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A NEW FAMILY OF WEIGHTED OPERATOR MEANS INCLUDING THE WEIGHTED HERON, LOGARITHMIC AND HEINZ MEANS

MASATOSHI ITO

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Abstract. The weighted power and Heron means are well known as generalizations of the weighted arithmetic, geometric and harmonic ones, and also some researchers investigate weighted means except them. Recently, Pal, Singh, Moslehian and Aujla introduced the weighted logarithmic mean of two positive numbers or operators.

In this paper, we propose the notion of a transpose symmetric path of weighted $\mathfrak{M}$-means for a symmetric operator mean $\mathfrak{M}$, and we introduce a new family of operator means including the weighted logarithmic mean by Pal et al.. This family newly produces the weighted Heinz mean. Moreover we obtain some relations among the weighted Heron, logarithmic and Heinz means.

1. Introduction

The means are researched and used in many branches. As fundamental ones, the arithmetic, geometric and harmonic means are defined by $\frac{a+b}{2}$, $\sqrt{ab}$ and $\frac{2ab}{a+b}$ for two positive real numbers a and b, respectively. It is also well known that these means have their weighted versions as follows: For $t \in [0,1]$,

$$A_t(a,b) = (1-t)a + tb \quad \text{(arithmetic mean)},$$
$$G_t(a,b) = a^{1-t}b^t \quad \text{(geometric mean)},$$
$$H_t(a,b) = \{(1-t)a^{-1} + tb^{-1}\}^{-1} \quad \text{(harmonic mean)}.$$

If the weight parameter t is equal to $\frac{1}{2}$, then the weighted means coincide with the original (non-weighted) ones, and then we abbreviate the weight t as $A(a,b) = A_{\frac{1}{2}}(a,b)$. It is well known that the inequalities $H_t(a,b) \leqslant G_t(a,b) \leqslant A_t(a,b)$ always hold.

As one-parameter generalizations including the weighted arithmetic, geometric and harmonic means, the following are known. For $t \in [0,1]$ and $q \in \mathbb{R}$,

$$P_{t,[q]}(a,b) = \begin{cases} \{(1-t)a^q + tb^q\}^{\frac{1}{q}} & \text{if } q \neq 0, \\ a^{1-t}b^t & \text{if } q = 0, \end{cases} \quad \text{(power mean)},$$

$$K_{t,[q]}(a,b) = (1-q)a^{1-t}b^t + q\{(1-t)a + tb\} \quad \text{(Heron mean)}.$$

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In this paper, we denote the power mean by $P_{t,[q]}(a,b)$ to distinguish the parameter q determining the mean from the weight parameter t, and also we use the notation for the non-weighted power mean that $P_{[q]}(a,b) = P_{\frac{1}{2},[q]}(a,b)$. We remark that the other means with parameters also obey these rules. The weighted power mean $P_{t,[q]}(a,b)$ and the weighted Heron mean $K_{t,[q]}(a,b)$ are monotone increasing on $q \in \mathbb{R}$, and also $A_t(a,b) = P_{t,[1]}(a,b) = K_{t,[1]}(a,b)$, $G_t(a,b) = P_{t,[0]}(a,b) = K_{t,[0]}(a,b)$ and $H_t(a,b) = P_{t,[-1]}(a,b)$ hold.

It is also known that the non-weighted arithmetic, geometric and harmonic means have many kinds of generalizations besides the power and Heron means. For example, for $q \in \mathbb{R}$,

$$J_{[q]}(a,b) = \frac{q}{q+1}\frac{a^{q+1}-b^{q+1}}{a^q-b^q} \; (q \neq 0,-1) \text{ (power difference mean)},$$

$$HZ_{[q]}(a,b) = \frac{a^q b^{1-q} + a^{1-q} b^q}{2} \text{ (Heinz mean)},$$

We note that $J_{[0]}(a,b)$ and $J_{[-1]}(a,b)$ can be defined as the limit, and also $J_{[q]}(a,b)$ is monotone increasing on $q \in \mathbb{R}$. On the Heinz mean, $HZ_{[q]}(a,b)$ is increasing for $q \geqslant \frac{1}{2}$ and decreasing for $q \leqslant \frac{1}{2}$. Moreover, the power difference mean $J_{[q]}(a,b)$ also includes the logarithmic mean $LM(a,b) = \frac{a-b}{\log a - \log b}$ as $LM(a,b) = J_{[0]}(a,b)$. We remark that many researchers investigate estimations of a parameterized mean by another parameterized one, for example [6, 7, 17, 18].

It seems that there are not familiar weighted means except the power and Heron means. Some researchers discussed the weighted logarithmic mean in their own way in [9, 10, 13, 14]. Recently, based on the Hermite-Hadamard inequality for convex functions, Pal, Singh, Moslehian and Aujla [10] introduced the weighted logarithmic mean $LM_t(a,b)$ for $t \in [0,1]$ by

$$LM_t(a,b) = \frac{1}{\log a - \log b}\left\{\frac{1-t}{t}a^{1-t}(a^t - b^t) + \frac{t}{1-t}b^t(a^{1-t} - b^{1-t})\right\},$$

and also they showed that the inequalities

$$H_t(a,b) \leqslant G_t(a,b) \leqslant LM_t(a,b) \leqslant K_{t,[\frac{1}{2}]}(a,b) \leqslant A_t(a,b) \tag{1.1}$$

always hold.

