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APPROXIMATION OF QUADRATIC LIE $*$–DERIVATIONS ON ρ–COMPLETE CONVEX MODULAR ALGEBRAS

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Abstract. In this paper, we investigate stable approximation of almost quadratic Lie $*$-derivations associated with approximate quadratic mappings on ρ-complete convex modular algebras χ_ρ by using Δ_2-condition via convex modular ρ.

1. Introduction

In 1940, S.M. Ulam gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the the question concerning the stability of group homomorphisms [18]. Let G be a group and let G' be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G \to G'$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $F : G \to G'$ with $d(f(x), F(x)) < \varepsilon$ for all $x \in G$? D.H. Hyers [6] has solved the problem of Ulam for the case of additive mappings in 1941. The result was generalized by T.Aoki [1] in 1950, by Th.M. Rassias [15] in 1978, by J.M. Rassias [14] in 1992, and by P. Găvruta [5] in 1994. Over the past few decades, many mathematicians have published the generalized Hyers–Ulam stability of functional equations [2, 3, 4, 11, 17, 21].

Now, we recall some basic definitions and remarks of modular spaces with modular functions, which are primitive notions corresponding to norms or metrics, as in the followings [8, 9, 19].

DEFINITION 1. Let χ be a linear space.

(1) A function $\rho : \chi \to [0,\infty]$ is called a modular if for arbitrary $x, y \in \chi$,

(m1) $\rho(x) = 0$ if and only if $x = 0$,

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(m2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,

(m3) $\rho(\alpha x + \beta y) \leqslant \rho(x) + \rho(y)$ for any scalars α, β, where $\alpha + \beta = 1$ and $\alpha, \beta \geqslant 0$,

(2) alternatively, if (m3) is replaced by

(m3)' $\rho(\alpha x + \beta y) \leqslant \alpha\rho(x) + \beta\rho(y)$ for every scalars α, β, where $\alpha + \beta = 1$ and $\alpha, \beta \geqslant 0$,

acting on the real linear space χ, then we say that ρ is a convex modular. Now, we define to extend the inequality (m3)' to the following inequality

(m3)" $\rho(\alpha x + \beta y) \leqslant |\alpha|\rho(x) + |\beta|\rho(y)$ for every scalars $\alpha, \beta \in \mathbb{C}$, where $|\alpha| + |\beta| = 1$,

acting on the complex linear space χ. Then, we remark a modular ρ defines a corresponding modular space, i.e., the linear space χ_ρ given by

$$\chi_\rho = \{x \in \chi : \rho(\lambda x) \to 0 \quad \text{as} \quad \lambda \to 0\}.$$

Let ρ be a modular on χ_ρ. Then we remark that $\rho(tx)$ is an increasing function in $t \geqslant 0$ for each fixed $x \in \chi$, that is, $\rho(ax) \leqslant \rho(bx)$ whenever $0 \leqslant a < b$. In addition, if ρ is a convex modular on χ, then $\rho(\alpha x) \leqslant \alpha\rho(x)$ for all $x \in \chi$ and for all α with $0 \leqslant \alpha \leqslant 1$. Moreover, we see that $\rho(\alpha x) \leqslant |\alpha|\rho(x)$ for all $x \in \chi$ and all α with $|\alpha| \leqslant 1$.

REMARK 1. (1) In general, we note that $\rho\left(\sum_{i=1}^n \alpha_i x_i\right) \leqslant \sum_{i=1}^n \alpha_i \rho(x_i)$ for all $x_i \in \chi$ and $\alpha_i \geqslant 0$ $(i = 1, \cdots, n)$ whenever $0 < \alpha := \sum_{i=1}^n \alpha_i \leqslant 1$ [9]. (2) Consequently, we lead to $\rho\left(\sum_{i=1}^n \alpha_i x_i\right) \leqslant \sum_{i=1}^n |\alpha_i|\rho(x_i)$ for all $x_i \in \chi$ and all $\alpha_i \in \mathbb{C}$ whenever $0 < \alpha := \sum_{i=1}^n |\alpha_i| \leqslant 1$.

DEFINITION 2. Let χ_ρ be a modular space and let $\{x_n\}$ be a sequence in χ_ρ. Then,

(1) $\{x_n\}$ is ρ-convergent to $x \in \chi_\rho$ and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(2) $\{x_n\}$ is called ρ-Cauchy in χ_ρ if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(3) A subset K of χ_ρ is called ρ-complete if and only if any ρ-Cauchy sequence is ρ-convergent to an element in K.

It is said that the modular ρ has the Fatou property if and only if $\rho(x) \leqslant \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ-convergent to x. A modular function ρ is said to satisfy the Δ_2-condition if there exists $\kappa > 0$ such that $\rho(2x) \leqslant \kappa\rho(x)$ for all $x \in \chi_\rho$.

The concept of modular spaces was first introduced by Nakano [13], and then by Musielak and Orlicz [12]. Concerning the stability theory in modular spaces, G. Sadeghi [16] has established generalized Hyers–Ulam stability via the fixed point method of a generalized Jensen functional equation $f(rx + sy) = rg(x) + sh(y)$ in convex modular spaces with the Fatou property satisfying the Δ_2-condition with $0 < \kappa \leqslant 2$. In [19], the authors have presented the generalized Hyers–Ulam stability of quadratic functional equations via the extensive studies of fixed point theory in the framework of modular spaces whose modular is convex, lower semicontinuous but does not satisfy any relatives of Δ_2-condition (see also [7, 10]). Recently, the stability problems of various functional equations in modular spaces have been intensively investigated by many authors [8, 9, 10].

