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UPPER BOUND OF THE THIRD HANKEL DETERMINANT FOR A SUBCLASS OF q–STARLIKE FUNCTIONS ASSOCIATED WITH THE LEMNISCATE OF BERNOULLI

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Abstract. In this paper, we use q-derivative operator to define a new subclass of starlike functions related with the lemniscate of Bernoulli. For this function class we obtain upper bound of the third Hankel determinant. For validity of our results, relevant connections with those in earlier works are also pointed out.

1. Introduction and Basic Definitions

Let by $\mathscr{H}(\mathbb{U})$ we denote the class of functions which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

where $\mathbb{C}$ is the set of complex numbers and let $\mathscr{A}$ be the class of analytic functions having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (\forall\, z \in \mathbb{U}), \tag{1.1}$$

in the open unit disk $\mathbb{U}$, centered at origin and normalized by the conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Also, let $\mathscr{S}$, the subclass of analytic function class $\mathscr{A}$, be the class of functions which are univalent in $\mathbb{U}$.

Furthermore, let the class of starlike functions in $\mathbb{U}$ will be denoted by $\mathscr{S}^*$, which consists of normalized functions $f \in \mathscr{A}$ that satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (\forall\, z \in \mathbb{U}).$$

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Moreover, by $\mathcal{SL}^*$, we denote the class of a function $f \in \mathcal{A}$ that satisfy the following inequality:

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \qquad (\forall\, z \in \mathbb{U}).$$

Thus a function $f \in \mathcal{SL}^*$ is such that $\frac{zf'(z)}{f(z)}$ lies in the region bounded by the right half of the lemniscate of Bernoulli given by the relation

$$\left| \mathfrak{w}^2 - 1 \right| < 1.$$

This class of function was introduced by Sokól and Stankiewicz (see [25]).

Next, If two functions f and g are analytic in $\mathbb{U}$, we say that the function f is subordinate to the function g and write as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function w which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1,$$

such that

$$f(z) = g(w(z)).$$

Furthermore, if the function g is univalent in $\mathbb{U}$ then it follows that:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \;\text{ and }\; f(\mathbb{U}) \subset g(\mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$ then we have the following equivalence (*cf., eg.,* [20], see also [21]):

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \;\text{ and }\; f(\mathbb{U}) \subset g(\mathbb{U}).$$

We next denote by $\mathcal{P}$ the class of analytic functions p which are normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.2}$$

such that

$$\Re(p(z)) > 0 \qquad (\forall\, z \in \mathbb{U}).$$

We now recollect some basic definitions and concept details of the q-calculus which are used in this paper. We suppose throughout the paper that $0 < q < 1$ and that

$$\mathbb{N} = \{1, 2, 3 \ldots\} = \mathbb{N}_0 \setminus \{0\} \qquad (\mathbb{N}_0 := \{0, 1, 2, \ldots\}).$$

DEFINITION 1. Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \ldots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

DEFINITION 2. (see [11] and [12]) Let $0 < q < 1$. The q-derivative (or q-difference) D_q of a function f defined is in a given subset of $\mathbb{C}$ by

$$(D_q f)(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0) \end{cases} \tag{1.3}$$

provided that $f'(0)$ exists.

We note from Definition 2 that

$$\lim_{q \to 1-} (D_q f)(z) = \lim_{q \to 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z),$$

for a differentiable function f in a given subset of $\mathbb{C}$. It is readily deduced from (1.1) and (1.3) that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{1.4}$$

In Geometric Function Theory, many subclasses of normalized analytic functions class $\mathscr{A}$ have been studied already in different aspect. The above defined q-calculus gives invaluable tools that have been extensively used in order to investigate several subclasses of class $\mathscr{A}$. Historically speaking, Ismail *et al.* [10] were the first who used the q-derivative operator D_q to study the q-calculus analogous of the class $\mathscr{S}^*$ of starlike functions in $\mathbb{U}$ (see Definition 3 below). A firm footing usage of the q-calculus in the context of Geometric Function Theory was presented mainly and basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [28, pp. 347 *et seq.*]). Subsequently, a great deal of work has been developed by many mathematicians, which has played an important role in the development of Geometric Function Theory. In particular, Srivastava and Bansal [29] studied the close-to-convexity of q-Mittag-Leffler functions, while also Srivastava *et al.* [32] introduced the generalized subfamilies of q-starlike functions related with the Janowski functions. On the other hand, Mahmood *et al.* [19] studied the class of q-starlike functions in the conic region. Also for some recent investigations regarding q-calculus in the context of Geometric Function Theory one may refer to [3, 1, 6, 7, 9, 17, 35].