We can extend above discussion for bounded linear operators on a Hilbert space $\mathscr{H}$. An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in \mathscr{H}$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. We denote the set of positive operators by $B^+(\mathscr{H})$. A real-valued function f defined on $J \subset \mathbb{R}$ is said to be operator monotone if $A \leqslant B$ implies $f(A) \leqslant f(B)$ for selfadjoint operators A and B whose spectra $\sigma(A), \sigma(B) \subset J$, where $A \leqslant B$ means $B - A \geqslant 0$.

Kubo and Ando [8] constructed the general theory of operator means. In [8], they obtained that there exists a one-to-one correspondence between an operator mean $\mathfrak{M}$

and an operator monotone function $f \geqslant 0$ on $[0,\infty)$ with $f(1)=1$ via $f(x)I = \mathfrak{M}(I,xI)$ as follows:

$$\mathfrak{M}(A,B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} \tag{1.2}$$

if $A>0$ and $B \geqslant 0$. We remark that f is called the representing function of $\mathfrak{M}$, and also it is permitted to consider binary operations given by (1.2) even if f is a general real-valued function. By (1.2), we can introduce the following weighted operator means for two strictly positive operators A and B. For example, for $t \in [0,1]$,

$$\mathfrak{A}_t(A,B) = (1-t)A + tB \quad \text{(arithmetic mean)},$$

$$\mathfrak{G}_t(A,B) = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^t A^{\frac{1}{2}} \quad \text{(geometric mean)},$$

$$\mathfrak{H}_t(A,B) = \{(1-t)A^{-1} + tB^{-1}\}^{-1} \quad \text{(harmonic mean)},$$

$$\mathfrak{P}_{t,[q]}(A,B) = \begin{cases} A^{\frac{1}{2}}\{(1-t)I + t(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^q\}^{\frac{1}{q}} A^{\frac{1}{2}} & \text{if } 0<|q| \leqslant 1, \\ A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^t A^{\frac{1}{2}} & \text{if } q=0, \end{cases} \quad \text{(power mean)}.$$

We remark that their representing functions are $A_t(1,x)$, $G_t(1,x)$ (denoted by $A_t(x)$, $G_t(x)$) and so on. Similarly, we can introduce the operator mean $\mathfrak{M}$ corresponding to the representing function $M(1,x)$ (denoted by $M(x)$) by the numerical mean M if $M(1,x)$ is operator monotone. Refer to [12] for more details on operator means. Here we also remark that the power difference mean $\mathfrak{J}_{[q]}(A,B)$ is an operator mean if $-2 \leqslant q \leqslant 1$ (see [3, 4, 5]).

In this paper, we discuss a new family of operator means. Firstly, from the viewpoint of the representing functions of operator means, we propose the notion of a transpose symmetric path of weighted $\mathfrak{M}$-means. Secondly, we introduce a new family of operator means including the weighted logarithmic mean by Pal et al. and the weighted Heron mean. This family newly produces the weighted Heinz mean, and we get some relations among the weighted Heron, logarithmic and Heinz means. Thirdly, we obtain the results on estimations of the weighted logarithmic mean via the power difference mean.

2. A transpose symmetric path of weighted $\mathfrak{M}$-means

In this section, we discuss the definition of weighted means. Throughout this paper, a function $M : [0,\infty) \times [0,\infty) \to [0,\infty)$ is called a (numerical) mean if M satisfies the following four properties.

(i) $M(a,b)$ is monotone increasing in both a and b (monotonicity),

(ii) $M(\alpha a, \alpha b) = \alpha M(a,b)$ for all $\alpha > 0$ (homogeneity),

(iii) $M(a,b)$ is continuous in a and b (continuity),

(iv) $M(a,a) = a$ for all a (normalization).

We remark that (i) and (iv) imply $\min\{a,b\} \leqslant M(a,b) \leqslant \max\{a,b\}$. We can easily verify that the means introduced in the previous section satisfy the above (i)–(iv). A mean M is said to be symmetric if $M(a,b) = M(b,a)$ (symmetry) holds. The weighted means in Section 1 are not symmetric except the case $t = \frac{1}{2}$, but they have the property that $M_t(a,b) = M_{1-t}(b,a)$ (transpose symmetry) holds for all $t \in [0,1]$ instead of symmetry. Transpose symmetry is called conjugate symmetry in [13].

