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ON APPROXIMATION OF FUNCTION IN GENERALIZED ZYGMUND CLASS USING $C^{\eta}T$ OPERATOR

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Abstract. In the present work, we, for the first time, study the error estimates of a function h (2π-periodic) in generalized Zygmund class Z_r^u $(r \geqslant 1)$ by Cesàro-Matrix $(C^{\eta}T)$ product means of its Fourier series (F. S.). The results obtained in the paper provide the best approximation of the function h in Z_q^u $(q \geqslant 1)$ class. Our Theorem 2.1 generalizes eight previously known results. Thus, the results of Singh and Srivastava [29], Lal and Kushwaha [25], Lal [23], Nigam and Sharma [10], Nigam [8], Lal [22] and Uğur Değer [28] become the particular cases of our Theorem 2.1. Several useful results in the form of corollaries are also achieved from the main theorems.

1. Introduction

The study of error approximation of a function h in Lipschitz classes and Hölder classes using different single means have been a centre of creative study for the researchers [2, 3, 5, 7, 13, 14, 16, 15, 17, 21, 24] in past few decades.

The study of error approximation of a function h in Lipschitz classes and Hölder class using different product means have also been a centre of creative study among the researchers [8, 9, 11, 23, 24, 25] .

Here, it is important to note that the results obtained in all the above works could not provide the best approximation of the function. This fact strongly motivates us to consider a more advanced class of a function in order to obtain the best error approximation of a function h by a trigonometric polynomial of degree, not more than j.

Therefore, in the present work, we, for the first time, obtain the results on the best error approximations of the function h in generalized Zygmund class Z_r^u, $(r \geqslant 1)$ by Cesàro-Matrix $(C^{\eta}T)$ product means of F. S. It is worthwhile to mention here that we have used the most generalized product operator for Cesàro-Matrix operator. In fact, we establish two theorems on the degree of approximation of a function h (2π-periodic) in generalized Zygmund class Z_r^u $(r \geqslant 1)$ by Cesàro-Matrix $(C^{\eta}T)$ product means of its F. S. Our Theorem 2.1 generalizes several previously known results. Thus, the results of [8, 10, 22, 23, 25, 28, 29] become the particular cases of our Theorem 2.1. Since $C^{\eta}T$ is the most generalized product means so we also deduce several corollaries from our main theorems.

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Let $C_{2\pi}$ is a Banach space of all periodic functions with period 2π and continuous on the interval $0 \leqslant x \leqslant 2\pi$ under the supremum norm.
The best j-order error approximation of a function $h \in C_{2\pi}$ (Bernstein [26]) is defined by

$$E_j(h) = \inf_{t_j} \|h - t_j\|,$$

where t_j is a trigonometric polynomial of degree j.
Let

$$L^q[0,2\pi] := \left\{ h : [0,2\pi] \to \mathbb{R} : \int_0^{2\pi} |h(x)|^q dx < \infty \right\}, q \geqslant 1,$$

be the space of all functions (2π-periodic and integrable).
We define $\|.\|_q$ by

$$\|h\|_q = \begin{cases} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |h(x)|^q \, dx \right\}^{\frac{1}{q}}, & \text{for } 1 \leqslant q < \infty \\ \operatorname{ess\,sup}_{h \in (0,2\pi)} |h(x)|, & \text{for } q = \infty. \end{cases}$$

As defined in Zygmund [1] $u : [0,2\pi] \to \mathbb{R}$ be an arbitrary function with $u(\xi) > 0$ for $0 < \xi \leqslant 2\pi$ and $\lim_{\xi \to 0^+} u(\xi) = u(0) = 0$.
Now we define,

$$Z_q^{(u)} := \left\{ h \in L^q[0,2\pi] : q \geqslant 1, \sup_{\xi \neq 0} \frac{\|h(.+\xi) + h(.-\xi) - 2h(.)\|_q}{u(\xi)} < \infty \right\}$$

and

$$\|h\|_q^{(u)} := \|h\|_q + \sup_{\xi \neq 0} \frac{\|h(.+\xi) + h(.-\xi) - 2h(.)\|_q}{u(\xi)}, \; q \geqslant 1.$$

Here, the space $Z_q^{(u)}$ is a Banach space under the norm $\|.\|_q^{(u)}$.
One can discuss the completeness of the space $Z_q^{(u)}$ by considering the completeness of L^q; $q \geqslant 1$.
We define

$$Z_q^{(v)} := \left\{ h \in L^q[0,2\pi] : q \geqslant 1, \sup_{\xi \neq 0} \frac{\|h(.+\xi) + h(.-\xi) - 2h(.)\|_q}{v(\xi)} < \infty \right\}$$

and

$$\|h\|_q^{(v)} := \|h\|_q + \sup_{\xi \neq 0} \frac{\|h(.+\xi) + h(.-\xi) - 2h(.)\|_q}{v(\xi)}, \; q \geqslant 1.$$

REMARK 1. $u(\xi)$ and $v(\xi)$ denote moduli of continuity of order 2 [1].
Considering $\frac{u(\xi)}{v(\xi)}$ as non-negative and non-decreasing, then

$$\|h\|_q^{(v)} \leqslant \max\left(1, \frac{u(2\pi)}{v(2\pi)}\right) \|h\|_q^{(u)} < \infty.$$

Thus,

$$Z_q^{(u)} \subset Z_q^{(v)} \subset L^q,\ q \geqslant 1.$$

REMARK 2.

(i) If we take $q \to \infty$ in $Z_q^{(u)}$ then $Z_q^{(u)}$ reduces to $Z^{(u)}$.

(ii) If we take $u(\xi) = \xi^{\alpha}$ in $Z^{(u)}$ then $Z^{(u)}$ reduces to Z_{α}.

(iii) If we take $u(\xi) = \xi^{\alpha}$ in $Z_q^{(u)}$ then $Z_q^{(u)}$ reduces to $Z_{\alpha,q}$.

(iv) If we take $q \to \infty$ in $Z_{\alpha,q}$ then $Z_{\alpha,q}$ reduces to Z_{α}.