Now, we introduce the concept of convex modular $*$-algebras. It is said that χ_ρ is called a convex modular $*$-algebra if the fundamental space X is a $*$-algebra with convex modular ρ subject to $\rho(ab) \leqslant \rho(a)\rho(b)$ and $\rho(c^*) = \rho(c)$ for all $a,b,c \in X$. A subset K of a convex modular algebra χ_ρ is called ρ-complete if and only if any ρ-Cauchy sequence in K is ρ-convergent to an element in K.

Throughout the paper, χ_ρ will be a ρ-complete convex modular $*$-algebra and the symbol $[a,b]$ will denote the commutator $ab - ba$. We say that a linear mapping f is called a Lie $*$-derivation if $f([x,y]) = [f(x),y] + [x, f(y)]$ and $f(z^*) = f(z)^*$ for all vectors x,y,z, where $[a,b] = ab - ba$. In a similar way, we say that a quadratic mapping f is quadratic homogeneous if $f(\lambda x) = \lambda^2 f(x)$ for all vectors x and all scalars λ, and a quadratic homogeneous mapping f is called a quadratic Lie $*$-derivation if $f([x,y]) = [f(x),y^2] + [x^2, f(y)]$ and $f(z^*) = f(z)^*$ for all vectors x,y,z. Concerning the stability theory of approximate quadratic Lie $*$-derivations in ρ-complete convex modular algebras, we first investigate generalized Hyers–Ulam stability via direct method of the equation

$$f(2x-y) + f(x+y) = f(x-y) + 4f(x) + f(y) \tag{1}$$

in ρ-complete convex modular algebras without using both Fatou property and Δ_2-condition, and then alternatively present generalized Hyers–Ulam stability of the equation (1) via direct method using necessarily Δ_2-condition but not using the Fatou property in ρ-complete convex modular algebras.

2. Approximate quadratic Lie $*$-derivations

We recall that the classical functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2}$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. First of all, we remark that the equation (1) is equivalent to the original quadratic functional equation (2), and so every solution of equation (1) is a quadratic mapping. For notational convenience, we denote the quadratic difference operator QE_f^λ of quadratic equation (1) and QD_f of quadratic derivation, respectively, as follows:

$$\begin{aligned} QE_f^\lambda(x,y) &:= f(2\lambda x - \lambda y) + f(\lambda x + \lambda y) \\ &\qquad\qquad -\lambda^2 f(x-y) - 4\lambda^2 f(x) - \lambda^2 f(y), \\ QD_f(x,y) &:= f([x,y]) - [f(x),y^2] - [x^2, f(y)] \end{aligned}$$

for all x,y in a linear space X and $\lambda \in \Lambda := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, which act as perturbing terms of quadratic Lie $*$-derivations. In the following, we present a generalized Hyers–Ulam stability of the equation (1) via direct method associated with approximate quadratic Lie $*$-derivations in ρ-complete convex modular algebras without using both Fatou property and Δ_2-condition.

THEOREM 1. *Suppose that a mapping $f : \chi_\rho \to \chi_\rho$ with $f(0) = 0$ satisfies*

$$\rho(QE_f^\lambda(x,y) + f(z^*) - f(z)^*) \leqslant \phi_1(x,y,z), \tag{3}$$

$$\rho(QD_f(x,y)) \leqslant \phi_2(x,y) \tag{4}$$

and $\phi_1 : \chi_\rho^3 \to [0,\infty)$ and $\phi_2 : \chi_\rho^2 \to [0,\infty)$ are mappings such that

$$\Phi(x,y,z) := \sum_{j=0}^{\infty} \frac{\phi_1(2^j x, 2^j y, , 2^j z)}{2^{2j}} < \infty, \ \lim_{n\to\infty} \frac{\phi_2(2^n x, 2^n y)}{4^{2n}} = 0 \tag{5}$$

for all $x,y,z \in \chi_\rho$ and $\lambda \in \Lambda$. If for each $x \in \chi_\rho$ the mapping $r \to f(rx)$ from $\mathbb{R}$ to χ_ρ is continuous, then there exists a unique quadratic Lie $*$-derivation $F_1 : \chi_\rho \to \chi_\rho$ which satisfies the equation (1) and

$$\rho(f(x) - F_1(x)) \leqslant \frac{1}{4}\Phi(x,x,0) \tag{6}$$

for all $x \in \chi_\rho$.

Proof. Putting $y := x$ and $z := 0$ in (3), we obtain

$$\rho(QE_f^1(x,x)) = \rho(f(2x) - 4f(x)) \leqslant \phi_1(x,x,0), \tag{7}$$

which yields

$$\rho\left(f(x) - \frac{f(2x)}{4}\right) \leqslant \frac{1}{4}\rho(f(2x) - 4f(x)) \leqslant \frac{1}{4}\phi_1(x,x,0)$$

for all $x \in \chi_\rho$. Since $\sum_{j=0}^{n-1} \frac{1}{4^{j+1}} \leqslant 1$, we prove the following functional inequality

$$\begin{aligned} \rho\left(f(x) - \frac{f(2^n x)}{2^{2n}}\right) &= \rho\Big[\sum_{j=0}^{n-1}\Big(\frac{f(2^j x)}{2^{2j}} - \frac{f(2^{j+1}x)}{2^{2(j+1)}}\Big)\Big] \\ &= \rho\Big[\sum_{j=0}^{n-1}\frac{1}{2^{2(j+1)}}\big(4f(2^j x) - f(2^{j+1}x)\big)\Big] \\ &\leqslant \sum_{j=0}^{n-1}\frac{1}{2^{2(j+1)}}\rho\left(4f(2^j x) - f(2^{j+1}x)\right) \\ &\leqslant \frac{1}{4}\sum_{j=0}^{n-1}\frac{\phi_1(2^j x, 2^j x, 0)}{2^{2j}} \end{aligned} \tag{8}$$

for all $x \in \chi_\rho$ by using the property of convex modular ρ.