Kubo and Ando [8] discussed an axiomatic approach to operator means. A binary operation $\mathfrak{M} : \mathscr{B}^+(\mathscr{H}) \times \mathscr{B}^+(\mathscr{H}) \to \mathscr{B}^+(\mathscr{H})$ is called an operator mean if the following conditions are satisfied:

(i) $A \leqslant C$ and $B \leqslant D$ imply $\mathfrak{M}(A,B) \leqslant \mathfrak{M}(C,D)$ (monotonicity),

(ii) $T^*\mathfrak{M}(A,B)T \leqslant \mathfrak{M}(T^*AT, T^*BT)$ for every operator T (transformer inequality),

(iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $\mathfrak{M}(A_n,B_n) \downarrow \mathfrak{M}(A,B)$ (upper semicontinuity),

(iv) $\mathfrak{M}(I,I) = I$ (normalization),

where $A_n \downarrow A$ means that $A_1 \geqslant A_2 \geqslant \cdots$ and $A_n \to A$ in the strong operator topology as $n \to \infty$. We remark that (ii) leads $T^*\mathfrak{M}(A,B)T = \mathfrak{M}(T^*AT, T^*BT)$ (transformer equality) if T is invertible, and the transformer equality ensures that $\mathfrak{M}(\alpha A, \alpha B) = \alpha\mathfrak{M}(A,B)$ (homogeneity) holds for all $\alpha > 0$. It is also obtained in [8] that there exists a one-to-one correspondence between an operator mean $\mathfrak{M}$ and its representing function f as (1.2). For two operator means $\mathfrak{M}$ and $\mathfrak{N}$, $\mathfrak{M} \leqslant \mathfrak{N}$ (resp. $\mathfrak{M} = \mathfrak{N}$) means that $\mathfrak{M}(A,B) \leqslant \mathfrak{N}(A,B)$ (resp. $\mathfrak{M}(A,B) = \mathfrak{N}(A,B)$) for all $A,B > 0$. We define symmetry and transpose symmetry for operator means similarly to the numerical case. For an operator mean $\mathfrak{M}$ and its representing function f, the operator mean whose representing function is $xf(x^{-1})$ is called transpose of $\mathfrak{M}$, and we denote it by $\mathfrak{M}^\circ$. We easily obtain that $\mathfrak{M}^\circ(A,B) = \mathfrak{M}(B,A)$ for $A,B > 0$, and also an operator mean $\mathfrak{M}$ is symmetric if and only if $\mathfrak{M} = \mathfrak{M}^\circ$ if and only if $f(x) = xf(x^{-1})$ for all $x > 0$.

From this argument, we can discuss numerical means and operator means simultaneously via the representing function, so we write definitions and theorems in terms of operator means from now on. We remark that we have to pay attention to operator monotonicity when we treat operator means.

Next, we discuss what is the natural definition of "weighted" means. We can consider plural weighted means from one symmetric mean. In fact, the weighted logarithmic mean is defined by several ways in [9, 10, 13, 14]. Moreover, in [1, 2, 11, 16], they discussed the algorithms to make weighted operator means from a given operator mean. In [16], Udagawa, Yamazaki and Yanagida introduced the α-weighted operator mean as an expression of weight parameters (see also [11]).

DEFINITION 2.1 ([16]) *The operator mean $\mathfrak{M}$ is said to be α-weighted if $f'(1) = \alpha$, where $f(x)$ is the representing function of $\mathfrak{M}$.*

The weighted means introduced in Section 1, $\mathfrak{A}_t(a,b)$, $\mathfrak{P}_{t,[q]}(a,b)$ and so on, are all t-weighted. We can easily obtain $0 \leqslant \alpha \leqslant 1$ in Definition 2.1 since $\min\{1,x\} \leqslant$

$f(x) \leqslant \max\{1,x\}$ holds for all $x>0$. Every symmetric mean is $\frac{1}{2}$-weighted, but the converse does not hold in general by the operator mean whose representing function is

$$f(x) = \frac{2}{3}x^{\frac{2}{3}} + \frac{1}{3}x^{\frac{1}{6}}.$$

Here we assume that $\mathfrak{M}$ is a symmetric $\frac{1}{2}$-weighted mean, and we consider a path of t-weighted means $\mathfrak{M}_t(A,B)$ for $t \in [0,1]$ from A (0-weighted mean) to B (1-weighted mean) such that $\mathfrak{M}_{\frac{1}{2}}(A,B) = \mathfrak{M}(A,B)$. Raïssouli and Sándor [14] introduced the notion of the weighted M-mean for a one-parameter family of numerical weighted means $\{M_t\}_{t\in[0,1]}$. Here, as a refinement of their definition considering Definition 2.1, we introduce the notion of a transpose symmetric path of weighted $\mathfrak{M}$-means.

DEFINITION 2.2 *Let $\mathfrak{M}$ be a symmetric operator mean and $A,B>0$. If the following conditions hold, then $\mathfrak{M}_t$ is said to be a weighted $\mathfrak{M}$-mean, and a one-parameter family $\{\mathfrak{M}_t\}_{t\in[0,1]}$ is said to be a transpose symmetric path of weighted $\mathfrak{M}$-means.*

(i) $\mathfrak{M}_t$ *is an operator mean for all fixed* $t\in[0,1]$.

(ii) $\mathfrak{M}_0(A,B) = A$, $\mathfrak{M}_{\frac{1}{2}}(A,B) = \mathfrak{M}(A,B)$ *and* $\mathfrak{M}_1(A,B) = B$.

