SOME INEQUALITIES FOR RECIPROCALLY (s,m)—CONVEX IN THE SECOND SENSE FUNCTIONS AND APPLICATIONS TO SPECIAL MEANS

MIREYA R. BRACAMONTE, JESÚS MEDINA-VILORIA AND MIGUEL VIVAS-CORTEZ*

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Abstract. We present the notion of reciprocally (s,m)—convex functions and present some examples and properties of them. We derive some inequalities for this new class of functions, specifically these inequalities are: Hermite–Hadamard and Fejér. In addition, we present some applications of our results to special media of positive real numbers.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers, $I \subseteq \mathbb{R}$ be an interval and $\mathbb{R}_+ = (0, +\infty)$. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex in the classical sense if it satisfies the following inequality

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y),$$

for all $x, y \in I$ and $t \in [0,1]$. Convexity theory has played an important and fundamental role in the developments of different fields of pure and applied sciences [63, 64, 71]. In recent years, it received considerable attention, in order to extend the validity of their results to large classes of optimization, these concepts have been generalized and extended in several directions using novel and innovative techniques.
The concepts of convex functions have been generalized in various directions using novel and innovative ideas, see [65, 73, 67, 4, 71, 53, 74] A classical inequality for convex functions is the Hermite–Hadamard inequality, this is given as follows:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{a+b}{2}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2},$$

where $f : I \to \mathbb{R}$ is a convex function and $ab \in I$ with $a < b$ (see [35]). In [43], it is obtain the class of (s,m)—convex functions in the second sense as the following.

DEFINITION 1. A function $f : [0,b] \to \mathbb{R}$ is said to be (s,m)–convex in the second sense, where $(s,m) \in (0,1]^2$, if for every $x,y \in [0,b]$ and $t \in [0,1]$ we have

$$f(tx+m(1-t)y) \leqslant t^s f(x) + m(1-t)^s f(y). \tag{1}$$

İ. İşcan in [5], gave the following definition of harmonically convex functions:

DEFINITION 2. Let I be an interval in $\mathbb{R} \setminus \{0\}$. A function $f : I \to \mathbb{R}$ is said be harmonically convex on I if the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant tf(y)+(1-t)f(x), \tag{2}$$

holds, for all $x,y \in I$ and $t \in [0,1]$.

The following Fejér inequality for harmonically convex functions holds true.

THEOREM 1. ([4]) *Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a,b \in I$ with $a < b$. If $f \in L(a,b)$, then one has*

$$f\left(\frac{2ab}{a+b}\right) \int_a^b \frac{p(x)}{x^2} dx \leqslant \int_a^b \frac{f(x)}{x^2} p(x) dx \leqslant \frac{f(a)+f(b)}{2} \int_a^b \frac{p(x)}{x^2} dx, \tag{3}$$

where $p : [a,b] \to \mathbb{R}$ is nonnegative and integrable and satisfies

$$p\left(\frac{ab}{x}\right) = p\left(\frac{ab}{a+b-x}\right).$$

In [44] define m–harmonic-arithmetically convex functions.

DEFINITION 3. Let $f : (0,b] \subseteq \mathbb{R}_+ \to \mathbb{R}$ and $m \in (0,1]$ be a constant. If

$$f\left(\frac{xy}{ty+m(1-t)x}\right) \leqslant tf(x)+m(1-t)f(y),$$

for all $x,y \in (0,b]$ and $t \in [0,1]$, then f is said to be an m–harmonic-arithmetically convex (or m–HA–convex) function.

In [6], gave the definition of harmonic s–convexity in the second sense as follows.

DEFINITION 4. A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonically s–convex in the second sense and $s \in (0,1]$ if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant t^s f(y)+(1-t)^s f(x), \tag{4}$$

is valid for all $x,y \in I$ and $t \in [0,1]$.

Following are some classic inequalities that have been demonstrated for this class of functions. In [6], İşcan also shows the following inequalities of Hermite–Hadamard for harmonically s–convex functions.

THEOREM 2. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be an harmonically s–convex function, $s \in (0,1]$ and $a,b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequalities hold:*

$$2^{s-1}f\left(\frac{2ab}{a+b}\right) \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leqslant \frac{f(a)+f(b)}{s+1}. \tag{5}$$

2. Reciprocally (s,m)–convex in the second sense functions

Now, we combine two definitions of m–harmonicity and harmonic s–convexity in the second sense and obtain the class of reciprocally (s,m)–convex in the second sense functions as the following.

