

IMPROVEMENT AND GENERALIZATION OF SOME JENSEN–MERCER–TYPE INEQUALITIES

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Abstract. The present paper is devoted to the study of Jensen-Mercer-type inequalities. Our results generalize and improve some earlier results in the literature.

1. Introduction

The well-known Jensen inequality for the convex functions states that if f is a convex function on the interval $[m, M]$, then

$$f\left(\sum_{i=1}^n w_i a_i\right) \leq \sum_{i=1}^n w_i f(a_i) \quad (1.1)$$

for all $a_i \in [m, M]$ and $w_i \in [0, 1]$ ($i = 1, \dots, n$) with $\sum_{i=1}^n w_i = 1$. Various inequalities improving and extending (1.1) have been studied in [1, 6, 7].

Mercer [5] proved that if f is a convex function on $[m, M]$, then

$$f\left(M + m - \sum_{i=1}^n w_i a_i\right) \leq f(M) + f(m) - \sum_{i=1}^n w_i f(a_i) \quad (1.2)$$

for all $a_i \in [m, M]$ and $w_i \in [0, 1]$ ($i = 1, \dots, n$) with $\sum_{i=1}^n w_i = 1$. Several refinements and generalizations of the inequality (1.2) have been given in [4, 8].

In [3, Theorem 2.1] it has been shown that if f is a convex function on the interval $[m, M]$, then

$$f\left(M + m - \frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(M + m - u) du \leq f(M) + f(m) - \frac{f(a) + f(b)}{2} \quad (1.3)$$

for all $a, b \in [m, M]$.

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In this paper we prove the following general result:

$$f(M+m-\bar{a}) \leq \sum_{i=1}^n \frac{w_i}{\bar{a}-a_i} \int_{M+m-\bar{a}}^{M+m-a_i} f(t) dt \leq f(M) + f(m) - \sum_{i=1}^n w_i f(a_i) \quad (1.4)$$

where $\bar{a} := \sum_{i=1}^n w_i a_i$. After that, we show a refinement of inequality (1.4) in the following form

$$\begin{aligned} f(M+m-\bar{a}) &\leq \sum_{i=1}^n w_i f\left(M+m-\frac{\bar{a}+a_i}{2}\right) \leq \sum_{i=1}^n \frac{w_i}{\bar{a}-a_i} \int_{M+m-\bar{a}}^{M+m-a_i} f(t) dt \\ &\leq f(M) + f(m) - \sum_{i=1}^n w_i f(a_i). \end{aligned}$$

Though we confine our discussion to scalars, by changing the convex function assumption with the operator convex, the inequalities we obtain in this paper can be extended in a natural way to Hilbert space operators.

2. Main results

The following lemma is well-known in [5, Lemma 1.3], but we prove it for the reader convenience.

LEMMA 2.1. *Let f be a convex function on $[m, M]$, then*

$$f(M+m-a_i) \leq f(M) + f(m) - f(a_i), \quad (m \leq a_i \leq M, i = 1, \dots, n).$$

Proof. If $f : [m, M] \rightarrow \mathbb{R}$ is a convex function, then for any $x, y \in [m, M]$ and $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y). \quad (2.1)$$

It can be verified that if $m \leq a_i \leq M$ ($i = 1, \dots, n$), then $\frac{M-a_i}{M-m}, \frac{a_i-m}{M-m} \leq 1$ and $\frac{M-a_i}{M-m} + \frac{a_i-m}{M-m} = 1$. Thanks to (2.1), we have

$$f(a_i) \leq \frac{M-a_i}{M-m} f(m) + \frac{a_i-m}{M-m} f(M). \quad (2.2)$$

One the other hand, $m \leq a_i \leq M$ ($i = 1, \dots, n$) implies $m \leq M+m-a_i \leq M$ ($i = 1, \dots, n$). Thus, from (2.2) we infer

$$f(M+m-a_i) \leq \frac{a_i-m}{M-m} f(m) + \frac{M-a_i}{M-m} f(M). \quad (2.3)$$

Summing up (2.2) and (2.3), we get the desired result.

Based on this, our first result can be stated as follows:

THEOREM 2.1. Let f be a convex function on $[m, M]$ and $t \in [0, 1]$. Then

$$f(M + m - \bar{a}) \leq \sum_{i=1}^n w_i f(M + m - ((1-t)\bar{a} + ta_i)) \leq f(M) + f(m) - \sum_{i=1}^n w_i f(a_i) \quad (2.4)$$

for all $a_i \in [m, M]$ and $w_i \in [0, 1]$ ($i = 1, \dots, n$) with $\sum_{i=1}^n w_i = 1$, where $\bar{a} := \sum_{i=1}^n w_i a_i$. Moreover, the function $F : [0, 1] \rightarrow \mathbb{R}$ defined by

$$F(t) = \sum_{i=1}^n w_i f(M + m - ((1-t)\bar{a} + ta_i))$$

is monotonically nondecreasing and convex on $[0, 1]$.