(iii) $\mathfrak{M}_t(A,B) = \mathfrak{M}_{1-t}(B,A)$ *for all* $t\in[0,1]$ *(transpose symmetry).*

(iv) $\mathfrak{M}_t$ *is t-weighted for all fixed* $t\in[0,1]$.

In Definition 2.2, (iii) holds if and only if $\mathfrak{M}_t = \mathfrak{M}^{\circ}_{1-t}$ if and only if $f_t(x) = xf_{1-t}(x^{-1})$ for all $x>0$, where $f_t(x)$ is the representing function of $\mathfrak{M}_t$. We note that (iii) ensures symmetry of $\mathfrak{M}_{\frac{1}{2}}$. The families of weighted means introduced in Section 1, $\{\mathfrak{A}_t\}_{t\in[0,1]}$, $\{\mathfrak{P}_{t,[q]}\}_{t\in[0,1]}$ and so on, are all transpose symmetric paths of weighted means. We remark that the weight parameter t in $\{\mathfrak{M}_t\}_{t\in[0,1]}$ does not always equal $f_t'(1)$ even if (i)–(iii) hold in Definition 2.2. For example, we consider an operator mean $\mathfrak{M}_t$ whose representing function is

$$f_t(x) = x^{2t-3t^2+2t^3}\,(= G_{2t-3t^2+2t^3}(x)).$$

Then $\mathfrak{M}_t$ satisfies (i)–(iii) in Definition 2.2, but $\mathfrak{M}_t$ is not t-weighted but $(2t-3t^2+2t^3)$-weighted.

3. General results

For a symmetric operator mean $\mathfrak{M}$ with a representing function ψ, we consider a weighted $\mathfrak{M}$-mean $\mathfrak{M}_t$ whose representing function is

$$\psi_t(x) = \begin{cases} 1-2t+2t\psi(x) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ (2t-1)x + 2(1-t)\psi(x) & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Then $\{\mathfrak{M}_t\}_{t\in[0,1]}$ is a transpose symmetric path of weighted $\mathfrak{M}$-means. But this path does not include fundamental weighted operator means except the weighted arithmetic mean. In this section, we construct a transpose symmetric path of weighted $\mathfrak{M}$-means including some fundamental weighted means stated in Section 1.

Let $\{\mathfrak{M}_t\}_{t\in[0,1]}$ be a transpose symmetric path of weighted $\mathfrak{M}$-means and

$$\mathscr{R} = \left\{\{f_t\}_{t\in[0,1]} : f_t \text{ is the representing function of } \mathfrak{M}_t \in \{\mathfrak{M}_t\}_{t\in[0,1]}\right\}.$$

We denote $\{f_t\}_{t\in[0,1]}$ by $\{f_t\}$ briefly. Now we consider the following function.

DEFINITION 3.1 *Let $\{\varphi_s\} \in \mathscr{R}$. Then we define a function $m_t[\varphi_s] : [0,\infty) \to [0,\infty)$ as*

$$m_t[\varphi_s](x) = (1-t)\varphi_{1-s}(x^t) + tx^t\varphi_s(x^{1-t}) \quad \textit{for } t,s \in [0,1].$$

In particular, if φ is the representing function of a symmetric mean, then we define

$$m_t[\varphi](x) = (1-t)\varphi(x^t) + tx^t\varphi(x^{1-t}) \quad \textit{for } t \in [0,1].$$

Then the function $m_t[\varphi_s]$ makes a transpose symmetric path of weighted $\mathfrak{M}[\varphi_s]$-means by the following Theorem 3.1. We recognize that t and s in $m_t[\varphi_s]$ express the weight parameter and the parameter determining the path, respectively. We remark that we do not have to consider a one-parameter family $\{\varphi_s\} \in \mathscr{R}$ if we choose $s = \frac{1}{2}$, the representing function φ of a symmetric mean, in Definition 3.1.

THEOREM 3.1 *Let $\{\varphi_s\} \in \mathscr{R}$ and $m_t[\varphi_s]$ be as in Definition 3.1. Let $\mathfrak{M}_t[\varphi_s]$ be the binary operation whose representing function is $m_t[\varphi_s]$, and also $\mathfrak{M}[\varphi_s] = \mathfrak{M}_{\frac{1}{2}}[\varphi_s]$. Then the family $\{\mathfrak{M}_t[\varphi_s]\}_{t\in[0,1]}$ is a transpose symmetric path of weighted $\mathfrak{M}[\varphi_s]$-means.*

Proof. We shall verify that $\mathfrak{M}_t[\varphi_s]$ has four properties in Definition 2.2 by using its representing function $m_t[\varphi_s]$.

(i) Operator monotonicity of $m_t[\varphi_s]$ is obtained since $\varphi_{1-s}(x^t)$, $\varphi_s(x^{1-t})^{\frac{1}{1-t}}$ and

$$x^t\varphi_s(x^{1-t}) = x^t\{\varphi_s(x^{1-t})^{\frac{1}{1-t}}\}^{1-t}$$

are operator monotone for $t \in [0,1]$ (e.g. [15]), and we have $m_t[\varphi_s](1) = 1$ obviously.