DEFINITION 3. (see [10]) A function $f \in \mathscr{A}$ is said to belong to the class $\mathscr{S}_q^*$ if

$$f(0) = f'(0) - 1 = 0 \tag{1.5}$$

and

$$\left| \frac{z}{f(z)} (D_q f) z - \frac{1}{1-q} \right| \leqslant \frac{1}{1-q}. \tag{1.6}$$

It is readily observe that as $q \to 1-$ the closed disk

$$\left| w - \frac{1}{1-q} \right| \leqslant \frac{1}{1-q}$$

becomes the right-half plane and the class $\mathcal{S}_q^*$ of q-starlike functions reduces to the familiar class $\mathcal{S}^*$. Equivalently, by using the principle of subordination between analytic functions, we can rewrite the conditions in (1.5) and (1.6) as follows (see [36]):

$$\frac{z}{f(z)}(D_q f)(z) \prec \widehat{p} \qquad \left(\widehat{p} = \frac{1+z}{1-qz}\right).$$

Motivating and inspired by the work of Srivastava *et al.* [27], Oçar [36] and from the above mentioned q-Calculus, we define the following.

DEFINITION 4. A function f of the form (1.1) is said to be in the class $\mathcal{SL}_q^*$ if and only if

$$\left| \left(\frac{z(D_q f)(z)}{f(z)} \right)^2 - \frac{1}{1-q} \right| < \frac{1}{1-q}, \tag{1.7}$$

or equivalently

$$\frac{z(D_q f)(z)}{f(z)} \prec \sqrt{\frac{2(1+z)}{2+(1-q)z}} \qquad (\forall z \in \mathbb{U}).$$

Thus a function $f \in \mathcal{SL}_q^*$ is such that $\frac{z(D_q f)(z)}{f(z)}$ lies in the region bounded by the right half of the generalized lemniscate of Bernoulli given by the relation

$$\left| w^2 - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

It is worthy of note that, if we let $q \to 1-$, in condition (1.7), we are led the well-known function class $\mathcal{SL}^*$, which was introduced by Sokól and Stankiewicz (see [25]) and further investigated by the many authors see for example [2, 23, 26].

Let $n \geqslant 0$ and $q \geqslant 1$. Then the q^{th} Hankel determinant is defined as:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_{n+q-1} \\ a_{n+1} & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{n+q-1} & \cdot & \cdot & \cdot & \cdot & a_{n+2(q-1)} \end{vmatrix}$$

The determinant has been studied by several authors. In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles [8, 13, 22, 24, 31, 33] for various classes of functions. It is well-known that the Fekete-Szegö functional $|a_3 - a_2^2| = H_2(1)$. This functional is further generalized as $|a_3 - \mu a_2^2|$ for some μ real as well as complex. Fekete and Szegö gave sharp estimates of $|a_3 - \mu a_2^2|$ for μ real and $f \in \mathcal{S}$, the class of univalent functions. It is also know that the functional $|a_2 a_4 - a_3^2|$ is equivalent to $H_2(2)$. Babalola [4] studied the Hankel determinant $H_3(1)$ for some subclasses of analytic functions. The Hankel determinant $H_3(1)$ has been also studied by the many authors see for example [18, 30, 34]. In the present investigation, our focus is on the Hankel determinant $H_3(1)$ for the function class $\mathcal{SL}_q^*$.

2. A Set of Lemmas

In order to prove our main results, we need the following lemmas.

LEMMA 1. (see [16]) *Let*

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots$$

is in the class $\mathscr{P}$ *of functions positive real part in* $\mathbb{U}$*, then for any complex number* υ

$$\left|c_2 - \upsilon c_1^2\right| \leqq \begin{cases} -4\upsilon + 2 & (\upsilon \leqq 0) \\ 2 & (0 \leqq \upsilon \leqq 1) \\ 4\upsilon - 2 & (\upsilon \geqq 1). \end{cases} \tag{2.1}$$

When $\upsilon < 0$ *or* $\upsilon > 1$*, equality holds true in* (2.1) *if and only if*

$$p(z) = \frac{1+z}{1-z}$$

or one of its rotations. If $0 < \upsilon < 1$*, then equality holds true in* (2.1) *if and only if*

$$p(z) = \frac{1+z^2}{1-z^2}$$

or one of its rotations. If $\upsilon = 0$*, equality holds true in* (2.1) *if and only if*

$$p(z) = \left(\frac{1+\rho}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\rho}{2}\right)\frac{1-z}{1+z} \qquad (0 \leqq \rho \leqq 1)$$

or one of its rotations. If $\upsilon = 1$*, then the equality in* (2.1) *holds true if* $p(z)$ *is a reciprocal of one of the functions such that the equality holds true in the case when* $\upsilon = 0$.