Now, replacing x by $2^m x$ in (8), we have

$$\rho\Big(\frac{f(2^m x)}{2^{2m}} - \frac{f(2^{m+n}x)}{2^{2(m+n)}}\Big) \leqslant \frac{1}{4}\sum_{j=m}^{m+n-1} \frac{\phi_1(2^j x, 2^j x, 0)}{2^{2j}}$$

which converges to zero as $m \to \infty$ by the assumption (5). Thus the above inequality implies that the sequence $\{\frac{f(2^n x)}{2^{2n}}\}$ is ρ-Cauchy for all $x \in \chi_\rho$ and so it is convergent in χ_ρ since the space χ_ρ is ρ-complete. Thus, we may define a mapping $F_1 : \chi_\rho \to \chi_\rho$ as

$$F_1(x) := \rho - \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}} \Longleftrightarrow \lim_{n\to\infty} \rho\Big(\frac{f(2^n x)}{2^{2n}} - F_1(x)\Big) = 0,$$

for all $x \in \chi_\rho$.

Now, we proclaim F_1 is a quadratic mapping satisfying the equation (1) and the approximation (6). In fact, if we put $(x,y,z) := (2^n x, 2^n y, 0)$ in (3), and then divide the resulting inequality by 2^{2n}, one obtains

$$\rho\Big(\frac{QE_f^\lambda(2^n x, 2^n y)}{2^{2n}}\Big) \leqslant \frac{\rho(QE_f^\lambda(2^n x, 2^n y))}{2^{2n}} \leqslant \frac{\phi_1(2^n x, 2^n y, 0)}{2^{2n}},$$

which implies

$$\begin{aligned}\rho\Big(\frac{QE_f^\lambda(2^n x, 2^n y)}{2^{2n}}\Big) &\leqslant \frac{\rho(QE_f^\lambda(2^n x, 2^n y))}{2^{2n}} \\ &\leqslant \frac{\phi_1(2^n x, 2^n y, 0)}{2^{2n}} \\ &\to 0\end{aligned}$$

for all $x,y \in \chi_\rho$ and all $\lambda \in \Lambda$. Thus, noting $\frac{6|\lambda^2|+3}{9} \leqslant 1$ we figure out

$$\begin{aligned}&\rho(\frac{1}{9}QE_{F_1}^\lambda(x,y)) \\ &= \rho\Big(\frac{1}{9}QE_{F_1}^\lambda(x,y) - \frac{QE_f^\lambda(2^n x, 2^n y)}{9\cdot 2^{2n}} + \frac{QE_f^\lambda(2^n x, 2^n y)}{9\cdot 2^{2n}}\Big) \\ &\leqslant \frac{1}{9}\rho\Big(F_1(2\lambda x - \lambda y) - \frac{f\big(2^n(2\lambda x - \lambda y)\big)}{2^{2n}}\Big) + \frac{1}{9}\rho\Big(F_1(\lambda x + \lambda y) - \frac{f\big(2^n(\lambda x + \lambda y)\big)}{2^{2n}}\Big) \\ &\quad + \frac{\lambda^2}{9}\rho\Big(F_1(x-y) - \frac{f\big(2^n(x-y)\big)}{2^{2n}}\Big) + \frac{4\lambda^2}{9}\rho\Big(F_1(x) - \frac{f(2^n x)}{2^{2n}}\Big) \\ &\quad + \frac{\lambda^2}{9}\rho\Big(F_1(y) - \frac{f(2^n y)}{2^{2n}}\Big) + \frac{1}{9}\rho\Big(\frac{QE_f^\lambda\big(2^n x, 2^n y\big)}{2^{2n}}\Big)\end{aligned}$$

for all $x,y \in \chi_\rho$ and all positive integers n by Remark 1. Taking the limit as $n \to \infty$, one obtains $\rho(\frac{1}{9}QE_{F_1}^\lambda(x,y)) = 0$, and so

$$QE_{F_1}^\lambda(x,y) = 0$$

for all $x,y \in \chi_\rho$ and all $\lambda \in \Lambda$. Hence F_1 satisfies the equation (1) and so it is quadratic.

On the other hand, since $\sum_{i=0}^{n} \frac{1}{2^{2(i+1)}} + \frac{1}{2^2} \leqslant 1$ for all $n \in \mathbb{N}$, it follows from (7) and Remark 1 that

$$\begin{aligned}
&\rho(f(x) - F_1(x)) \\
&= \rho\left(\sum_{i=0}^{n} \frac{1}{2^{2(i+1)}}\left(2^2 f(2^i x) - f(2^{i+1}x)\right) + \frac{f(2^{n+1}x)}{2^{2(n+1)}} - \frac{F_1(2x)}{2^2}\right) \\
&\leqslant \sum_{i=0}^{n} \frac{1}{2^{2(i+1)}} \rho\left(QE_f^{\lambda}(2^i x, 2^i x)\right) + \frac{1}{2^2}\rho\left(\frac{f(2^{n+1}x)}{2^{2n}} - F_1(2x)\right) \\
&\leqslant \sum_{i=0}^{n} \frac{1}{2^{2(i+1)}} \phi_1(2^i x, 2^i x, 0) + \frac{1}{2^2}\rho\left(\frac{f(2^n \cdot 2x)}{2^{2n}} - F_1(2x)\right),
\end{aligned}$$

without applying Fatou property of the modular ρ for all $x \in \chi_\rho$ and all $n \in \mathbb{N}$, from which we obtain the approximation (6) of f by the quadratic mapping F_1 by taking $n \to \infty$ in the last inequality.