(ii) We easily get that $m_0[\varphi_s](x) = 1$ and $m_1[\varphi_s](x) = x$, and also $m_{\frac{1}{2}}[\varphi_s](x)$ is the representing function of $\mathfrak{M}[\varphi_s]$ obviously.

(iii) The equality $m_t[\varphi_s](x) = xm_{1-t}[\varphi_s](x^{-1})$ holds since

$$\begin{aligned} xm_{1-t}[\varphi_s](x^{-1}) &= x\left[\{1-(1-t)\}\varphi_{1-s}(x^{-(1-t)}) + (1-t)x^{-(1-t)}\varphi_s(x^{-\{1-(1-t)\}})\right] \\ &= tx^tx^{1-t}\varphi_{1-s}(x^{-(1-t)}) + (1-t)x^t\varphi_s(x^{-t}) \\ &= tx^t\varphi_s(x^{1-t}) + (1-t)\varphi_{1-s}(x^t) \\ &= m_t[\varphi_s](x). \end{aligned}$$

(iv) We obtain $m_t'[\varphi_s](1) = t$ since

$$m_t'[\varphi_s](x) = t(1-t)x^{t-1}\varphi_{1-s}'(x^t) + t\{tx^{t-1}\varphi_s(x^{1-t}) + (1-t)\varphi_s'(x^{1-t})\}$$

holds. □

We have the following fundamental properties of the weighted operator mean $\mathfrak{M}_t[\varphi_s]$ in Theorem 3.1. For $A, B > 0$ and binary operations $\mathfrak{M}, \mathfrak{N}$ given by (1.2), we note that $(\mathfrak{M} + \mathfrak{N})(A,B) \equiv \mathfrak{M}(A,B) + \mathfrak{N}(A,B)$ and $(\alpha\mathfrak{M})(A,B) \equiv \alpha\mathfrak{M}(A,B)$ $(\alpha > 0)$.

THEOREM 3.2 *Let* $\{\varphi_s\}, \{\psi_s\} \in \mathscr{R}$*. If* $\varphi_s \leqslant \psi_s$ *for each* $s \in [0,1]$*, then* $\mathfrak{M}_t[\varphi_s] \leqslant \mathfrak{M}_t[\psi_s]$ *for all* $t \in [0,1]$*.*

Proof. It is immediately obtained by Definition 3.1. □

THEOREM 3.3 *Let* $\{\psi_s^{(j)}\} \in \mathscr{R}$ *and* $\alpha_j > 0$ *for* $j = 1, \ldots, n$ *with* $\alpha_1 + \cdots + \alpha_n = 1$*. If* $\varphi_s = \alpha_1 \psi_s^{(1)} + \cdots + \alpha_n \psi_s^{(n)}$ *for each* $s \in [0,1]$*, then*

$$\mathfrak{M}_t[\varphi_s] = \alpha_1 \mathfrak{M}_t[\psi_s^{(1)}] + \cdots + \alpha_n \mathfrak{M}_t[\psi_s^{(n)}]$$

holds for $t \in [0,1]$*, and also* $\{\mathfrak{M}_t[\varphi_s]\}_{t \in [0,1]}$ *is a transpose symmetric path of weighted* $\mathfrak{M}[\varphi_s]$*-means.*

Proof. Since $\{\varphi_s\} \in \mathscr{R}$ holds, we have the desired result by Theorem 3.1 and by considering representing functions of $\mathfrak{M}_t[\varphi_s]$ and $\mathfrak{M}_t[\psi_s^{(j)}]$. □

4. Examples

A transpose symmetric path of weighted $\mathfrak{M}[\varphi_s]$-means given in Theorem 3.1 includes some weighted operator means, for example, the weighted Heron mean, the weighted logarithmic mean and the weighted Heinz mean.

Firstly we discuss $\{\mathfrak{M}_t[\varphi_s]\}_{t \in [0,1]}$ for the representing function φ of the logarithmic mean. Let $\varphi(x) = LM(x)$. Then the representing function of $\mathfrak{M}_t[LM]$ is

$$m_t[LM](x) = \frac{1}{\log x}\left\{\frac{1-t}{t}(x^t - 1) + \frac{t}{1-t}x^t(x^{1-t} - 1)\right\} = LM_t(x),$$

in particular

$$m_{\frac{1}{2}}[LM](x) = \frac{x-1}{\log x} = LM(x).$$

Therefore $m_t[LM]$ makes a transpose symmetric path of weighted $\mathfrak{LM}$-means. This weighted $\mathfrak{LM}$-mean coincides with the weighted logarithmic mean $\mathfrak{LM}_t$ introduced by Pal, Singh, Moslehian and Aujla [10], that is, $\mathfrak{M}_t[LM] = \mathfrak{LM}_t$.