(v) Let $0 \leqslant \alpha_1 < \alpha_2 < 1$. If $u(\xi) = \xi^{\alpha_2}$ and $v(\xi) = \xi^{\alpha_1}$, then $\frac{u(\xi)}{v(\xi)}$ is increasing, but $\frac{u(\xi)}{\xi v(\xi)}$ is a decreasing function of ξ.

One can find the detailed work on F. S. in [1].
Let $\sum_{j=0}^{\infty} d_j$ be an infinite series such that $s_k = \sum_{v=0}^{k} d_v$.
The j^{th} partial sum of the F. S. is denoted by $s_j(h;x)$ and is given by

$$s_j(h;x) - h(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(x,\xi) \frac{\sin(j+\frac{1}{2})i}{\sin(\frac{\xi}{2})} d\xi$$

[1].
Let $T = (a_{j,k})$ be an infinite triangular matrix satisfying the conditions of regularity [20] i.e.,

$$\begin{cases} \sum_{k=0}^{j} a_{j,k} = 1, & \text{as} \quad j \to \infty \\ a_{j,k} = 0, & \text{for} \quad j < k \\ \sum_{k=0}^{j} |a_{j,k}| \leqslant M, & \text{a finite constant.} \end{cases} \tag{1}$$

The sequence-to-sequence transformation

$$t_j^T(h;.) := \sum_{k=0}^{j} a_{j,k} s_k = \sum_{k=0}^{j} a_{j,j-k} s_{j-k}$$

defines the sequence $t_j^T(h;.)$ of a triangular matrix means of the sequence $\{s_j\}$ generated by the sequence of coefficients $(a_{j,k})$.
If $t_j^T(h;x) \to s$ as $j \to \infty$ then the infinite series $\sum_{j=0}^{\infty} d_j$ or the sequence $\{s_j\}$ is summable to s by a triangular matrix (T-method) [1].
Following [6], let us write $S_j^0 = s_j$, $S_j^{\eta} = S_0^1 + S_1^{\eta-1} + \ldots + S_j^{\eta-1}$ and E_j^{η} for the value of S_j^{η} when $d_0 = 1$ and $d_j = 0$ for $j > 0$, that is, when $s_j = 1$ for all j.
If $C_j^{\eta} = \frac{S_j^{\eta}}{E_j^{\eta}} \to s$, when $j \to \infty$, then we say that $\sum d_j$ is summable C^{η} (Cesàro means of order $\eta > -1$) to sum s, where $S_j^{\eta} = \sum \binom{j-v+\eta-1}{\eta-1} s_v$ and $E_j^{\eta} = \binom{j+\eta}{\eta}$.

NOTE 1. $(C,0)$ means is an ordinary convergence.
The product of C^{η} means with T means defines $C^{\eta}T$ means and is given by

$$t_j^{C^{\eta}.T}(h;x) := \sum_{r=0}^{j} \frac{\binom{j-r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}} \sum_{k=0}^{r} a_{r,k} s_k(h;x). \tag{2}$$

If $t_j^{C^{\eta}T}(h;x) \to s$ as $j \to \infty$, then the series $\sum_{j=0}^{\infty} d_j$ is said to be summable to s by $C^{\eta}T$ method.

NOTE 2. The regularity of C^{η} and T methods implies the regularity of $C^{\eta}T$ product method.

Particular cases of $C^{\eta}T$ means: $C^{\eta}T$ means reduces to

(i) $C^{\eta}(H, \frac{1}{r+1})$ or $C^{\eta}H$ means if $a_{r,m} = \frac{1}{(r-m+1)\log(r+1)}$.

(ii) $C^{\eta}(N, p_r)$ or $C^{\eta}N_p$ means if $a_{r,m} = \frac{p_{r-m}}{P_r}$ where $P_r = \sum_{m=0}^{r} p_m \neq 0$.

(iii) $(C,\eta)(\bar{N}, p_r)$ or $C^{\eta}\bar{N}_p$ means if $a_{r,m} = \frac{p_m}{P_r}$.

(iv) $(C,\eta)(E,q)$ or $C^{\eta}E^q$ means when $a_{r,m} = \frac{1}{(1+q)^r}\binom{r}{m}q^{r-m}$

(v) $(C,\eta)(E,1)$ or $C^{\eta}E^1$ when $a_{r,m} = \frac{1}{2^r}\binom{r}{m}$

(vi) $(C,\eta)(N,p,q)$ or $C^{\eta}N_{pq}$ means if $a_{r,m} = \frac{p_{r-m}q_m}{R_r}$ where $R_r = \sum_{m=0}^{r} p_m q_{r-m}$.

In above particular case (ii), (iii), and (vi) p_r and q_r are two non-negative monotonic non-increasing sequences of real constants.

REMARK 3. C^1H, C^1N_p, C^1N_{pq}, C^1E^q and C^1E^1 are also the particular cases of $C^{\eta}T$ for $\eta = 1$.

REMARK 4: (EXAMPLE). We consider

$$1 - 4038 \sum_{j=1}^{\infty} (-4037)^{j-1}. \tag{3}$$

The j^{th} partial sum of the above series is given by

$$s_j = (-4037)^j$$

we take $a_{j,k} = \frac{1}{(2019)^j}\binom{j}{k}2018^{j-k}$, then

$$\begin{aligned} t_j^T &= a_{j,0}s_0 + a_{j,1}s_1 + \ldots + a_{j,j}s_j \\ &= \frac{1}{(2019)^j}\left[\binom{j}{0}2018^j - \binom{j}{1}(2018)^{j-1}.4037 + \ldots + \binom{j}{j}(-4037)^j\right] \\ &= \frac{1}{(2019)^j}(-2019)^j = (-1)^j. \end{aligned}$$

Since,

$$(-1)^j = \begin{cases} 1, & \text{when } j \text{ is even number} \\ -1, & \text{when } j \text{ is odd number.} \end{cases} \tag{4}$$

Then, the series (3) is not summable by T means. The series (3) is also not summable by C^η means for $\eta = 1$. But the series (3) is summable by $C^\eta T$ means for $\eta = 1$ as (4) is summable by C^η for $\eta = 1$. Thus, we can observe that product means are more effective than the individual single means.
We write,

$$K_j^{C^\eta T}(\xi) = \frac{1}{2\pi} \sum_{r=0}^{j} \frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}} \sum_{k=0}^{r} a_{r,r-k} \frac{\sin(r-k+\frac{1}{2})\xi}{\sin\frac{\xi}{2}}$$

$$\tau = \text{ integral part of} \left(\frac{1}{\xi}\right)$$

$$\phi(x,\xi) = h(x+\xi) + h(x-\xi) - 2h(x).$$

We mention here some standard inequalities which are used in the paper.

$$\frac{1}{\sin(\frac{\xi}{2})} \leqslant \frac{\pi}{\xi}, \;\; 0 < \xi \leqslant \pi \tag{5}$$

$$\sin\xi \leqslant \xi, \;\; \xi \geqslant 0 \tag{6}$$

Zygmund ([1]).