LEMMA 2. [14, 15] *Let*

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots$$

is in the class $\mathscr{P}$ *of functions positive real part in* $\mathbb{U}$*, then*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right)$$

for some x, $|x| \leqslant 1$ *and*

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - \left(4 - c_1^2\right)c_1 x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z$$

for some z, $|z| \leqslant 1$.

LEMMA 3. [5] *Let*

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots$$

is in the class $\mathscr{P}$ *of functions positive real part in* $\mathbb{U}$*, then*

$$|c_k| \leqslant 2 \qquad (k \in \mathbb{N})$$

and the inequality is sharp.

3. Main Results and their Demonstrations

In this section, we will prove our main results. Throughout our discussion we will assume that, $q \in (0,1)$.

THEOREM 1. *Let* $f \in \mathscr{SL}_q^*$ *and be of the form* (1.1)*, then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \begin{cases} \frac{\left(3q^2-3q+2\right)-2(1+q)^2\mu}{32q^2} & \left(\mu < \frac{3q^2+11q+2}{2(1+q)^2}\right) \\ \frac{1}{4q} & \left(\frac{3q^2+11q+2}{2(1+q)^2} \leqslant \mu \leqslant \frac{3q^2+5q+2}{2(1+q)^2}\right) \\ \frac{2(1+q)^2\mu-\left(3q^2-3q+2\right)}{32q^2} & \left(\mu > \frac{3q^2+5q+2}{2(1+q)^2}\right). \end{cases}$$

These results are sharp.

Proof. If $f \in \mathscr{SL}_q^*$ then it follows from definition that

$$\frac{z\left(D_q f\right)(z)}{f(z)} \prec \phi(z), \tag{3.1}$$

where

$$\phi(z) = \left(\frac{2(1+z)}{2+(1-q)z}\right)^{\frac{1}{2}}.$$

Define a function

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \dots$$

it is clear that $p \in \mathscr{P}$. This implies that

$$w(z) = \frac{p(z)-1}{p(z)+1}.$$

From (3.1) we have

$$\frac{z\left(D_q f\right)(z)}{f(z)} \prec \phi(z),$$

with

$$\phi(w(z)) = \left(\frac{4p}{(1+q)+(3-q)p}\right)^{\frac{1}{2}}.$$

Now

$$\begin{aligned}&\left(\frac{4p}{(1+q)+(3-q)p}\right)^{\frac{1}{2}}\\ &= 1+\frac{(1+q)}{8}c_1 z+\frac{(1+q)}{128}\left[16c_2+(3q-13)c_1^2\right]z^2\\ &+\frac{(1+q)}{1024}\left[128c_3+(48q-208)c_1c_2+\left(5q^2-38q+85\right)c_1^3\right]z^3+\ldots\end{aligned}$$

Similarly

$$\begin{aligned}\frac{z(D_q f)(z)}{f(z)} &= 1+qa_2 z+\left\{\left(q+q^2\right)a_3-qa_2^2\right\}z^2+\left\{\left(q+q^2+q^3\right)a_4\right.\\ &\quad\left.-\left(2q+q^2\right)a_2a_3+qa_2^3\right\}z^3+\ldots\end{aligned}$$

Therefore

$$a_2 = \frac{(1+q)}{8q}c_1 \tag{3.2}$$

$$a_3 = \frac{1}{64q}\left[8c_2+\frac{3q^2-11q+2}{2q}c_1^2\right] \tag{3.3}$$

$$a_4 = \frac{(1+q)}{1024(q+q^2+q^3)}\left[128c_3+\frac{16\left(3q^2-12q+2\right)}{q}c_1c_2+\zeta c_1^3\right]. \tag{3.4}$$

Where

$$\zeta = \frac{5q^4-35q^3+78q^2-24q+2}{q^2}.$$

Thus

$$\left|a_3-\mu a_2^2\right| = \frac{1}{8q}\left|c_2-\kappa c_1^2\right|, \tag{3.5}$$

where

$$\kappa = \frac{2(1+q)^2\mu-\left(3q^2-11q+2\right)}{16q}$$

Using Lemma 1 on (3.5), we obtain the required result.