Proof. Firstly, we have

$$\begin{aligned} \sum_{i=1}^n w_i f(M + m - ((1-t)\bar{a} + ta_i)) &\geq f\left(\sum_{i=1}^n w_i (M + m - ((1-t)\bar{a} + ta_i))\right) \\ &= f(M + m - \bar{a}). \end{aligned} \quad (2.5)$$

On the other hand,

$$\begin{aligned} &\sum_{i=1}^n w_i f(M + m - ((1-t)\bar{a} + ta_i)) \\ &= \sum_{i=1}^n w_i f((1-t)(M + m - \bar{a}) + t(M + m - a_i)) \\ &\leq \sum_{i=1}^n w_i ((1-t)f(M + m - \bar{a}) + tf(M + m - a_i)) \\ &\leq \sum_{i=1}^n w_i \left((1-t) \left(f(M) + f(m) - \sum_{j=1}^n w_j f(a_j) \right) + t(f(M) + f(m) - f(a_i)) \right) \\ &= f(M) + f(m) - \sum_{i=1}^n w_i f(a_i). \end{aligned}$$

For the convexity of F , we have

$$\begin{aligned} &F\left(\frac{t+s}{2}\right) \\ &= \sum_{i=1}^n w_i f\left(M + m - \left(\left(1 - \frac{t+s}{2}\right)\bar{a} + \frac{t+s}{2}a_i\right)\right) \\ &= \sum_{i=1}^n w_i f\left(M + m - \left(\frac{(1-t)\bar{a} + ta_i + (1-s)\bar{a} + sa_i}{2}\right)\right) \\ &= \sum_{i=1}^n w_i f\left(\frac{M + m - ((1-t)\bar{a} + ta_i) + M + m - ((1-s)\bar{a} + sa_i)}{2}\right) \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2} \left[\sum_{i=1}^n w_i f(M+m - ((1-t)\bar{a} + ta_i)) + \sum_{i=1}^n w_i f(M+m - ((1-s)\bar{a} + sa_i)) \right] \\ &= \frac{F(t) + F(s)}{2}. \end{aligned}$$

Now, if $0 < s < t < 1$, then $s = \frac{t-s}{t} \cdot 0 + \frac{s}{t} \cdot t$ and hence the convexity of F implies

$$F(s) = F\left(\frac{t-s}{t} \cdot 0 + \frac{s}{t} \cdot t\right) \leq \frac{t-s}{t} F(0) + \frac{s}{t} F(t) \leq \frac{t-s}{t} F(t) + \frac{s}{t} F(t) = F(t).$$

We remark that the second inequality in the above follows from (2.5) and the fact

$$F(0) = \sum_{i=1}^n w_i f(M+m-\bar{a}) = f(M+m-\bar{a}).$$

Therefore F is monotonically nondecreasing on $[0, 1]$.

COROLLARY 2.1. *Let all the assumptions of Theorem 2.1 hold, then*

$$f(M+m-\bar{a}) \leq \sum_{i=1}^n \frac{w_i}{\bar{a}-a_i} \int_{M+m-\bar{a}}^{M+m-a_i} f(t) dt \leq f(M) + f(m) - \sum_{i=1}^n w_i f(a_i). \tag{2.6}$$

Proof. Integrating the inequality (2.4) over $t \in [0, 1]$, we get the inequalities (2.6). Here we used the fact

$$\begin{aligned} \int_0^1 f(M+m - ((1-t)\bar{a} + ta_i)) dt &= \int_0^1 f((1-t)(M+m-\bar{a}) + t(M+m-a_i)) dt \\ &= \int_0^1 f(t(M+m-\bar{a}) + (1-t)(M+m-a_i)) dt \\ &= \frac{1}{\bar{a}-a_i} \int_{M+m-\bar{a}}^{M+m-a_i} f(t) dt. \end{aligned}$$

REMARK 2.1. Put $n = 2$, $w_1 = w_2 = 1/2$, $a_1 = a$, and $a_2 = b$ in Corollary 2.1, then

$$f\left(M+m - \frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(M+m-u) du \leq f(M) + f(m) - \frac{f(a) + f(b)}{2}$$

which shows that our inequalities (2.6) generalize inequalities (1.3).

We give a more precise estimate in the next theorem.

THEOREM 2.2. *Let f be a convex function on $[m, M]$. Then*

$$\begin{aligned} f(M+m-\bar{a}) &\leq \sum_{i=1}^n w_i f\left(M+m - \frac{\bar{a}+a_i}{2}\right) \leq \sum_{i=1}^n \frac{w_i}{\bar{a}-a_i} \int_{M+m-\bar{a}}^{M+m-a_i} f(t) dt \\ &\leq f(M) + f(m) - \sum_{i=1}^n w_i f(a_i) \end{aligned} \tag{2.7}$$

for all $a_i \in [m, M]$ and $w_i \in [0, 1]$ ($i = 1, \dots, n$) with $\sum_{i=1}^n w_i = 1$.