NOTE 3. We also use the following condition in our main theorems

$$\begin{cases} a_{j,j-k} - a_{j+1,j+1-k} \geqslant 0, \;\; \text{for } 0 \leqslant k \leqslant j \\ A_{j,k} = \sum_{r=k}^{j} a_{j,j-r} \;\text{ and }\; A_{j,0} = 1, \forall j \in \mathbb{N}_0. \end{cases} \tag{7}$$

REMARK 5. Considering the matrix $T = (a_{j,k})$ as

$$a_{j,k} = \begin{cases} \frac{2\times 3^k}{3^{j+1}-1}, & 0 \leqslant k \leqslant j \\ 0, & k > j, \end{cases}$$

we can check all conditions of T method as defined in (1) and also satisfies condition (7).

2. Main results

THEOREM 2.1. *If a function h is 2π-periodic and Lebesgue integrable then the error approximation of h in $Z_q^{(u)}$, $q \geqslant 1$ class by $C^\eta T$ product means of its F. S. is given by*

$$E_j(h) = \inf_{t_j^{C^\eta T}} \|t_j^{C^\eta T}(h;x) - h(x)\|_q^{(v)} = O\left[\frac{1}{(j+1)} \int_{\frac{\pi}{j+1}}^{\pi} \frac{u(\xi)}{\xi^2 v(\xi)} d\xi\right],$$

where $C^{\eta}T$ is as defined in (2), u and v denote moduli of continuity of order two such that $\frac{u(\xi)}{v(\xi)}$ is non-negative and non-decreasing provided (7) holds.

THEOREM 2.2. *If a function h is 2π-periodic and Lebesgue integrable then the error approximation of h in $Z_q^{(u)}$, $q \geqslant 1$ class by $C^{\eta}T$ product means of its F. S. is given by*

$$E_j(h) = \inf_{t_j^{C^{\eta}T}} \|t_j^{C^{\eta}\ T}(h;x) - h(x)\|_q^{(v)} = O\left(\frac{u\left(\frac{1}{j+1}\right)}{v\left(\frac{1}{j+1}\right)} \log(j+1)\right),$$

where $C^{\eta}T$ is as defined in (2), u and v denote moduli of continuity of order two such that $\frac{u(\xi)}{\xi v(\xi)}$ is non-negative and decreasing provided (7) holds.

3. Lemmas

LEMMA 3.1. *If the conditions of (1) and (7) holds for $\{a_{j,k}\}$, then*

$$|K_j^{C^{\eta}T}(\xi)| = O(j+1) \ \ \forall\, \eta \geqslant 1, \ \ 0 < \xi \leqslant \frac{\pi}{j+1}.$$

Proof. For $0 < \xi \leqslant \frac{\pi}{j+1}$, using (5) and (6)

$$\begin{aligned}
|K_j^{C^{\eta}T}(\xi)| &= \frac{1}{2\pi}\left|\sum_{r=0}^{j}\frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\sum_{k=0}^{r} a_{r,r-k}\frac{\sin(r-k+\frac{1}{2})\xi}{\sin\frac{\xi}{2}}\right| \\
&\leqslant \frac{1}{2\pi}\sum_{r=0}^{j}\frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\sum_{k=0}^{r} a_{r,r-k}\left|\frac{\sin(r-k+\frac{1}{2})\xi}{\sin\frac{\xi}{2}}\right| \\
&\leqslant \frac{1}{4}\sum_{r=0}^{j}\frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\sum_{k=0}^{r} a_{r,r-k}(2r-2k+1) \\
&\leqslant \frac{1}{4}\sum_{r=0}^{j}\frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}(2r+1)A_{r,0} \\
&= \frac{1}{4}\sum_{r=0}^{j}\frac{(r+\eta-1)!}{(\eta-1)!\,r!}\times\frac{\eta!\,j!}{(\eta+j)!}(2r+1) \qquad \text{since } A_{r,0}=1 \\
&= \frac{j!}{4.(\eta+1)\ldots(\eta+j)\,\eta!}\times\sum_{r=0}^{j}\frac{(2r+1)(r+\eta-1)!\,\eta}{r!} \\
&= \frac{j!}{4.(\eta+1)\ldots(\eta+j)}\sum_{r=0}^{j}\frac{(2r+1)\eta(\eta+1)\ldots(\eta+r-1)}{r!} \\
&= \frac{j!}{4(\eta+1)\ldots(\eta+j)}\left[1+3\eta+\ldots+\frac{(2j+1)\eta(\eta+1)\ldots(\eta+j-1)}{j!}\right]
\end{aligned}$$

$$\leqslant \frac{j!}{4(\eta+1)\dots(\eta+j)}\left[(j+1)\times\frac{(2j+1)\eta(\eta+1)\dots(\eta+j-1)}{j!}\right]$$
$$=\frac{(j+1)(2j+1)\eta}{4(\eta+j)}\leqslant\frac{(2j+1)\eta}{4}\leqslant\frac{(j+1)\eta}{2}=O(j+1)\ \text{ for all } \eta\geqslant 1.$$

LEMMA 3.2. *If the conditions of (1) and (7) holds for* $\{a_{j,k}\}$*, then*

$$\left|K_j^{C^\eta T}(\xi)\right| = O\left(\frac{1}{(j+1)\xi^2}\right)\ \ \forall\, \eta\geqslant 1,\ \frac{\pi}{n+1}\leqslant\xi\leqslant\pi.$$