Next, we claim that F_1 is a quadratic Lie $*$-derivation. By (9), we have $QE_{F_1}^{\lambda}(x,x) = 0$ which yields $F_1(2\lambda x) = 4\lambda^2 F_1(x)$ for all $x \in \chi_\rho$ and $\lambda \in \Lambda$. From the assumption that for each $x \in \chi_\rho$ the mapping $r \to f(rx)$ from $\mathbb{R}$ to χ_ρ is continuous, it follows that $F_1(rx) = r^2 F_1(x)$ for all $x \in \chi_\rho$ and $r \in \mathbb{R}$ by the same argument as in the paper [15]. Thus, for any nonzero $\lambda \in \mathbb{C}$

$$\begin{aligned}
F_1(\lambda x) &= F_1\left(2\frac{\lambda}{|\lambda|}\frac{|\lambda|}{2}x\right) = 4\left(\frac{\lambda}{|\lambda|}\right)^2 F_1\left(\frac{|\lambda|}{2}x\right) \\
&= 4\left(\frac{\lambda}{|\lambda|}\right)^2 \left(\frac{|\lambda|}{2}\right)^2 F_1(x) = \lambda^2 F_1(x)
\end{aligned}$$

for all $x \in \chi_\rho$ and $\lambda \in \mathbb{C}$, which concludes that F_1 is quadratic homogeneous. In addition, in view of the inequality in (4) and the second condition in (5), we arrive at

$$\begin{aligned}
&\rho(\frac{1}{4}QD_{F_1}(x,y)) \\
&= \rho\left(\frac{1}{4}QD_{F_1}(x,y) - \frac{QD_f(2^n x, 2^n y)}{4 \cdot 4^{2n}} + \frac{QD_f(2^n x, 2^n y)}{4 \cdot 4^{2n}}\right) \\
&\leqslant \frac{1}{4}\rho\left(F_1([x,y]) - \frac{f(2^{2n}[x,y])}{4^{2n}}\right) + \frac{1}{4}\rho\left(\frac{[x^2, f(2^n y)]}{4^n} - [x^2, F_1(y)]\right) \\
&\quad + \frac{1}{4}\rho\left(\frac{[f(2^n x), y^2]}{4^n} - [F_1(x), y^2]\right) + \frac{1}{4 \cdot 4^{2n}}\rho\left(QD_f(2^n x, 2^n y)\right)
\end{aligned}$$

for all $x,y \in \chi_\rho$, which tends to zero as n tends to ∞. Therefore, one obtains $\rho(\frac{1}{4}QD_{F_1}(x,y)) = 0$, and so F_1 is a quadratic Lie derivation. In addition, we get the following inequality

$$\begin{aligned}\rho\Big(\frac{1}{3}\Big(F_1(z^*)-F_1(z)^*\Big)\Big) &\leqslant \frac{1}{3}\rho\Big(F_1(z^*)-\frac{f(2^nz^*)}{4^n}\Big)\\ &\quad+\frac{1}{3}\rho\Big(\frac{f(2^nz)}{4^n}^*-F_1(z)^*\Big)+\frac{1}{3}\rho\Big(\frac{f(2^nz^*)}{4^n}-\frac{f(2^nz)}{4^n}^*\Big)\\ &\leqslant \frac{1}{3}\rho\Big(F_1(z^*)-\frac{f(2^nz^*)}{4^n}\Big)\\ &\quad+\frac{1}{3}\rho\Big(\frac{f(2^nz)}{4^n}^*-F_1(z)^*\Big)+\frac{\phi_1(0,0,2^nz)}{3\cdot 4^n}\end{aligned}$$

for all vector z. Taking $n\to\infty$, one concludes F_1 is a Lie $*$-derivation.

Finally, applying the same argument as in the proof of Theorem [9], we prove the uniqueness of F_1 satisfying the approximation (6) near f. Therefore, one concludes that the mapping F_1 is a unique quadratic Lie derivation near f satisfying the approximation (6) in the modular algebra χ_ρ.

As a corollary of Theorem 1, we obtain the following stability result of approximate quadratic Lie $*$-derivations on complete normed algebras χ, which may be considered as χ_ρ equipped with norm $\|\cdot\|=\rho(\cdot)$.

COROLLARY 1. *Let χ_ρ be a complete normed $*$-algebra. For given nonnegative real numbers θ_i,ϑ_i together with $r_i<2(i=1,2,3)$ and a,b with $a+b<2$, suppose that a mapping $f:\chi_\rho\to\chi_\rho$ with $f(0)=0$ satisfies*

$$\begin{aligned}\|QE_f^\lambda(x,y)+f(z^*)-f(z)^*\| &\leqslant \theta_1\|x\|^{r_1}+\theta_2\|y\|^{r_2}+\theta_3(\|x\|^a\|y\|^b+\|z\|^{r_3}),\\ \|QD_f(x,y)\| &\leqslant \vartheta_1\|x\|^{2r_1}+\vartheta_2\|y\|^{2r_2}+\vartheta_3\|x\|^{2a}\|y\|^{2b}\end{aligned}$$

for all $x,y,z\in\chi_\rho$ and all $\lambda\in\Lambda$. If for each $x\in\chi_\rho$ the mapping $r\to f(rx)$ from $\mathbb{R}$ to χ_ρ is continuous, then there exists a unique quadratic Lie $*$-derivation $F_1:\chi_\rho\to\chi_\rho$ such that

$$\|f(x)-F_1(x)\|\leqslant\frac{\theta_1\|x\|^{r_1}}{2^2-2^{r_1}}+\frac{\theta_2\|x\|^{r_2}}{2^2-2^{r_2}}+\frac{\theta_3\|x\|^{a+b}}{2^2-2^{a+b}}$$

for all $x\in\chi_\rho$.