The results are sharp for the functions $H_1(z)$ and $H_2(z)$, such that

$$\frac{z\left(D_qH_1\right)(z)}{H_1(z)} = \sqrt{\frac{2(1+z)}{2+(1-q)z}},\text{ if } \mu < \frac{3q^2+11q+2}{2(1+q)^2} \text{ or } \mu > \frac{3q^2+5q+2}{2(1+q)^2}$$

$$\frac{z\left(D_qH_2\right)(z)}{H_2(z)} = \sqrt{\frac{2\left(2+2(1-q)z+\left(1+q^2\right)z^2\right)}{(2+(1-q)z)^2}},\text{ if } \frac{3q^2+11q+2}{2(1+q)^2} < \mu < \frac{3q^2+5q+2}{2(1+q)^2}.$$

THEOREM 2. *Let $f \in \mathscr{SL}_q^*$ and of the form (1.1), then*

$$\left|a_2a_4 - a_3^2\right| \leqslant \frac{1}{16q^2}. \tag{3.6}$$

This result is sharp.

Proof. Making use of (3.2), (3.3) and (3.4), we have

$$a_2a_4 - a_3^2 = \frac{(1+q)^2}{64q^2(1+q+q^2)}c_1c_3 + \frac{2q^2-13q+1}{512q^2(1+q+q^2)}c_1^2c_2 + \frac{q^4+7q^3-50q^2+171q-29}{16384q^2(1+q+q^2)}c_1^4.$$

Putting the value of c_2 and c_3 from Lemma 2, using triangular inequality and replacing $|x| < 1$ by ρ and c_1 by c, we have

$$\begin{aligned}|a_2a_4 - a_3^2| \leqslant \frac{1}{\Upsilon(q)}\Big[&\Lambda(q)c^4 + 128(1+q)^2\left(4-c^2\right) + 16\Omega(q)c^2\left(4-c^2\right)\rho \\ &+ \left(64c^2q + 4\left(1+q+q^2\right) - 128(1+q)^2\right)\left(4-c^2\right)\rho^2\Big] \\ =\,& F(c,\rho),\end{aligned} \tag{3.7}$$

where

$$\Upsilon(q) = 16384q^2\left(1+q+q^2\right),$$
$$\Lambda(q) = \left|q^4+7q^3-18q^2-27q-13\right|$$

and

$$\Omega(q) = \left|2q^2-5q+1\right|.$$

Differentiating (3.7) with respect to ρ, we have

$$\frac{\partial F}{\partial \rho} = \frac{1}{\Upsilon(q)}\Big[16\Omega(q)c^2\left(4-c^2\right) + 2\left(64c^2q + 4\left(1+q+q^2\right) - 128(1+q)^2\right)\left(4-c^2\right)\rho\Big].$$

It is clear that $\frac{\partial F(c,\rho)}{\partial \rho} > 0$.

Which shows that $F(c,\rho)$ is an increasing function on the closed interval $[0,1]$. This implies that maximum value occurs at $\rho = 1$. That is for maximum of $F(c,\rho) = F(c,1) = G(c)$ say.

Now

$$\begin{aligned}G(c) = \frac{1}{\Upsilon(q)}\Big[&(\Lambda(q) - 16\Omega(q) - 64)c^4 \\ &+ (64\Omega(q) - 256q(1+q))c^2 + 512\left((1+q)^2 + \left(1+q^2\right)\right)\Big].\end{aligned} \tag{3.8}$$

Differentiating (3.8) with respect to c we have

$$G'(c) = \frac{1}{\Upsilon(q)}\left[-4\left(\Lambda(q) - 16\Omega(q) - 64\right)c^3 + 2\left(64\Omega(q) - 256q(1+q)\right)c\right.$$

Differentiating again above equation with respect to c we have

$$G''(c) = \frac{1}{\Upsilon(q)}\left[-12\left(\Lambda(q) - 16\Omega(q) - 64\right)c^2 + 2\left(64\Omega(q) - 256q(1+q)\right)\right].$$

For $c = 0$ this shows that the $\max G(c)$ occurs at $c = 0$. Hence we obtain

$$\left|a_2a_4 - a_3^2\right| \leqslant \frac{1}{16q^2}.$$

This result is sharp for the functions

$$\frac{z(D_qf)(z)}{f(z)} = \sqrt{\frac{2(1+z)}{2+(1-q)z}}$$

or

$$\frac{z(D_qf)(z)}{f(z)} = \sqrt{\frac{2\left(2+2(1-q)z+(1+q^2)z^2\right)}{\left(2+(1-q)z\right)^2}}.$$