Proof. For $\frac{\pi}{j+1}\leqslant\xi\leqslant\pi$, using (5)

$$|K_j^{C^\eta T}(\xi)| = \frac{1}{2\pi}\left|\sum_{r=0}^{j}\frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\sum_{k=0}^{r}a_{r,r-k}\frac{\sin(r-k+\frac{1}{2})\xi}{\sin\frac{\xi}{2}}\right|$$
$$= O\left(\frac{1}{\xi}\right)\left|\operatorname{Im}\sum_{r=0}^{j}\frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\sum_{k=0}^{r}a_{r,r-k}e^{\iota(r-k+\frac{1}{2})\xi}\right|. \tag{8}$$

Now we take,

$$\left|\sum_{r=0}^{j}\frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\sum_{k=0}^{r}a_{r,r-k}e^{i(r-k+\frac{1}{2})\xi}\right| \leqslant \left|\sum_{r=0}^{r}\frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\sum_{k=0}^{\tau}a_{r,r-k}e^{i(r-k)\xi}\right|$$
$$+\left|\sum_{r=\tau+1}^{j}\frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\sum_{k=0}^{\tau}a_{r,r-k}e^{i(r-k)\xi}\right|$$
$$+\left|\sum_{r=\tau+1}^{j}\frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\sum_{k=\tau+1}^{r}a_{r,r-k}e^{i(r-k)\xi}\right|$$
$$\leqslant M_1+M_2+M_3,\quad \text{(say)}. \tag{9}$$

Now,

$$M_1\leqslant\sum_{r=0}^{\tau}\frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\sum_{k=0}^{r}a_{r,r-k}\left|e^{i(r-k)\xi}\right|\leqslant\sum_{r=0}^{\tau}\frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}A_{r,0}$$
$$=\sum_{r=0}^{\tau}\frac{(r+\eta-1)!}{(\eta-1)!\,r!}\times\frac{\eta!\,j!}{(\eta+j)!}\qquad\text{since, } A_{r,0}=1$$
$$=\frac{j!}{(\eta+1)\dots(\eta+j)\ \eta!}\sum_{r=0}^{\tau}\frac{(r+\eta-1)!\ \eta}{r!}$$
$$=\frac{j!}{(\eta+1)\dots(\eta+\tau)\dots(\eta+j)}\left[1+\eta+\frac{\eta(\eta+1)}{2!}+\dots+\frac{\tau+\eta-1}{\tau!(\eta-1)!}\right]$$
$$\leqslant\frac{j!}{(\eta+1)\dots(\eta+\tau)\dots(\eta+j)}\left[(\tau+1)\frac{\eta(\eta+1)\dots(\eta+\tau-1)}{\tau!}\right]$$

$$= \frac{\tau!(\tau+1)\dots j}{(\eta+\tau)\dots(\eta+j-1)} \times \frac{\eta}{\eta+j} \times \frac{\tau+1}{\tau!} \leqslant \frac{\eta}{\eta+j} \times (\tau+1) \leqslant \frac{\eta}{\eta+j} \times \left(\frac{1}{\xi}+1\right)$$
$$= O\left(\frac{1+\xi}{\xi(j+1)}\right) \text{ for all } \eta \geqslant 1.$$

Changing the order of summation and using Abel's transformation in M_2, we have

$$M_2 = \left|\sum_{k=0}^{\tau} \sum_{r=\tau+1}^{j} \frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}} a_{r,r-k} e^{i(r-k)\xi}\right|$$
$$= \frac{1}{\binom{\eta+j}{\eta}} \left|\sum_{k=0}^{\tau}\left[\left\{\sum_{r=\tau+1}^{j-1}\left(\binom{r+\eta-1}{\eta-1} a_{r,r-k} - \binom{r+\eta}{\eta-1} a_{r+1,r+1-k}\right)\sum_{\nu=0}^{r} e^{i(\nu-k)\xi}\right\}\right.\right.$$
$$\left.\left. + \binom{j+\eta-1}{\eta-1} a_{j,j-k} \sum_{\nu=0}^{j} e^{i(\nu-k)\xi} - \binom{\eta+\tau}{\eta-1} a_{\tau+1,\tau+1-k} \sum_{\nu=0}^{\tau} e^{i(\nu-k)\xi}\right]\right|$$
$$= O(\xi^{-1}) \frac{1}{\binom{\eta+n}{\eta}} \sum_{k=0}^{\tau}\left[\left|\sum_{r=\tau+1}^{j-1}\left(\binom{r+\eta-1}{\eta-1} a_{r,r-k} - \binom{r+\eta}{\eta-1} a_{r+1,r+1-k}\right)\right|\right.$$
$$\left. + \left|\binom{j+\eta-1}{\eta-1} a_{j,j-k}\right| + \left|\binom{\eta+\tau}{\eta-1} a_{\tau+1,\tau+1-k}\right|\right]$$
$$= O(\xi^{-1}) \frac{1}{\binom{\eta+j}{\eta}} \sum_{k=0}^{\tau}\left[\binom{\eta+\tau}{\eta-1} a_{\tau+1,\tau+1-k} + \binom{\eta+j-1}{\eta-1} a_{j,j-k} + \binom{\eta+j-1}{\eta-1} a_{j,j-k}\right.$$
$$\left. + \binom{\eta+\tau}{\eta-1} a_{\tau+1,\tau+1-k}\right]$$
$$= O(\xi^{-1}) \frac{1}{\binom{\eta+j}{\eta}} \sum_{k=0}^{\tau}\left[\binom{\eta+\tau}{\eta-1} a_{\tau+1,\tau+1-k} + \binom{\eta+j-1}{\eta-1} a_{j,j-k}\right]$$
$$= O(\xi^{-1}) \frac{j!}{(\eta+1)\dots(\eta+j)} \sum_{k=0}^{\tau}\left[\frac{\eta(\eta+1)\dots(\eta+\tau)\dots(\eta+j)}{(\eta+\tau+1)\dots(\eta+j)(\tau+1)!} a_{\tau+1,\tau+1-k}\right.$$
$$\left. + \frac{\eta(\eta+1)\dots(\eta+j-1)(\eta+j)}{(\eta+j)\, j!} a_{j,j-k}\right]$$
$$= O(\xi^{-1}) \frac{j!}{(\eta+1)\dots(\eta+j)} \times \frac{\eta(\eta+1)\dots(\eta+j)}{(j+1)!} \sum_{k=0}^{\tau} (a_{\tau,\tau-k} + a_{j,j-k})$$
$$= O\left(\frac{1}{\xi(j+1)}(A_{\tau,0} + A_{j,0})\right) = O\left(\frac{1}{\xi(j+1)}\right).$$