DEFINITION 5. The function $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be reciprocally (s,m)–convex in the second sense, where $(s,m) \in (0,1]^2$ if

$$f\left(\frac{xy}{ty+m(1-t)x}\right) \leqslant t^s f(x) + m(1-t)^s f(y), \tag{6}$$

for all $x,y \in \mathbb{R}_+$ and $t \in [0,1]$.

REMARK 1. 1. Note that if $s = 1$, then the reciprocally (s,m)–convex functions in the second sense is m–harmonic-arithmetically convex functions.

2. If $m = 1$, then the reciprocally (s,m)–convex functions in the second sense is harmonically s–convex functions in the second sense.

3. Note that if $s,m = 1$, then the reciprocally (s,m)–convex functions in the second sense is harmonically convex functions.

EXAMPLE 1. Let $s,m \in (0,1]$, $p \in [1,+\infty)$ and $f : \mathbb{R}_+ \to \mathbb{R}$ defined by $f(x) = \frac{1}{x^p}$, then f is reciprocally (s,m)–convex.

Proof. Let $x,y \in \mathbb{R}_+$ and $s,m \in (0,1]$,

$$f\left(\frac{xy}{ty+m(1-t)x}\right) = \left[\frac{ty+m(1-t)x}{xy}\right]^p = \frac{[ty+(1-t)mx]^p}{(xy)^p} \leqslant \frac{ty^p + m^p(1-t)x^p}{(xy)^p}$$

$$= t\frac{1}{x^p} + m^p(1-t)\frac{1}{y^p} \leqslant t^s\frac{1}{x^p} + m(1-t)^s\frac{1}{y^p}$$

$$= t^s f(x) + m(1-t)^s f(y).$$

Thus, f is a reciprocally (s,m)–convex function.

The next result is a characterization of the functions reciprocally (s,m)–convex.

THEOREM 3. *Let $(s,m) \in (0,1]^2$ and if we consider the function $g : \mathbb{R}_+ \to \mathbb{R}$, defined by $g(t) = f\left(\dfrac{1}{t}\right)$, then f is reciprocally (s,m)–convex function if and only if g is (s,m)–convex in the second sense.*

Proof. For the proof, we will make use of the following equivalences:

$$f\left(\frac{xy}{ty + m(1-t)x}\right) \leqslant t^s f(x) + m(1-t)^s f(y), \quad \forall x,y \in \mathbb{R}^+, t \in [0,1] \text{ y } s,m \in (0,1],$$

$$f\left(\frac{1}{t\dfrac{1}{x} + m(1-t)\dfrac{1}{y}}\right) \leqslant t^s f(x) + m(1-t)^s f(y), \quad \forall x,y \in \mathbb{R}^+, t \in [0,1] \text{ y } s,m \in (0,1],$$

$$f\left(\frac{1}{tu + m(1-t)w}\right) \leqslant t^s f\left(\frac{1}{u}\right) + m(1-t)^s f\left(\frac{1}{w}\right),$$

donde $u = \dfrac{1}{x}, w = \dfrac{1}{y} \in \mathbb{R}^+, t \in [0,1]$ y $s,m \in (0,1]$,

$$g(tu + m(1-t)w) \leqslant t^s g(u) + m(1-t)^s g(w), \quad \forall u,w \in \mathbb{R}^+, t \in [0,1] \text{ y } s,m \in (0,1].$$

This completes the proof.

EXAMPLE 2. Let $s \in (0,1]$ and $f : \mathbb{R}_+ \to \mathbb{R}$, $f(x) = \dfrac{1}{x^s}$, then f is a reciprocally $(s,1)$–convex function.

In effect, let $h : \mathbb{R} \to R$, defined by $h(x) = x^s$, then h is $(s,1)$–convex function in the second sense (see [45]).Thus, $f : \mathbb{R}_+ \to \mathbb{R}$, defined by $f(x) = h\left(\dfrac{1}{x}\right) = \dfrac{1}{x^s}$ is a reciprocally $(s,1)$–convex function (by Theorem 3).

The following theorems are properties of the reciprocally (s,m)–convex in the second sense functions.