As a corollary, we obtain a stability result by strictly quadratical contractive conditions of control functions for perturbing terms QE_f^λ, QD_f of quadratic Lie $*$-derivations.

COROLLARY 2. *Suppose there exist two functions $\phi_1:\chi_\rho^3\to[0,\infty)$ and $\phi_2:\chi_\rho^2\to[0,\infty)$ and two constant l_i with $0<l_i<1$ $(i=1,2)$ for which a mapping $f:\chi_\rho\to\chi_\rho$ with $f(0)=0$ satisfies*

$$\begin{aligned}\rho(QE_f^\lambda(x,y)+f(z^*)-f(z)^*) &\leqslant \phi_1(x,y,z),\ \phi_1(2x,2y,2z)\leqslant 4l_1\phi_1(x,y,z),\\ \rho(QD_f(x,y)) &\leqslant \phi_2(x,y),\quad \phi_2(2x,2y)\leqslant 16l_2\phi_2(x,y)\end{aligned}$$

for all $x,y,z \in \chi_\rho$ and all $\lambda \in \Lambda$. If for each $x \in \chi_\rho$ the mapping $r \to f(rx)$ from $\mathbb{R}$ to χ_ρ is continuous, then there exists a unique quadratic Lie $*$-derivation $F_1 : \chi_\rho \to \chi_\rho$ which satisfies the equation (1) and

$$\rho(f(x) - F_1(x)) \leqslant \frac{1}{4(1-l_1)}\phi_1(x,x,0)$$

for all $x \in \chi_\rho$.

Proof. In view of strict quadratical contractive conditions for control functions ϕ_1 and ϕ_2, one leads to $\phi_1(2^n x, 2^n y, 2^n z) \leqslant (4l_1)^n \phi_1(x,y,z)$ and $\phi_2(2^n x, 2^n y) \leqslant (16l_2)^n \phi_2(x,y)$ for all $x,y,z \in \chi_\rho$ and all $\lambda \in \Lambda$. Hence, applying Theorem 1 to the theorem, we obtain the required approximation.

We recall that if the modular ρ satisfies the Δ_2-condition, then $\kappa \geqslant 1$ for nontrivial modular ρ, and $\kappa \geqslant 2$ for nontrivial convex modular ρ. See references [8, 9, 16, 19]. Now, we are going to investigate alternatively generalized Hyers–Ulam stability of the equation (1) associated with approximate quadratic Lie $*$-derivations via direct method using necessarily Δ_2-condition but not using the Fatou property in ρ-complete convex modular algebras.

THEOREM 2. *Let χ_ρ be a ρ-complete convex modular $*$-algebra with Δ_2-condition. Suppose there exist two functions $\varphi_1 : \chi_\rho^3 \to [0,\infty)$ and $\varphi_2 : \chi_\rho^2 \to [0,\infty)$ for which a mapping $f : \chi_\rho \to \chi_\rho$ satisfies*

$$\rho(QE_f^\lambda(x,y) + f(z^*) - f(z)^*) \leqslant \varphi_1(x,y,z), \tag{9}$$

$$\sum_{j=1}^{\infty} \frac{\kappa^{3j}}{2^j}\varphi_1\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) := \Psi(x,y,z) < \infty,$$

$$\rho(QD_f(x,y)) \leqslant \varphi_2(x,y), \tag{10}$$

$$\lim_{n\to\infty} \kappa^{4n}\varphi_2(2^{-n}x, 2^{-n}y) = 0$$

for all $x,y,z \in \chi_\rho$ and all $\lambda \in \Lambda$. If for each $x \in \chi_\rho$ the mapping $r \to f(rx)$ from $\mathbb{R}$ to χ_ρ is continuous, then there exists a unique quadratic Lie $*$-derivation $F_2 : \chi_\rho \to \chi_\rho$ satisfies the equation (1) and

$$\rho(f(x) - F_2(x)) \leqslant \frac{1}{2\kappa}\Psi(x,x,0) \tag{11}$$

for all $x \in \chi_\rho$.