Using Abel's transformation in M_3, we have

$$M_3 = \left|\sum_{r=\tau+1}^{j} \frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\left[\sum_{k=\tau+1}^{r-1} (a_{r,r-k} - a_{r,r-k+1}) \sum_{\nu=0}^{k} e^{i(r-\nu)\xi}\right.\right.$$
$$\left.\left. + a_{r,0} \sum_{\nu=0}^{r} e^{i(r-\nu)\xi} - a_{r,r-\tau-1} \sum_{\nu=0}^{\tau} e^{i(r-\nu)\xi}\right|\right]$$

$$
\begin{aligned}
&= O(\xi^{-1}) \sum_{r=\tau+1}^{j} \frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}} \left[\left| \sum_{k=\tau+1}^{r-1} (a_{r,r-k} - a_{r,r-k+1}) \right| + a_{r,0} + a_{r,r-\tau-1} \right] \\
&= O(\xi^{-1}) \sum_{r=\tau+1}^{j} \frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}} \left[\left| -a_{r,r-\tau} + a_{r,1} \right| + a_{r,0} + a_{r,r-\tau-1} \right] \\
&= O(\xi^{-1}) \frac{1}{\binom{\eta+j}{\eta}} \sum_{r=\tau+1}^{j} \binom{r+\eta-1}{\eta-1} a_{r,r-\tau} \\
&= O(\xi^{-1}) \frac{j!}{(\eta+1)\ldots(\eta+j)} \times \frac{\eta(\eta+1)\ldots(\eta+j)}{j!(\eta+j)} \left[\frac{\eta\ldots(\eta+\tau)}{(\tau+1)!} a_{\tau+1,1} + \ldots \right. \\
&\qquad \left. + \frac{\eta\ldots(\eta+j-1)}{j!} a_{j,j-\tau} \right] \\
&= O(\xi^{-1}) \frac{j!}{(\eta+1)\ldots(\eta+j)} \times \frac{\eta(\eta+1)\ldots(\eta+j)}{j!(\eta+j)} \left[a_{\tau+1,1} + a_{\tau+2,2} + \ldots a_{j,j-\tau} \right] \\
&= O(\xi^{-1}) \frac{\eta}{\eta+j} \left[a_{\tau+1,1} + a_{\tau+1,2} + \ldots + a_{\tau+1,j-\tau} \right] = O\left(\frac{1}{\xi(j+1)} \right) A_{\tau+1,1} \\
&= O\left(\frac{1}{\xi(j+1)} \right).
\end{aligned}
$$

(In view of $a_{r,r-k} \geqslant a_{r+1,r+1-k} \geqslant a_{r+1,r-k}$ and $A_{\tau+1,0} = 1$.)
Combining M_1, M_2 and M_3 we have,

$$
\begin{aligned}
M_1 + M_2 + M_3 &= O\left[\frac{1}{\xi(j+1)} \times (1+\xi) \right] + O\left[\frac{1}{\xi(j+1)} \right] + O\left[\frac{1}{\xi(j+1)} \right] \\
&= O\left[\frac{1}{(j+1)} \left(1 + \frac{3}{\xi} \right) \right] \\
&= O\left(\frac{1}{j+1} \times \frac{3+\pi}{\xi} \right) \qquad (10)
\end{aligned}
$$

(Let $1+\frac{3}{\xi} \leqslant \frac{k}{\xi}$ for ξ fixed $\implies k^{min} = 3+\pi$.)
Now, from (8), (9) and (10) we get

$$
\left| K_j^{C^{\eta.T}}(\xi) \right| = O\left(\frac{1}{(j+1)\xi^2} \right).
$$

LEMMA 3.3. *Let $h \in Z_q^{(u)}$, for $0 < \xi \leqslant \pi$,*
If $u(\xi)$ and $v(xi)$ are defined as in Theorem 2.1, then $\|\phi(.+y,\xi) + \phi(.-y,\xi) - 2\phi(.,\xi)\|_q = O\left(v(|y|) \frac{u(\xi)}{v(\xi)} \right)$.

Proof. We can prove this lemma along the same lines of the proof of [24, p.93].

4. Proof of the theorems

4.1. Proof of Theorem 2.1

Proof. Following Zygmund [1], we have

$$s_j(h;x)-h(x)=\frac{1}{2\pi}\int_0^{\pi}\phi(x,\xi)\frac{\sin(j+\frac{1}{2})\xi}{\sin(\frac{\xi}{2})}d\xi.$$

Denoting $C^{\eta}T$ means of $\{s_j(h;x)\}$ by $t_j^{C^{\eta}.T}(h;x)$, we get

$$\begin{aligned}
t_j^{C^{\eta}.T}(h;x)-h(x) &= \sum_{r=0}^{j}\frac{\binom{j-r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\sum_{k=0}^{r}a_{r,k}[s_k(h;x)-h(x)]\\
&=\frac{1}{2\pi}\int_0^{\pi}\phi(x,\xi)\sum_{r=0}^{j}\frac{\binom{r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\sum_{k=0}^{r}a_{r,r-k}\frac{\sin(r-k+\frac{1}{2})\xi}{\sin(\frac{\xi}{2})}d\xi\\
&=\int_0^{\pi}\phi(x,\xi)K_j^{C^{\eta}T}(\xi)d\xi. \qquad (11)
\end{aligned}$$

Let

$$R_j(x)=t_n^{C^{\eta}T}(h;x)-h(x)=\int_0^{\pi}\phi(x,\xi)K_j^{C^{\eta}T}(\xi)d\xi. \qquad (12)$$