Proof. If $f : [m, M] \rightarrow \mathbb{R}$ is a convex function, then we have for any $a, b \in [m, M]$

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b+tb+(1-t)a}{2}\right) \\ &\leq \frac{f(ta+(1-t)b)+f(tb+(1-t)a)}{2} \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

Replacing a and b by $M+m-a$ and $M+m-b$, respectively, we get

$$\begin{aligned} f\left(M+m-\frac{a+b}{2}\right) &\leq \frac{f(M+m-(ta+(1-t)b))+f(M+m-(tb+(1-t)a))}{2} \\ &\leq \frac{f(M+m-a)+f(M+m-b)}{2}. \end{aligned}$$

Integrating the inequality over $t \in [0, 1]$, and using the fact

$$\int_0^1 f(tx+(1-t)y) dt = \int_0^1 f(ty+(1-t)x) dt,$$

we infer that

$$\begin{aligned} f\left(M+m-\frac{a+b}{2}\right) &\leq \int_0^1 f(M+m-(ta+(1-t)b)) dt \\ &\leq \frac{f(M+m-a)+f(M+m-b)}{2}. \end{aligned}$$

Since $a_i, \bar{a} \in [m, M]$, we can write

$$f\left(M+m-\frac{\bar{a}+a_i}{2}\right) \leq \frac{1}{\bar{a}-a_i} \int_{M+m-\bar{a}}^{M+m-a_i} f(t) dt \leq \frac{f(M+m-\bar{a})+f(M+m-a_i)}{2},$$

due to

$$\int_0^1 f(M+m-(t\bar{a}+(1-t)a_i)) dt = \frac{1}{\bar{a}-a_i} \int_{M+m-\bar{a}}^{M+m-a_i} f(t) dt.$$

Multiplying by $w_i > 0$ ($i = 1, \dots, n$) and summing over i from 1 to n we may deduce

$$\begin{aligned} \sum_{i=1}^n w_i f\left(M+m-\frac{\bar{a}+a_i}{2}\right) &\leq \sum_{i=1}^n \frac{w_i}{\bar{a}-a_i} \int_{M+m-\bar{a}}^{M+m-a_i} f(t) dt \\ &\leq \frac{f(M+m-\bar{a})+\sum_{i=1}^n w_i f(M+m-a_i)}{2}. \end{aligned} \tag{2.8}$$

On the other hand, by (1.1)

$$f(M + m - \bar{a}) = f\left(\sum_{i=1}^n w_i \left(M + m - \frac{\bar{a} + a_i}{2}\right)\right) \leq \sum_{i=1}^n w_i f\left(M + m - \frac{\bar{a} + a_i}{2}\right) \tag{2.9}$$

and by Lemma 2.1

$$\begin{aligned} & \frac{f(M + m - \bar{a}) + \sum_{i=1}^n w_i f(M + m - a_i)}{2} \\ & \leq \frac{f(M) + f(m) - \sum_{j=1}^n w_j f(a_j) + f(M) + f(m) - \sum_{i=1}^n w_i f(a_i)}{2} \\ & = f(M) + f(m) - \sum_{i=1}^n w_i f(a_i). \end{aligned} \tag{2.10}$$

Combining (2.8), (2.9), and (2.10), we get (2.7).

COROLLARY 2.2. Let $a_i \in [m, M]$ and $w_i \in [0, 1]$ ($i = 1, \dots, n$) with $\sum_{i=1}^n w_i = 1$.

Then

$$\begin{aligned} \frac{Mm}{\prod_{i=1}^n a_i^{w_i}} & \leq \exp \left[\sum_{i=1}^n \frac{w_i}{\bar{a} - a_i} \int_{M+m-\bar{a}}^{M+m-a_i} \log t \, dt \right] \leq \prod_{i=1}^n \left(M + m - \frac{\bar{a} + a_i}{2} \right)^{w_i} \\ & \leq M + m - \sum_{i=1}^n w_i a_i. \end{aligned}$$

Proof. Put $f(t) = -\log t$, ($0 < t \leq 1$) in Theorem 2.2.

REMARK 2.2. If we set $n = 2$, $a_1 = m, a_2 = M$ and $w_1 = w_2 = 1/2$ in Corollary 2.2, then we have

$$\sqrt{Mm} \leq \frac{M^{M-m}}{e m^{M-m}} \leq \frac{1}{4} \sqrt{(M + 3m)(m + 3M)} \leq \frac{1}{2} (M + m).$$

One can obtain the inequalities for the weighted parameter in means of two variables m and M by elementary calculations. We leave it to the interested readers.

REMARK 2.3. Let all the assumptions of Theorem 2.2 hold, then

$$\begin{aligned} f(\bar{a}) & \leq \sum_{i=1}^n w_i f\left(\frac{\bar{a} + a_i}{2}\right) \leq \sum_{i=1}^n \frac{w_i}{a_i - \bar{a}} \int_{\bar{a}}^{a_i} f(t) \, dt \leq \frac{f(\bar{a}) + \sum_{i=1}^n w_i f(a_i)}{2} \\ & \leq \sum_{i=1}^n w_i f(a_i). \end{aligned}$$

The proof is in the same spirit as that of Theorem 2.2 (see also [2, Corollary 3]).

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