Proof. First, note that $\sum_{j=1}^{\infty} \frac{\kappa^{3j}}{2^j}\varphi_1(0,0,0) = \Psi(0,0,0) < \infty$ and $\rho(QE_f^1(0,0)) \leqslant \varphi_1(0,0,0)$ lead to $\varphi_1(0,0,0) = 0$, $QE_f^1(0,0) = 0$ and so $f(0) = 0$. Thus, it follows from (7) that

$$\rho\left(f(x) - 4f\left(\frac{x}{2}\right)\right) \leqslant \varphi_1\left(\frac{x}{2}, \frac{x}{2}, 0\right)$$

for all $x \in \chi_\rho$. Thus, one obtains the following inequality by the convexity of the modular ρ and Δ_2-condition

$$\begin{aligned}\rho(f(x)-4^2f(\frac{x}{2^2})) &\leqslant \frac{1}{2}\rho\Big(2f(x)-2\cdot 4f(\frac{x}{2})\Big)+\frac{1}{2^2}\rho\Big(2^2\cdot 4f(\frac{x}{2})-2^2\cdot 4^2f(\frac{x}{2^2})\Big)\\ &\leqslant \frac{\kappa}{2}\varphi_1\Big(\frac{x}{2},\frac{x}{2},0\Big)+\frac{\kappa^4}{2^2}\varphi_1\Big(\frac{x}{2^2},\frac{x}{2^2},0\Big)\end{aligned}$$

for all $x \in \chi_\rho$. Then using the repeating process for any $n \geqslant 2$, we prove the following functional inequality

$$\rho(f(x)-4^nf(\frac{x}{2^n})) \leqslant \frac{1}{\kappa^2}\sum_{j=1}^{n}\frac{\kappa^{3j}}{2^j}\varphi\Big(\frac{x}{2^j},\frac{x}{2^j},0\Big) \tag{12}$$

for all $x \in \chi_\rho$. In fact, it is true for $j=2$. Assume that the inequality (12) holds true for n. Thus, using the convexity of the modular ρ, we deduce

$$\begin{aligned}&\rho(f(x)-4^{n+1}f(\frac{x}{2^{n+1}}))\\ &=\rho\Big(\frac{1}{2}\{2f(x)-2\cdot 4f(\frac{x}{2})\}+\frac{1}{2}\{2\cdot 4f(\frac{x}{2})-2\cdot 4^{n+1}f(\frac{x}{2^{n+1}})\}\Big)\\ &\leqslant \frac{\kappa}{2}\rho\Big(f(x)-4f(\frac{x}{2})\Big)+\frac{\kappa^3}{2}\rho\Big(f(\frac{x}{2})-4^nf(\frac{x}{2^{n+1}})\Big)\\ &\leqslant \frac{\kappa}{2}\varphi_1\Big(\frac{x}{2},\frac{x}{2},0\Big)+\frac{\kappa^3}{2}\cdot\frac{1}{\kappa^2}\sum_{j=1}^{n}\frac{\kappa^{3j}}{2^j}\varphi_1\Big(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},0\Big)\\ &=\frac{\kappa}{2}\varphi_1\Big(\frac{x}{2},\frac{x}{2},0\Big)+\frac{1}{\kappa^2}\sum_{j=1}^{n}\frac{\kappa^{3(j+1)}}{2^{j+1}}\varphi_1\Big(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},0\Big)\\ &=\frac{1}{\kappa^2}\sum_{j=1}^{n+1}\frac{\kappa^{3j}}{2^j}\varphi_1\Big(\frac{x}{2^j},\frac{x}{2^j},0\Big),\end{aligned}$$

which proves (12) for $n+1$. Now, replacing x by $2^{-m}x$ in (12), we have

$$\begin{aligned}\rho\Big(2^{2m}f(\frac{x}{2^m})-2^{2(m+n)}f(\frac{x}{2^{m+n}})\Big) &\leqslant \kappa^{2m}\rho\Big(f(\frac{x}{2^m})-2^{2n}f(\frac{x}{2^{m+n}})\Big)\\ &\leqslant \frac{\kappa^{2m}}{\kappa^2}\sum_{j=1}^{n}\frac{\kappa^{3j}}{2^j}\varphi_1\Big(\frac{x}{2^{j+m}},\frac{x}{2^{j+m}},0\Big)\\ &\leqslant \frac{\kappa^{2m}}{\kappa^2}\sum_{j=1}^{n}\frac{\kappa^{3j}}{2^j}\varphi_1\Big(\frac{x}{2^{j+m}},\frac{x}{2^{j+m}},0\Big)\cdot\frac{\kappa^m}{2^m}\\ &=\frac{1}{\kappa^2}\sum_{j=1}^{n}\frac{\kappa^{3(j+m)}}{2^{j+m}}\varphi_1\Big(\frac{x}{2^{j+m}},\frac{x}{2^{j+m}},0\Big)\\ &=\frac{1}{\kappa^2}\sum_{j=m+1}^{m+n}\frac{\kappa^{3j}}{2^j}\varphi_1\Big(\frac{x}{2^j},\frac{x}{2^j},0\Big),\end{aligned}$$

which converges to zero as $m \to \infty$ by the assumption (10). Thus, the sequence $\{4^n f(\frac{x}{2^n})\}$ is ρ-Cauchy for all $x \in \chi_\rho$ and so it is ρ-convergent in χ_ρ since the space χ_ρ is ρ-complete. Thus, we may define a mapping $F_2 : \chi_\rho \to \chi_\rho$ as

$$F_2(x) := \rho - \lim_{n\to\infty} 4^n f(\frac{x}{2^n}) \Longleftrightarrow \lim_{n\to\infty} \rho\left(4^n f(\frac{x}{2^n}) - F_2(x)\right) = 0,$$

for all $x \in \chi_\rho$.