Moreover, we consider $\{\mathfrak{M}_t[\varphi_s]\}_{t\in[0,1]}$ for the representing function of the weighted logarithmic mean, that is, $\varphi_s(x) = LM_s(x)$. The representing function of $\mathfrak{M}_t[LM_s]$ is

$$m_t[LM_s](x) = \frac{1-t}{t\log x}\left\{\frac{s}{1-s}(x^{t(1-s)}-1)+\frac{1-s}{s}(x^t - x^{t(1-s)})\right\} + \frac{tx^t}{(1-t)\log x}\left\{\frac{1-s}{s}(x^{(1-t)s}-1)+\frac{s}{1-s}(x^{1-t}-x^{(1-t)s})\right\},$$

in particular

$$m_{\frac{1}{2}}[LM_s](x) = \frac{1}{\log x}\left\{\frac{s}{1-s}(x^{\frac{1-s}{2}}-1)+\frac{1-s}{s}(x^{\frac{1}{2}}-x^{\frac{1-s}{2}})\right\} + \frac{x^{\frac{1}{2}}}{\log x}\left\{\frac{1-s}{s}(x^{\frac{s}{2}}-1)+\frac{s}{1-s}(x^{\frac{1}{2}}-x^{\frac{s}{2}})\right\}.$$

Next, we discuss $\{\mathfrak{M}_t[\varphi_s]\}_{t\in[0,1]}$ for the representing function φ_s of the weighted power mean. Let $\varphi_s(x) = P_{s,[q]}(x) = \{(1-s)+sx^q\}^{\frac{1}{q}}$ for $q \in [-1,1]$. Then the representing function of $\mathfrak{M}_t[P_{s,[q]}]$ is

$$m_t[P_{s,[q]}](x) = (1-t)\left\{s+(1-s)x^{tq}\right\}^{\frac{1}{q}} + tx^t\left\{(1-s)+sx^{(1-t)q}\right\}^{\frac{1}{q}}, \quad (4.1)$$

in particular

$$m_{\frac{1}{2}}[P_{s,[q]}](x) = \frac{1}{2}\left\{s+(1-s)x^{\frac{q}{2}}\right\}^{\frac{1}{q}} + \frac{1}{2}x^{\frac{1}{2}}\left\{(1-s)+sx^{\frac{q}{2}}\right\}^{\frac{1}{q}}.$$

This transpose symmetric path includes the following weighted means. Here, we can newly introduce the weighted Heinz mean.

(i) Weighted Heron mean: By putting $q = 1$ in (4.1), we have $P_{s,[1]} = A_s$ and the representing function of $\mathfrak{M}_t[A_s]$ is

$$m_t[A_s](x) = (1-s)x^t + s\{(1-t)+tx\} = K_{t,[s]}(x),$$

in particular

$$m_{\frac{1}{2}}[A_s](x) = (1-s)x^{\frac{1}{2}} + s\frac{x+1}{2} = K_{[s]}(x).$$

Therefore $m_t[A_s]$ makes a transpose symmetric path of weighted $\mathfrak{K}_{[s]}$-means. This weighted $\mathfrak{K}_{[s]}$-mean coincides with the weighted Heron mean $\mathfrak{K}_{t,[s]}$, that is, $\mathfrak{M}_t[A_s] = \mathfrak{K}_{t,[s]}$. We remark that $K_{[\frac{1}{2}]}(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 = P_{[\frac{1}{2}]}(a,b)$ is sometimes called the square-root mean, and $\mathfrak{M}_t[A]$ is regarded as the weighted square-root mean in the sense of the Heron mean.

(ii) Weighted Heinz mean: Here, we replace the parameter q of $HZ_{[q]}(a,b)$ as $\overline{HZ}_{[s]}(a,b) = HZ_{[\frac{1-s}{2}]}(a,b)$. Then $\overline{HZ}_{[0]}(a,b) = G(a,b)$ and $\overline{HZ}_{[1]}(a,b) = A(a,b)$ hold, and also $\overline{HZ}_{[s]}(a,b)$ is increasing for $s \geqslant 0$. By putting $q=0$ in (4.1), we have $P_{s,[0]} = G_s$ and the representing function of $\mathfrak{M}_t[G_s]$ is

$$m_t[G_s](x) = (1-t)x^{(1-s)t} + tx^{t+s(1-t)} = x^{(1-s)t}\{(1-t)+tx^s\},$$

in particular

$$m_{\frac{1}{2}}[G_s](x) = \frac{x^{\frac{1-s}{2}} + x^{\frac{1+s}{2}}}{2} = \overline{HZ}_{[s]}(x).$$

Therefore $m_t[G_s]$ makes a transpose symmetric path of weighted $\overline{\mathfrak{H}\mathfrak{Z}}_{[s]}$-means, so that we can define the weighted Heinz mean $\overline{\mathfrak{H}\mathfrak{Z}}_{t,[s]}$ and its representing function $\overline{HZ}_{t,[s]}(x)$ as

$$\begin{aligned}
&\overline{\mathfrak{H}\mathfrak{Z}}_{t,[s]} = \mathfrak{M}_t[G_s],\\
&\overline{HZ}_{t,[s]}(x) = m_t[G_s](x) = (1-t)x^{t-st} + tx^{t+s(1-t)}.
\end{aligned}$$