THEOREM 3. *Let $f \in \mathcal{SL}_q^*$ and of the form* (1.1), *then*

$$|a_2a_3 - a_4| \leqslant \frac{(1+q)}{4q(1+q+q^2)}.$$

Proof. Using the values of (3.2), (3.3) and (3.4), we have

$$a_2a_3 - a_4 = \frac{(1+q)}{\lambda(q)}\left[-128q^2c_3 - 16\left(3q^4 - 13q^3 + q^2 - q\right)c_1c_2 - \left(2q^4 - 27q^3 + 84q^2 - 15q + 21\right)c_1^3\right],$$

where

$$\lambda(q) = 1024q^3\left(1+q+q^2\right).$$

Using Lemma 2 and since $c_1 \leqslant 2$ by Lemma 3, let $c_1 = c$ and assume without restriction that $c \in [0,2]$. Taking absolute and applying the triangle inequality with $\rho = |x|$ we obtain

$$\begin{aligned}|a_2a_3 - a_4| &\leqslant \frac{(1+q)}{\lambda(q)}\left[\chi(q)c^3 + 64q^2(4-c^2) + \psi(q)\right.\\ &\quad\left.(4-c^2)c\rho + 32q^2(c-2)\left(4-c^2\right)\rho^2\right]\\ &= F(\rho) \text{ say,}\end{aligned}$$

where

$$\chi(q) = \left|-26q^4 + 131q^3 - 124q^2 + 32q - 21\right|$$

and

$$\psi(q) = \left|24q^4 - 104q^3 + 72q^2 - 8q\right|.$$

Differentiating $F(\rho)$ with respect to ρ we have

$$F'(\rho) = \frac{(1+q)}{\lambda(q)}\left[\psi(q)(4-c^2)c + 64q^2(c-2)\left(4-c^2\right)\rho\right] > 0.$$

This implies that $F(\rho)$ is an increasing function of ρ on the closed interval $[0,1]$.

Hence $F(\rho) \leqslant F(0)$ for all $\rho \in [0,1]$ that is

$$\begin{aligned} F(\rho) &\leqslant \frac{(1+q)}{\lambda(q)}\left[\chi(q)c^3 + 64q^2(4-c^2)\right] \\ &= G(c) \text{ say.} \end{aligned}$$

Differentiating $G(c)$ with respect to c we have

$$G'(c) = \frac{(1+q)}{\lambda(q)}\left[3\chi(q)c^2 - 128q^2c\right].$$

Again differentiating the above equation with respect to c we have

$$G''(c) = \frac{(1+q)}{\lambda(q)}\left[6\chi(q)c - 128q^2\right].$$

Since $c \in [0,2]$, by the assumption, it follows that $G(c)$ attains maximum at $c = 0$, which corresponds to $\rho = 0$ and it is the desired upper bound.

In its special case, if we let $q \to 1-$, Theorem 3 reduce to the following known result.

COROLLARY 1. (see [23]) *If a function f of the form (1.1) is in the class $\mathcal{SL}^*$, then*

$$|a_2a_3 - a_4| \leqslant \frac{1}{6}.$$

To prove Theorem 4, we need the following Lemma (Lemma 4).

LEMMA 4. *If a function f of the form (1.1) is in the class $\mathcal{SL}_q^*$, then*

$$\begin{aligned} |a_2| &\leqslant \frac{(1+q)}{4q} \\ |a_3| &\leqslant \frac{1}{4q} \\ |a_4| &\leqslant \frac{(1+q)}{4q(1+q+q^2)} \\ |a_5| &\leqslant \frac{(1+q)}{4q(1+q+q^2+q^3)}. \end{aligned}$$

Proof. The proof of Lemma 4 is similar to that of result which has been already proved by Sokól (see [26]), therefore we here choice to omit the detial of the proof of Lemma 4.

THEOREM 4. *Let* $f \in \mathcal{SL}_q^*$ *and of the form* (1.1), *then*

$$H_3(1) \leqslant \frac{q^6+10q^5+20q^4+24q^3+20q^2+10q+1}{64q^3\left(q^2+1\right)\left(q^2+q+1\right)^2}.$$

Proof. Since

$$H_3(1) \leqslant |a_3|\left|\left(a_2a_4-a_3^2\right)\right|+|a_4|\left|\left(a_2a_3-a_4\right)\right|+|a_5|\left|\left(a_1a_3-a_2^2\right)\right|.$$

Using the fact that $a_1 = 1$, with Theorem 1, Theorem 2, Theorem 3 and Lemma 4 we have the required result.

If we let $q \to 1-$, in Theorem 4, we are led the following known result.

COROLLARY 2. (see [23]) *If a function* f *of the form* (1.1) *is in the class* $\mathcal{SL}^*$, *then*

$$H_3(1) \leqslant \frac{43}{576}.$$

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