Now, we prove the mapping F_2 satisfies the equation (1). Letting $z := 0$ and setting $(x,y) := (2^{-n}x, 2^{-n}y)$ in (9), and then multiplying the resulting inequality by 4^n, we get

$$\begin{aligned} \rho(2^{2n} QE_f^\lambda(2^{-n}x, 2^{-n}y)) &\leqslant \kappa^{2n}\varphi_1(2^{-n}x, 2^{-n}y, 0) \\ &\leqslant \kappa^{2n}\varphi_1(2^{-n}x, 2^{-n}y, 0) \cdot \frac{\kappa^n}{2^n} \\ &= \frac{\kappa^{3n}}{2^n}\varphi_1(2^{-n}x, 2^{-n}y, 0), \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x, y \in \chi_\rho$. Thus, it follows from Remark 1 that

$$\begin{aligned} &\rho(\frac{1}{9} QE_{F_2}^\lambda(x,y)) \\ &= \rho\left(\frac{1}{9} QE_{F_2}^\lambda(x,y) - \frac{1}{9} 2^{2n} QE_f^\lambda(\frac{x}{2^n}, \frac{y}{2^n}) + \frac{1}{9} 2^{2n} QE_f^\lambda(\frac{x}{2^n}, \frac{y}{2^n})\right) \\ &\leqslant \frac{1}{9}\rho\left(F_2(2\lambda x - \lambda y) - 2^{2n} f(\frac{2\lambda x - \lambda y}{2^n})\right) + \frac{1}{9}\rho\left(F_2(\lambda x + \lambda y) - 2^{2n} f(\frac{\lambda x + \lambda y}{2^n})\right) \\ &\quad + \frac{\lambda^2}{9}\rho\left(F_2(x-y) - 2^{2n} f(\frac{x-y}{2^n})\right) + \frac{4\lambda^2}{9}\rho\left(F_2(x) - 2^{2n} f(\frac{x}{2^n})\right) \\ &\quad + \frac{\lambda^2}{9}\rho\left(F_2(y) - 2^{2n} f(\frac{y}{2^n})\right) + \frac{1}{9}\rho\left(2^{2n} QE_f^\lambda(\frac{x}{2^n}, \frac{y}{2^n})\right) \end{aligned}$$

for all $x, y \in \chi_\rho$ and all positive integers n. Taking the limit as $n \to \infty$, one obtains

$$QE_{F_2}^\lambda(x,y) = 0$$

for all $x, y \in \chi_\rho$ and all $\lambda \in \Lambda$. Hence F_2 satisfies the equation (1), and so it is quadratic.

Now, we prove that F_2 is a quadratic Lie $*$-derivation. It is easy to see that the mapping F_2 is quadratic homogeneous by the same reasoning as in Theorem 1. From the last inequality in (10) and the last condition in (9), it follows that

$$\begin{aligned}
&\rho\Big(\frac{1}{4}QD_{F_2}(x,y)\Big)\\
&=\rho\Big(\frac{1}{4}QD_{F_2}(x,y)-4^{2n}\frac{QD_f(2^{-n}x,2^{-n}y)}{4}+4^{2n}\frac{QD_f(2^{-n}x,2^{-n}y)}{4}\Big)\\
&\leqslant\frac{1}{4}\rho\Big(F_2([x,y])-4^{2n}f(2^{-2n}[x,y])\Big)+\frac{1}{4}\rho\Big([x^2,4^nf(2^{-n}y)]-[x^2,F_2(y)]\Big)\\
&\quad+\frac{1}{4}\rho\Big([4^nf(2^{-n}x),y^2]-[F_2(x),y^2]\Big)+\frac{1}{4}\rho\Big(4^{2n}QD_f(2^{-n}x,2^{-n}y)\Big)\\
&\leqslant\frac{1}{4}\rho\Big(F_2([x,y])-4^{2n}f(2^{-2n}[x,y])\Big)+\frac{1}{4}\rho\Big([x^2,4^nf(2^{-n}y)-F_2(y)]\Big)\\
&\quad+\frac{1}{4}\rho\Big([4^nf(2^{-n}x)-F_2(x),y^2]\Big)+\frac{\kappa^{4n}}{4}\varphi_2\Big(2^{-n}x,2^{-n}y\Big)
\end{aligned}$$

for all $x,y\in\chi_\rho$, from which $QD_{F_2}(x,y)=0$ by taking $n\to\infty$, and so F_2 is a quadratic Lie derivation. In addition, it follows from the definition of F_2 that the following inequality

$$\begin{aligned}
\rho\Big(\frac{1}{3}\Big(F_2(z^*)-F_2(z)^*\Big)\Big)&\leqslant\frac{1}{3}\rho\Big(F_2(z^*)-4^nf\Big(\frac{z^*}{2^n}\Big)\Big)\\
&\quad+\frac{1}{3}\rho\Big(4^nf\Big(\frac{z}{2^n}\Big)^*-F_2(z)^*\Big)+\frac{1}{3}\rho\Big(4^nf\Big(\frac{z^*}{2^n}\Big)-4^nf\Big(\frac{z}{2^n}\Big)^*\Big)\\
&\leqslant\frac{1}{3}\rho\Big(F_2(z^*)-4^nf\Big(\frac{z^*}{2^n}\Big)\Big)\\
&\quad+\frac{1}{3}\rho\Big(4^nf\Big(\frac{z}{2^n}\Big)^*-F_2(z)^*\Big)+\frac{\kappa^{2n}}{3}\varphi_1\Big(0,0,\frac{z}{2^n}\Big)\cdot\frac{\kappa^n}{2^n}
\end{aligned}$$

holds for all vectors z, which goes to zero as $n\to\infty$. Hence, one concludes F_2 is a quadratic Lie $*$-derivation.

