-z_0)^{s-1}\phi(z) \\
&= (z-z_0)^s D_\alpha \phi(z) + s(\alpha - z_0)(z-z_0)^{s-1}\phi(z),
\end{aligned}
$$

which implies

$$
\begin{aligned}
(z-z_0)D_\alpha P(z) &= (z-z_0)^{s+1}D_\alpha\phi(z) + s(\alpha - z_0)(z-z_0)^s\phi(z) \\
&= (z-z_0)^{s+1}D_\alpha\phi(z) + s(\alpha - z_0)P(z). \qquad (2.2)
\end{aligned}
$$

Hence for $|z| = 1$, we get from (2.2) that

$$
\max_{|z|=1}|z-z_0||D_\alpha P(z)| \leqslant s|\alpha - z_0|\max_{|z|=1}|P(z)| + \max_{|z|=1}|z-z_0|^{s+1}|D_\alpha\phi(z)|. \qquad (2.3)
$$

For $|z| = 1$, we have

$$
|z-z_0| \geqslant |z| - |z_0| = 1 - |z_0|,
$$

and

$$
|z-z_0| \leqslant |z| + |z_0| = 1 + |z_0|.
$$

We obtain from (2.3) that

$$
(1-|z_0|)\max_{|z|=1}|D_\alpha P(z)| \leqslant s(|\alpha| + |z_0|)\max_{|z|=1}|P(z)| + (1+|z_0|)^{s+1}\max_{|z|=1}|D_\alpha\phi(z)|. \quad (2.4)
$$

Inequality (2.4) when combined with (2.1), gives

$$
\begin{aligned}
&(1-|z_0|)\max_{|z|=1}|D_\alpha P(z)| \\
&\qquad \leqslant (1+|z_0|)^{s+1}\left(\frac{n-s}{1+k^\mu}\right)\left\{(|\alpha| + k^\mu)\max_{|z|=1}|\phi(z)| - (|\alpha| - 1)\min_{|z|=k}|\phi(z)|\right\} \\
&\qquad\qquad + s(|\alpha| + |z_0|)\max_{|z|=1}|P(z)|. \qquad (2.5)
\end{aligned}
$$

The relation between $\phi(z)$ and $P(z)$ yields

$$
\min_{|z|=k}|\phi(z)| = \min_{|z|=k}\left[\frac{1}{|z-z_0|^s}|P(z)|\right] \geqslant \frac{1}{(k+|z_0|)^s}\min_{|z|=k}|P(z)|,
$$

and

$$\max_{|z|=1}|\phi(z)| = \max_{|z|=1}\left|\left[\frac{1}{|z-z_0|^s}|P(z)|\right]\right| \leqslant \frac{1}{(1-|z_0|)^s}\max_{|z|=1}|P(z)|.$$

Applying these relations in (2.5), we obtain

$$\max_{|z|=1}|D_\alpha P(z)| \leqslant \left[\frac{s(|\alpha|+|z_0|)}{(1-|z_0|)} + \frac{A(|\alpha|+k^\mu)}{(1-|z_0|)^s}\right]\max_{|z|=1}|P(z)| - \frac{A(|\alpha|-1)}{(k+|z_0|)^s}\min_{|z|=k}|P(z)|,$$

where A is defined by (1.9) and this proves Theorem 1 completely.

2.2. Some comments on Theorem B. Going through the proof of Theorem B, we notice that it uses the following result of Dewan and Hans [4], which the authors call Theorem 3 (see [8], page 144). Unfortunately, this result of Dewan and Hans is false and a counter example in support of our claim is presented below after the statement of the result.

THEOREM 2. *If $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n having all its zeros zeros on $|z| = k$, $k \leqslant 1$, then*

$$\max_{|z|=1}|P'(z)| \leqslant \frac{n}{k^{n-2\mu+1}+k^{n-\mu+1}}\max_{|z|=1}|P(z)|. \tag{2.6}$$

To see that Theorem 2 is false, let us consider the polynomial $P(z) = z^n + k^n$, which is a polynomial of the form $a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leqslant \mu \leqslant n$ with

$$a_0 = k^n,\ \ a_\nu = 0 \text{ for } \nu = \mu,\ \mu+1,\ \ldots, n-1 \text{ and } a_n = 1,$$

where $1 \leqslant \mu \leqslant n$.

Clearly, $P(z)$ has all its zeros on $|z| = k$, $\max_{|z|=1}|P(z)| = 1 + k^n$ and $\max_{|z|=1}|P'(z)| = n$.

Thus if (2.6) is true, then we would have

$$n \leqslant \frac{n}{k^{n-2\mu+1}+k^{n-\mu+1}}(1+k^n),$$

for any $\mu \in \{1,2,\ldots,n\}$. For convenience, we take $n = 2\mu - 1$, then it amounts to saying that $1 + k^\mu \leqslant 1 + k^{2\mu-1}$. For $k < 1$, this is obviously false except when $\mu = 1$.

Thus the example $z^n + k^n$ shows that Theorem 2 is false for $2 \leqslant \mu \leqslant n$ and $k < 1$.

This further implies that Theorem B is false for $2 \leqslant \mu \leqslant n-s$, $s \geqslant 0$.

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