THEOREM 4. *Let $f,g : \mathbb{R}_+ \to \mathbb{R}$ be two functions and $k \geqslant 0$,*

1. *If f,g are reciprocally (s,m)–convex functions, then $f+g$ is reciprocally (s,m)–convex function.*

2. *If f is reciprocally (s,m)–convex functions, then kf is reciprocally (s,m)–convex function.*

Proof.

1. Let $x,y \in \mathbb{R}_+$ and $t \in [0,1]$,

$$
\begin{aligned}
(f+g)\left(\frac{xy}{ty+m(1-t)x}\right) &= f\left(\frac{xy}{ty+m(1-t)x}\right) + g\left(\frac{xy}{ty+m(1-t)x}\right) \\
&\leqslant t^s f(x) + m(1-t)^s f(y) + t^s g(x) + m(1-t)^s g(y) \\
&= t^s (f+g)(x) + m(1-t)^s (f+g)(y).
\end{aligned}
$$

Thus, $f+g$ is reciprocally (s,m)–convex functions.

2. Let $x,y \in \mathbb{R}_+$ and $t \in [0,1]$,

$$
(kf)\left(\frac{xy}{ty+m(1-t)x}\right) = kf\left(\frac{xy}{ty+(1-t)x}\right) \leqslant kt^s f(x) + km(1-t)^s f(y).
$$

Verifying that kf is reciprocally (s,m)–convex function.

THEOREM 5. *If $f_1,f_2 : \mathbb{R}_+ \to \mathbb{R}$ are reciprocally (s,m)–convex functions, then $f := \max\{f_1,f_2\}$ so too is.*

Proof. Let $x,y \in$ and $t \in [0,1]$. We have

$$
f_1\left(\frac{xy}{ty+m(1-t)x}\right) \leqslant t^s f_1(x) + m(1-t)^s f_1(x) \leqslant t^s f(x) + m(1-t)^s f(y),
$$

and

$$
f_2\left(\frac{xy}{ty+(1-t)x}\right) \leqslant t^s f_2(x) + (1-t)^s f_2(y) \leqslant t^s f(x) + m(1-t)^s f(y).
$$

From which we obtain that

$$
\begin{aligned}
f\left(\frac{xy}{tx+(1-t)y}\right) &= \max\left\{ f_1\left(\frac{xy}{ty+(1-t)x}\right), f_2\left(\frac{xy}{ty+(1-t)x}\right) \right\} \\
&\leqslant t^s f(x) + m(1-t)^s f(y).
\end{aligned}
$$

Thus, f is reciprocally (s,m)–convex functions.

THEOREM 6. *If $f_n : \mathbb{R}_+ \to \mathbb{R}$ is a sequence of reciprocally (s,m)–convex in the second sense functions, converging pointwise to a function f on $\mathbb{R}_+$, then f is reciprocally (s,m)–convex in the second sense function.*

Proof. Let $x,y \in \mathbb{R}_+$ and $t \in [0,1]$

$$
\begin{aligned}
f\left(\frac{xy}{ty+m(1-t)x}\right) &= \lim_{n\to\infty} f_n\left(\frac{xy}{ty+m(1-t)x}\right) \leqslant \lim_{n\to\infty} [t^s f_n(x) + m(1-t)^s f_n(y)] \\
&= t^s f(x) + m(1-t)^s f(y).
\end{aligned}
$$

This completes the demonstration.

Now our interest is to determine the conditions under which the composition of functions belongs to the class studied. Thus, these conditions are presented in the following theorem.

THEOREM 7. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a m–HA–convex function with $m \in (0,1]$ and $g : \mathbb{R}_+ \to \mathbb{R}$ be a nondecreasing and (s,m)–convex in the second sense, such that $f(\mathbb{R}_+) \subseteq \mathbb{R}_+$, then $g \circ f$ is reciprocally (s,m)–convex in the second sense.*

Proof. Since f is a m–HA–convex function we have, for any $x,y \in \mathbb{R}_+$ and $t \in [0,1]$ and $m \in (0,1]$, we obtain,

$$f\left(\frac{xy}{ty + m(1-t)x}\right) \leqslant tf(x) + m(1-t)f(y)$$

In addition, g is a nondecreasing function and is a reciprocally (s,m)–convex in the second sense, therefore

$$g\left(f\left(\frac{xy}{ty + m(1-t)x}\right)\right) \leqslant g(tf(x) + m(1-t)f(y)) \leqslant t^s g(f(x)) + m(1-t)^s g(f(y)).$$

Thus, $g \circ f$ is reciprocally (s,m)–convex.