On the other hand, one can see the following inequality

$$\begin{aligned}
\rho(f(x)-F_2(x))&=\rho\Big(\frac{1}{2}\{2f(x)-2\cdot4^nf(\frac{x}{2^n})\}+\frac{1}{2}\{2\cdot4^nf(\frac{x}{2^n})-2F_2(x)\}\Big)\\
&\leqslant\frac{\kappa}{2}\rho\Big(f(x)-4^nf(\frac{x}{2^n})\Big)+\frac{\kappa}{2}\rho\Big(4^nf(\frac{x}{2^n})-F_2(x)\Big)\\
&\leqslant\frac{\kappa}{2}\cdot\frac{1}{\kappa^2}\sum_{j=1}^{n}\frac{\kappa^{3j}}{2^j}\varphi_1(\frac{x}{2^j},\frac{x}{2^j},0)+\frac{\kappa}{2}\rho\Big(4^nf(\frac{x}{2^n})-F_2(x)\Big)\\
&\leqslant\frac{1}{2\kappa}\sum_{j=1}^{\infty}\frac{\kappa^{3j}}{2^j}\varphi_1(\frac{x}{2^j},\frac{x}{2^j},0)=\frac{1}{2\kappa}\Psi(x,x,0),
\end{aligned}$$

by Δ_2-condition without using the Fatou property for all positive integers n, which yields the approximation (11) by taking $n\to\infty$.

Finally, applying the same argument as in the proof of Theorem [9], we prove the uniqueness of F_2 satisfying the approximation (11) near f. Hence, the mapping F_2 is a unique quadratic Lie $*$-derivation satisfying the estimation (11) near f.

REMARK 2. In Theorem 2, if χ_ρ is a Banach $*$-algebra with norm $\|\cdot\| := \rho$, and so $\rho(2x) = 2\rho(x)$, $\kappa := 2$, then we see from (9) and (10) that there exists a unique quadratic Lie $*$-derivation $F_2 : \chi_\rho \to \chi_\rho$, defined as $F_2(x) = \lim_{n\to\infty} 2^{2n} f(\frac{x}{2^n})$, $x \in \chi_\rho$, which satisfies the equation (1) and

$$\rho(f(x) - F_2(x)) \leqslant \frac{1}{4}\sum_{j=1}^{\infty} 2^{2j}\varphi_1(\frac{x}{2^j}, \frac{x}{2^j}, 0)$$

for all $x \in \chi_\rho$.

As a corollary of Theorem 2, we obtain the following stability result of the equation (1) associated with quadratic Lie $*$-derivations, which generalizes stability result in normed $*$-algebras.

COROLLARY 3. *Let χ_ρ be a complete normed $*$-algebra. For given nonnegative real numbers θ_i, ϑ_i together with $2 < r_i$ $(i = 1,2,3)$ and a,b with $2 < a+b$, suppose a mapping $f : \chi_\rho \to \chi_\rho$ satisfies*

$$\begin{aligned} \|QE_f^\lambda(x,y) + f(z^*) - f(z)^*\| &\leqslant \theta_1\|x\|^{r_1} + \theta_2\|y\|^{r_2} + \theta_3(\|x\|^a\|y\|^b + \|z\|^{r_3}), \\ \|QD_f(x,y)\| &\leqslant \vartheta_1\|x\|^{2r_1} + \vartheta_2\|y\|^{2r_2} + \vartheta_3\|x\|^{2a}\|y\|^{2b} \end{aligned}$$

for all $x,y,z \in \chi_\rho$ and all $\lambda \in \Lambda$. If for each $x \in \chi_\rho$ the mapping $r \to f(rx)$ from $\mathbb{R}$ to χ_ρ is continuous, then there exists a unique quadratic Lie $*$-derivation $F_2 : \chi_\rho \to \chi_\rho$ such that

$$\rho(f(x) - F_2(x)) \leqslant \frac{\theta_1\|x\|^{r_1}}{2^{r_1} - 2^2} + \frac{\theta_2\|x\|^{r_2}}{2^{r_2} - 2^2} + \frac{\theta_3\|x\|^{a+b}}{2^{a+b} - 2^2}$$

for all $x \in \chi_\rho$.

COROLLARY 4. *Let χ_ρ be a ρ-complete convex modular $*$-algebra with Δ_2-condition. Suppose there exist two functions $\varphi_1 : \chi_\rho^3 \to [0,\infty)$ and $\varphi_2 : \chi_\rho^2 \to [0,\infty)$ and two constant l_i with $0 < l_1 < \frac{8}{\kappa^3}$ and $0 < l_2 < \frac{16}{\kappa^4}$ for which a mapping $f : \chi_\rho \to \chi_\rho$ satisfies*

$$\begin{aligned} \rho(QE_f^\lambda(x,y) + f(z^*) - f(z)^*) &\leqslant \varphi_1(x,y,z), \ \varphi_1(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) \leqslant \frac{l_1}{4}\varphi_1(x,y,z), \\ \rho(QD_f(x,y)) &\leqslant \varphi_2(x,y), \quad \varphi_2(\frac{x}{2}, \frac{y}{2}) \leqslant \frac{l_2}{16}\varphi_2(x,y) \end{aligned}$$

for all $x,y,z \in \chi_\rho$ and all $\lambda \in \Lambda$. If for each $x \in \chi_\rho$ the mapping $r \to f(rx)$ from $\mathbb{R}$ to χ_ρ is continuous, then there exists a unique quadratic Lie $*$-derivation $F_2 : \chi_\rho \to \chi_\rho$ satisfies the equation (1) and

$$\rho(f(x) - F_2(x)) \leqslant \frac{\kappa^2 l_1}{2(8 - \kappa^3 l_1)}\varphi_1(x,x,0)$$

for all $x \in \chi_\rho$.

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