Then,

$$R_j(x+y)+R_j(x-y)-2R_j(x)=\int_0^{\pi}[\phi(x+y,\xi)+\phi(x-y,\xi)-2\phi(x,\xi)]K_j^{C^{\eta}.T}(\xi)d\xi.$$

Using the generalized Minkowski inequality [4, p. 31], we get

$$\begin{aligned}
&\|R_j(.+y)+R_j(.-y)-2R_j(.)\|_q\\
\leqslant&\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|R_j(x+y)+R_j(x-y)-2R_j(x)\right|^q dx\right\}^{\frac{1}{q}}\\
\leqslant&\int_0^{\pi}\|\phi(.+y,\xi)+\phi(.-y,\xi)-2\phi(.,\xi)\|_q|K_j^{C^{\eta}T}(\xi)|d\xi\\
=&\int_0^{\frac{\pi}{j+1}}\|\phi(.+y,\xi)+\phi(.-y,\xi)-2\phi(.,\xi)\|_q|K_j^{C^{\eta}T}(\xi)|d\xi\\
&+\int_{\frac{\pi}{j+1}}^{\pi}\|\phi(.+y,\xi)+\phi(.-y,\xi)-2\phi(.,\xi)\|_q|K_j^{C^{\eta}T}(\xi)|d\xi\\
=&I_1+I_2 \text{ (say)}. \qquad (13)
\end{aligned}$$

Using Lemmas 3.1, 3.3 and the monotonicity of $\frac{u(\xi)}{v(\xi)}$ with respect to ξ, we have

$$I_1=\int_0^{\frac{\pi}{j+1}}\|\phi(.+y,\xi)+\phi(.-y,\xi)-2\phi(.,\xi)\|_q|K_j^{C^{\eta}T}(\xi)|d\xi$$

$$= O\left(\int_0^{\frac{\pi}{j+1}} v(|y|)\frac{u(\xi)}{v(\xi)}(j+1)d\xi\right) = O\left((j+1)\, v(|y|)\int_0^{\frac{\pi}{j+1}} \frac{u(\xi)}{v(\xi)}d\xi\right)$$
$$= O\left((j+1)\, v(|y|)\frac{u(\frac{\pi}{j+1})}{v(\frac{\pi}{j+1})}\int_0^{\frac{\pi}{j+1}} 1\, d\xi\right)$$
$$= O\left(v(|y|)\frac{u(\frac{\pi}{j+1})}{v(\frac{\pi}{j+1})}\right). \tag{14}$$

Using Lemmas 3.2 and 3.3, we get

$$I_2 = \int_{\frac{\pi}{j+1}}^{\pi} \|\phi(.+y,\xi)+\phi(.-y,\xi)-2\phi(.,\xi)\|_q |K_j^{C^\eta T}(\xi)|d\xi$$
$$= O\left(\int_{\frac{\pi}{j+1}}^{\pi} v(|y|)\frac{u(\xi)}{v(\xi)}(j+1)^{-1}\xi^{-2}d\xi\right)$$
$$= O\left((j+1)^{-1}v(|y|)\int_{\frac{\pi}{j+1}}^{\pi} \frac{u(\xi)}{v(\xi)}\xi^{-2}d\xi\right). \tag{15}$$

From (13), (14) and (15), we get

$$\|R_j(.+y)+R_j(.-y)-2R_j(.)\|_q = O\left(v(|y|)\frac{u(\frac{\pi}{j+1})}{v(\frac{\pi}{j+1})}\right)$$
$$+ O\left((j+1)^{-1}v(|y|)\int_{\frac{\pi}{j+1}}^{\pi} \frac{u(\xi)}{v(\xi)}\xi^{-2}d\xi\right)$$

$$\sup_{y\neq 0}\frac{\|R_j(.+y)+R_j(.-y)-2R_j(.)\|_q}{v(|y|)} = O\left(\frac{u(\frac{\pi}{j+1})}{v(\frac{\pi}{j+1})}\right)+O\left((j+1)^{-1}\int_{\frac{\pi}{j+1}}^{\pi} \frac{u(\xi)}{v(\xi)}\xi^{-2}d\xi\right). \tag{16}$$

Again using Lemmas 3.1, 3.2 and $\|\phi(.,\xi)\|_q = O(u(\xi))$

$$\|R_j(.)\|_q \leqslant \left(\int_0^{\frac{\pi}{j+1}} + \int_{\frac{\pi}{j+1}}^{\pi}\right)\|\phi(.,\xi)\|_q |K_j^{C^\eta T}(\xi)|d\xi$$
$$= O\left((j+1)\int_0^{\frac{\pi}{j+1}} u(\xi)d\xi\right) + O\left(\frac{\pi}{j+1}\int_{\frac{\pi}{j+1}}^{\pi} \xi^{-2}u(\xi)d\xi\right)$$
$$= O\left(u\left(\frac{\pi}{j+1}\right)\right) + O\left(\frac{\pi}{j+1}\int_{\frac{\pi}{j+1}}^{\pi} \xi^{-2}u(\xi)d\xi\right). \tag{17}$$

Now, we have

$$\|R_j(.)\|_q^{(v)} = \|R_j(.)\|_q + \sup_{y\neq 0}\frac{\|R_j(.+y)+R_j(.-y)-2R_j(.)\|_q}{v(y)}.$$

Using (16) and (17) we obtain,

$$\|R_j(.)\|_q^{(v)} = O\left(u\left(\frac{\pi}{j+1}\right)\right) + O\left(\frac{\pi}{j+1}\int_{\frac{\pi}{j+1}}^{\pi} \xi^{-2}u(\xi)d\xi\right)$$
$$+ O\left(\frac{u\left(\frac{\pi}{j+1}\right)}{v\left(\frac{\pi}{j+1}\right)}\right) + O\left(\frac{\pi}{j+1}\int_{\frac{\pi}{j+1}}^{\pi} \xi^{-2}\frac{u(\xi)}{v(\xi)}d\xi\right). \tag{18}$$

In view of the monotonicity of $v(\xi)$, we have

$$u(\xi) = \frac{u(\xi)}{v(\xi)}v(\xi) \leqslant v(\pi)\frac{u(\xi)}{v(\xi)} = O\left(\frac{u(\xi)}{v(\xi)}\right) \text{ for } 0 < \xi \leqslant \pi.$$