(iii) By putting $q=-1$ in (4.1), we have $P_{s,[-1]} = H_s$ and the representing function of $\mathfrak{M}_t[H_s]$ is

$$m_t[H_s](x) = (1-t)\left\{s+(1-s)x^{-t}\right\}^{-1} + tx^t\left\{(1-s)+sx^{-(1-t)}\right\}^{-1},$$

in particular

$$m_{\frac{1}{2}}[H_s](x) = \frac{1}{2}\left\{s+(1-s)x^{\frac{-1}{2}}\right\}^{-1} + \frac{1}{2}x^{\frac{1}{2}}\left\{(1-s)+sx^{\frac{-1}{2}}\right\}^{-1}.$$

We remark that if $s=\frac{1}{2}$, then the representing function of $\mathfrak{M}_t[H]$ is

$$m_t[H](x) = (1-t)\left(\frac{x^{-t}+1}{2}\right)^{-1} + tx^t\left(\frac{x^{-(1-t)}+1}{2}\right)^{-1},$$

in particular

$$m_{\frac{1}{2}}[H](x) = x^{\frac{1}{2}} = G(x).$$

Therefore $m_t[H]$ makes a transpose symmetric path of weighted $\mathfrak{G}$-means. Of course, this is a different weighted $\mathfrak{G}$-mean from the standard weighted geometric mean $\mathfrak{G}_t$.

By Theorem 3.2, we have

$$\mathfrak{M}_t[H_s] \leqslant \mathfrak{M}_t[G_s] \leqslant \mathfrak{M}_t[LM_s] \leqslant \mathfrak{M}_t[A_s] \tag{4.2}$$

for $t,s \in [0,1]$. (4.2) ensures the following inequalities since $x^t \leqslant m_t[G_s](x)$ holds for all $x>0$, which is a refinement of (1.1).

THEOREM 4.1 *For $t, s \in [0,1]$, the inequalities*

$$\mathfrak{H}_t \leqslant \mathfrak{G}_t \leqslant \overline{\mathfrak{HZ}}_{t,[s]} \leqslant \mathfrak{M}_t[LM_s] \leqslant \mathfrak{K}_{t,[s]} \leqslant \mathfrak{A}_t$$

hold. In particular, we have

$$\mathfrak{H}_t \leqslant \mathfrak{G}_t \leqslant \overline{\mathfrak{HZ}}_{t,[\frac{1}{2}]} \leqslant \mathfrak{LM}_t \leqslant \mathfrak{K}_{t,[\frac{1}{2}]} \leqslant \mathfrak{A}_t. \tag{4.3}$$

5. Estimations of the weighted logarithmic mean

In this section, we obtain the following estimations of $\mathfrak{LM}_t$ via the power difference mean, which are more precise than (4.3).

THEOREM 5.1 *For $t \in [0,1]$ and natural numbers n such that $n \geqslant 2$, the inequalities*

$$\begin{aligned}\overline{\mathfrak{HZ}}_{t,[\frac{1}{2}]} &\leqslant \mathfrak{M}_t[J_{[\frac{-1}{3}]}] \leqslant \cdots \leqslant \mathfrak{M}_t[J_{[\frac{-1}{n}]}] \leqslant \mathfrak{M}_t[J_{[\frac{-1}{n+1}]}] \leqslant \cdots \\ &\leqslant \mathfrak{LM}_t \leqslant \cdots \leqslant \mathfrak{M}_t[J_{[\frac{1}{n+1}]}] \leqslant \mathfrak{M}_t[J_{[\frac{1}{n}]}] \leqslant \cdots \leqslant \mathfrak{M}_t[J_{[\frac{1}{2}]}] \leqslant \mathfrak{K}_{t,[\frac{1}{2}]}\end{aligned}$$

hold, where $\mathfrak{M}_t[J_{[\frac{1}{n}]}]$ and $\mathfrak{M}_t[J_{[\frac{-1}{n}]}]$ are the weighted operator means such that

$$\mathfrak{M}_t[J_{[\frac{1}{n}]}] = \frac{1}{n+1}\left(\mathfrak{A}_t + \overline{\mathfrak{HZ}}_{t,[\frac{n-1}{n}]} + \cdots + \overline{\mathfrak{HZ}}_{t,[\frac{1}{n}]} + \mathfrak{G}_t\right),$$
$$\mathfrak{M}_t[J_{[\frac{-1}{n}]}] = \frac{1}{n-1}\left(\overline{\mathfrak{HZ}}_{t,[\frac{n-1}{n}]} + \overline{\mathfrak{HZ}}_{t,[\frac{n-2}{n}]} + \cdots + \overline{\mathfrak{HZ}}_{t,[\frac{2}{n}]} + \overline{\mathfrak{HZ}}_{t,[\frac{1}{n}]}\right).$$