Hence, for $\xi = \frac{\pi}{j+1}$, we have

$$O\left(u\left(\frac{\pi}{j+1}\right)\right) = O\left(\frac{u\left(\frac{\pi}{j+1}\right)}{v\left(\frac{\pi}{j+1}\right)}\right) \tag{19}$$

and by the monotonicity of $v(\xi)$, we have

$$\frac{\pi}{j+1}\int_{\frac{\pi}{j+1}}^{\pi} \frac{\xi^{-2}u(\xi)}{v(\xi)}v(\xi)d\xi \leqslant \frac{\pi}{j+1}v(\pi)\int_{\frac{\pi}{j+1}}^{\pi} \frac{\xi^{-2}u(\xi)}{v(\xi)}d\xi = O\left(\frac{\pi}{j+1}\int_{\frac{\pi}{j+1}}^{\pi} \xi^{-2}\frac{u(\xi)}{v(\xi)}d\xi\right). \tag{20}$$

Now, using (19) and (20) in (18), we get

$$\|R_j(.)\|_q^{(v)} = O\left(\frac{u\left(\frac{\pi}{j+1}\right)}{v\left(\frac{\pi}{j+1}\right)}\right) + O\left(\frac{\pi}{j+1}\int_{\frac{\pi}{j+1}}^{\pi} \xi^{-2}\frac{u(\xi)}{v(\xi)}d\xi\right). \tag{21}$$

Using the fact that $\frac{u(\xi)}{v(\xi)}$ is non-negative and non-decreasing, we have

$$\frac{\pi}{j+1}\int_{\frac{\pi}{j+1}}^{\pi} \frac{\xi^{-2}u(\xi)}{v(\xi)}d\xi \geqslant \frac{u(\frac{\pi}{j+1})}{v(\frac{\pi}{j+1})}\frac{\pi}{j+1}\int_{\frac{\pi}{j+1}}^{\pi} \xi^{-2}d\xi \geqslant \frac{u(\frac{\pi}{j+1})}{v(\frac{\pi}{j+1})}$$
$$\frac{u(\frac{\pi}{j+1})}{v(\frac{\pi}{j+1})} = O\left(\frac{\pi}{j+1}\int_{\frac{\pi}{j+1}}^{\pi} \frac{\xi^{-2}u(\xi)}{v(\xi)}d\xi\right). \tag{22}$$

From (20) and (21), we have

$$\|R_j(.)\|_q^{(v)} = O\left(\frac{\pi}{j+1}\int_{\frac{\pi}{j+1}}^{\pi} \frac{u(\xi)}{\xi^2 v(\xi)}d\xi\right).$$

Hence,

$$E_j(h) = \inf_{t_j^{C^\eta T}} \|R_j(.)\|_q^{(v)} = O\left(\frac{1}{j+1}\int_{\frac{\pi}{j+1}}^{\pi} \frac{u(\xi)}{\xi^2 v(\xi)}d\xi\right).$$

4.2. Proof of the Theorem 2.2

Proof. From Theorem 2.1, we have

$$\inf_{t_n^{C^\eta.T}} \|t_j^{C^\eta T}(h;.)-h(.)\|_q^{(v)} = O\left[\frac{1}{(j+1)}\int_{\frac{\pi}{j+1}}^{\pi}\frac{u(\xi)}{\xi^2 v(\xi)}d\xi\right].$$

For non-negative and non-increasing $\frac{u(\xi)}{\xi v(\xi)}$ with ξ, we get

$$\begin{aligned}\inf_{t_j^{C^\eta.T}} \|t_j^{C^\eta T}(h;.)-h(.)\|_q^{(v)} &= O\left(\frac{1}{j+1}(j+1)\frac{u(\frac{1}{j+1})}{v(\frac{1}{j+1})}\int_{\frac{\pi}{j+1}}^{\pi}\frac{d\xi}{\xi}\right)\\ &= O\left(\frac{u(\frac{1}{j+1})}{v(\frac{1}{j+1})}\log(j+1)\right).\end{aligned}$$

5. Corollaries

COROLLARY 1. *If a function h is 2π-periodic and Lebesgue integrable then the error approximation of h in $Z_q^{(u)}$, $q \geqslant 1$ class by $C^\eta T$ product means of its F. S. is given by*

$$\inf_{t_n^{C^\eta T}} \|t_n^{C^\eta T}(h;x)-h(x)\|_q^{(v)} = \begin{cases} O\left((j+1)\right)^{\alpha_1-\alpha_2}, & 0\leqslant \alpha_1<\alpha_2<1\\ O\left((j+1)^{-1}\right)\{\log(j+1)\}, & \alpha_1=0,\alpha_2=1,\end{cases}$$

where $C^\eta T$ is as defined in (2), u and v denote the Zygmund moduli of continuity [1] such that $\frac{u(\xi)}{v(\xi)}$ is non-negative and non-decreasing provided (7) holds.

Proof. Taking $u(\xi)=\xi^{\alpha_2}, v(\xi)=\xi^{\alpha_1}$ in Theorem 2.1, proof of this Corollary can be obtained.

COROLLARY 2. *The error approximation of a function $h\in Z_q^{(u)}$ by $C^\eta H$ means*

$$t_n^{C^\eta H} = \sum_{r=0}^{n}\frac{\binom{j-r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}}\frac{1}{\log(r+1)}\sum_{k=0}^{r}\frac{1}{(r-k+1)}s_k,$$

of the F. S. is given by

$$E_j(h) = \inf_{t_j^{C^\eta H}} \|t_j^{C^\eta H}(h;.)-h(.)\|_q^{(v)} = O\left[\frac{1}{(j+1)}\int_{\frac{\pi}{j+1}}^{\pi}\frac{u(\xi)}{\xi^2 v(\xi)}d\xi\right].$$