Proof. We consider $\{\mathfrak{M}_t[\varphi]\}_{t\in[0,1]}$ for the representing function of the power difference mean, that is, $\varphi(x) = J_{[q]}(x) = \dfrac{q}{q+1}\dfrac{x^{q+1}-1}{x^q-1}$ for $q \in [-2,1]$. The representing function of $\mathfrak{M}_t[J_{[q]}]$ is

$$m_t[J_{[q]}](x) = (1-t)\frac{q}{q+1}\frac{x^{t(q+1)}-1}{x^{tq}-1} + tx^t\frac{q}{q+1}\frac{x^{(1-t)(q+1)}-1}{x^{(1-t)q}-1}, \tag{5.1}$$

in particular

$$m_{\frac{1}{2}}[J_{[q]}](x) = \frac{q}{q+1}\frac{x^{\frac{q+1}{2}}-1}{x^{\frac{q}{2}}-1}\frac{1+x^{\frac{1}{2}}}{2}.$$

Put $q = \frac{1}{n}$ in (5.1) for a natural number n. Since

$$\begin{aligned}J_{[\frac{1}{n}]}(x) &= \frac{1}{n+1}\frac{x^{\frac{n+1}{n}}-1}{x^{\frac{1}{n}}-1} = \frac{1}{n+1}\left(x + x^{\frac{n-1}{n}} + \cdots + x^{\frac{1}{n}} + 1\right) \\ &= \frac{1}{n+1}\left(G_1(x) + G_{\frac{n-1}{n}}(x) + \cdots + G_{\frac{1}{n}}(x) + G_0(x)\right),\end{aligned}$$

we have

$$\begin{aligned} m_t[J_{[\frac{1}{n}]}](x) &= \frac{1}{n+1}\Big(m_t[G_1](x)+m_t[G_{\frac{n-1}{n}}](x)+\cdots+m_t[G_{\frac{1}{n}}](x)+m_t[G_0](x)\Big) \\ &= \frac{1}{n+1}\Big(A_t(x)+\overline{HZ}_{t,[\frac{n-1}{n}]}(x)+\cdots+\overline{HZ}_{t,[\frac{1}{n}]}(x)+G_t(x)\Big) \end{aligned}$$

by Theorem 3.3, and also Theorem 3.2 ensures

$$\begin{aligned} LM_t(x) = m_t[LM](x) &\leqslant \cdots \leqslant m_t[J_{[\frac{1}{n+1}]}](x) \leqslant m_t[J_{[\frac{1}{n}]}](x) \\ &\leqslant \cdots \leqslant m_t[J_{[\frac{1}{2}]}](x) \leqslant m_t[A](x) = K_{t,[\frac{1}{2}]}(x) \end{aligned}$$

since $J_{[1]} = A$, $J_{[0]} = LM$ and $J_{[q]}$ is increasing for $q \in \mathbb{R}$.

Put $q = \frac{-1}{n}$ in (5.1) for a natural number n such that $n \geqslant 2$. Since

$$\begin{aligned} J_{[\frac{-1}{n}]}(x) &= \frac{-1}{n-1}\frac{x^{\frac{n-1}{n}}-1}{x^{\frac{-1}{n}}-1} = \frac{1}{n-1}x^{\frac{1}{n}}\Big(x^{\frac{n-2}{n}}+x^{\frac{n-3}{n}}+\cdots+x^{\frac{1}{n}}+1\Big) \\ &= \frac{1}{n-1}\Big(x^{\frac{n-1}{n}}+x^{\frac{n-2}{n}}+\cdots+x^{\frac{2}{n}}+x^{\frac{1}{n}}\Big) \\ &= \frac{1}{n-1}\Big(G_{\frac{n-1}{n}}(x)+G_{\frac{n-2}{n}}(x)+\cdots+G_{\frac{2}{n}}(x)+G_{\frac{1}{n}}(x)\Big), \end{aligned}$$

we have

$$\begin{aligned} & m_t[J_{[\frac{-1}{n}]}](x) \\ &= \frac{1}{n-1}\Big(m_t[G_{\frac{n-1}{n}}](x)+m_t[G_{\frac{n-2}{n}}](x)+\cdots+m_t[G_{\frac{2}{n}}](x)+m_t[G_{\frac{1}{n}}](x)\Big) \\ &= \frac{1}{n-1}\Big(\overline{HZ}_{t,[\frac{n-1}{n}]}(x)+\overline{HZ}_{t,[\frac{n-2}{n}]}(x)+\cdots+\overline{HZ}_{t,[\frac{2}{n}]}(x)+\overline{HZ}_{t,[\frac{1}{n}]}(x)\Big) \end{aligned}$$

by Theorem 3.3, and also Theorem 3.2 ensures

$$\begin{aligned} \overline{HZ}_{t,[\frac{1}{2}]}(x) = m_t[G](x) \leqslant m_t[J_{[\frac{-1}{3}]}](x) &\leqslant \cdots \leqslant m_t[J_{[\frac{-1}{n}]}](x) \leqslant m_t[J_{[\frac{-1}{n+1}]}](x) \\ &\leqslant \cdots \leqslant m_t[LM](x) = LM_t(x) \end{aligned}$$

since $J_{[0]} = LM$, $J_{[\frac{-1}{2}]} = G$ and $J_{[q]}$ is increasing for $q \in \mathbb{R}$. □

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Masatoshi Ito
Maebashi Institute of Technology
460-1 Kamisadorimachi, Maebashi
Gunma 371-0816, JAPAN
e-mail: m-ito@maebashi-it.ac.jp