COROLLARY 3. *The error approximation of a function* $h \in Z_q^{(u)}$ *by* $C^\eta N_p$ *means*

$$t_j^{C^\eta N_p} = \sum_{r=0}^{j} \frac{\binom{j-r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}} \frac{1}{P_r} \sum_{k=0}^{r} p_{r-k} s_k,$$

of the F. S. is given by

$$E_j(h) = \inf_{t_j^{C^\eta N_p}} \|t_j^{C^\eta N_p}(h;.) - h(.)\|_q^{(v)} = O\left[\frac{1}{(j+1)} \int_{\frac{\pi}{j+1}}^{\pi} \frac{u(\xi)}{\xi^2 v(\xi)} d\xi\right].$$

COROLLARY 4. *The error approximation of a function* $h \in Z_q^{(u)}$ *by* $C^\delta.N_{pq}$ *means*

$$t_j^{C^\eta N_{pq}} = \sum_{r=0}^{j} \frac{\binom{j-r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}} \frac{1}{R_r} \sum_{k=0}^{r} p_{r-k} q_k s_k,$$

of the F. S. is given by

$$E_j(h) = \inf_{t_j^{C^\eta N_{pq}}} \|t_j^{C^\eta N_{pq}}(h;.) - h(.)\|_q^{(v)} = O\left[\frac{1}{(j+1)} \int_{\frac{\pi}{j+1}}^{\pi} \frac{u(\xi)}{\xi^2 v(\xi)} d\xi\right].$$

COROLLARY 5. *The error approximation of a function* $h \in Z_q^{(u)}$ *by* $C^\eta \bar{N}_p$ *means*

$$t_j^{C^\eta \bar{N}_p} = \sum_{r=0}^{j} \frac{\binom{j-r+\eta-1}{\delta-1}}{\binom{\eta+j}{\eta}} \frac{1}{P_r} \sum_{k=0}^{r} p_k s_k,$$

of the F. S. is given by

$$E_j(h) = \inf_{t_j^{C^{\eta(\bar{N_p})}}} \|t_j^{C^\eta \bar{N}_p}(h;x) - h(x)\|_q^{(v)} = O\left[\frac{1}{(j+1)} \int_{\frac{\pi}{j+1}}^{\pi} \frac{u(\xi)}{\xi^2 v(\xi)} d\xi\right].$$

COROLLARY 6. *The error approximation of a function* $h \in Z_q^{(u)}$ *by* $C^\eta E^q$ *means*

$$t_j^{C^\eta E^q} = \sum_{r=0}^{j} \frac{\binom{j-r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}} \frac{1}{(1+q)^r} \sum_{k=0}^{r} \binom{r}{k} q^{r-k} s_k,$$

of the F. S. is given by

$$E_j(h) = \inf_{t_j^{C^{\eta E^q}}} \|t_j^{C^\eta E^q}(h;.) - h(.)\|_q^{(v)} = O\left[\frac{1}{(j+1)} \int_{\frac{\pi}{j+1}}^{\pi} \frac{u(\xi)}{\xi^2 v(\xi)} d\xi\right].$$

COROLLARY 7. *The error approximation of a function $h \in Z_q^{(u)}$ by $C^\eta.E^1$ means*

$$t_j^{C^\eta E^1} = \sum_{r=0}^{j} \frac{\binom{j-r+\eta-1}{\eta-1}}{\binom{\eta+j}{\eta}} \frac{1}{2^r} \sum_{k=0}^{r} \binom{r}{k} s_k,$$

of the F. S. is given by

$$E_j(h) = \inf_{t_j^{C^\eta E^1}} \|t_j^{C^\eta E^1}(h;.) - h(.)\|_q^{(v)} = O\left[\frac{1}{(j+1)} \int_{\frac{\pi}{j+1}}^{\pi} \frac{u(\xi)}{\xi^2 v(\xi)} d\xi\right].$$

REMARK 6. The above corollaries can also be obtained for the particular cases C^1H, C^1N_p,
C^1N_{pq}, $C^1\tilde{N}_p, C^1E^q$ and C^1E^1 in view of Remark 3.

REMARK 7. If we take $\beta = 0$, $\xi(t) = t^\alpha$ and $r \to \infty$ then $W(L_r, \xi(t))$ class reduces to Lip α class and thus, the results of [8] and [22] reduces for Lip α class.

6. Particular cases

Some particular cases of our main results are:

(i) If $\eta = 1$, $u(\xi) = \xi^\alpha$ and $v(\xi) = \xi^{\alpha_1}$ in Theorem 2.1 and also if $q \to \infty$ and $\alpha_1 = 0$, then the Theorem 2.1 of [29] become a particular case of our result.

(ii) Let us take $\eta = 1$ and reduce matrix means T to the (E,q) in Theorem 2.1. Further to this, if $u(\xi) = \xi^\alpha$ and $v(\xi) = \xi^{\alpha_1}$ in Theorem 2.1 and also if $q \to \infty$ and $\alpha_1 = 0$, then the main Theorem of [25] become a particular case of our result.

(iii) Let us take $\eta = 1$ and reduce matrix means T to the N_p in Theorem 2.1. Further to this, if $u(\xi) = \xi^\alpha$ and $v(\xi) = \xi^{\alpha_1}$ in Theorem 2.1 and also if $q \to \infty$ and $\alpha_1 = 0$, then the Theorem 1 of [23] become a particular case of our result.

(iv) Let us take $\eta = 1$ and reduce matrix means T to (E,q) means in Theorem 2.1. Further to this, if $u(\xi) = \xi^\alpha$ and $v(\xi) = \xi^{\alpha_1}$ in Theorem 2.1 and also if $q \to \infty$ and $\alpha_1 = 0$, then in view of remark 7, the results of [8] become a particular case of our result.

(v) Let us take $\eta = 1$ and reduce matrix means T to $(E,1)$ means in Theorem 2.1. Further to this, if $u(\xi) = \xi^\alpha$ and $v(\xi) = \xi^{\alpha_1}$ in Theorem 2.1 and also if $q \to \infty$ and $\alpha_1 = 0$, then in view of remark 7, the results of [22] become a particular case of our result.

(vi) Let us take $\eta = 1$ in Theorem 2.1. Further to this, if $u(\xi) = \xi^\alpha$ and $v(\xi) = \xi^{\alpha_1}$ in Theorem 2.1 and also if $q \to \infty$ and $\alpha_1 = 0$, then the Theorem 2.3 of [28] become a particular case of our result.

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