3. Hermite–Hadamard type inequalities

In the next section we demonstrate the important results of this article, in which we demonstrate some classic inequalities that were generalized for the new classes of functions.

THEOREM 8. (Hermite–Hadamard type left–inequality) *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a reciprocally (s,m)–convex function with $s,m \in (0,1]$. If $a,b \in \mathbb{R}_+$ with $a < b$ and $f \in L[a,b]$, then*

$$2^s f\left(\frac{2ab}{a+b}\right) \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx + \frac{abm}{b-a}\int_a^b \frac{f(mx)}{x^2}dx. \tag{7}$$

Proof. Since f is a reciprocally (s,m)–convex, we obtain

$$f\left(\frac{2ab}{a+b}\right) = f\left(\cfrac{1}{\cfrac{1}{2}\cfrac{1}{\cfrac{ab}{ta+(1-t)b}} + \cfrac{1}{2}m\cfrac{1}{\cfrac{abm}{tb+(1-t)a}}}\right)$$

$$\leqslant \frac{1}{2^s}f\left(\frac{ab}{ta+(1-t)b}\right) + \frac{m}{2^s}f\left(\frac{abm}{tb+(1-t)a}\right).$$

Thus,

$$2^s f\left(\frac{2ab}{a+b}\right) \leqslant f\left(\frac{ab}{ta+(1-t)b}\right) + mf\left(\frac{abm}{tb+(1-t)a}\right). \tag{8}$$

Integrating over the interval $[0,1]$ the inequality (8), we have

$$2^s f\left(\frac{2ab}{a+b}\right) \leqslant \int_0^1 \left[f\left(\frac{ab}{ta+(1-t)b}\right) + mf\left(\frac{abm}{tb+(1-t)a}\right)\right] dt. \qquad (9)$$

We know that,

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right) dt = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx, \qquad (10)$$

and for $\int_0^1 f\left(\frac{abm}{tb+(1-t)a}\right) dt$, it is done $x = \dfrac{ab}{tb+(1-t)a}$, we get

$$\int_0^1 f\left(\frac{abm}{tb+(1-t)a}\right) dt = \frac{ab}{b-a}\int_a^b \frac{f(mx)}{x^2}dx. \qquad (11)$$

Substituting (10) and (11) in (9), we obtain

$$2^s f\left(\frac{2ab}{a+b}\right) \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx + \frac{abm}{b-a}\int_a^b \frac{f(mx)}{x^2}dx,$$

thus obtaining the inequality (7).

THEOREM 9. (Hermite–Hadamard type right–inequality) *Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be a reciprocally* $(s.m)$*–convex function with* $s,m \in (0,1]$. *If* $a,b \in \mathbb{R}_+$ *with* $a < b$ *and* $f \in L[a,b]$, *then*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leqslant \min\left\{\frac{f(a)+mf(mb)}{s+1}, \frac{f(b)+mf(ma)}{s+1}\right\}. \qquad (12)$$

Proof. Since f is reciprocally (s,m)–convex, we have for all $x,y \in \mathbb{R}_+$ and $t \in [a,b]$, then

$$f\left(\frac{xy}{ty+(1-t)x}\right) \leqslant t^s f(x) + m(1-t)^s f(my). \qquad (13)$$

Since (13) is valid for any $x,y \in \mathbb{R}^+$, it is particularly true for $x = a$ and $y = b$, that is,

$$f\left(\frac{ab}{tb+(1-t)a}\right) \leqslant t^s f(a) + m(1-t)^s f(mb).$$

Integrating on $[0,1]$, we obtain

$$\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right) dt \leqslant \int_0^1 [t^s f(a) + m(1-t)^s f(mb)] dt. \qquad (14)$$

Resolving each of the integrals of (14), using changes of variables, we obtain

$$\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right) dt = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx, \qquad (15)$$

$$\int_0^1 t^s dt = \left[\frac{t^{s+1}}{s+1}\right]_0^1 = \frac{1}{s+1}, \tag{16}$$

$$\int_0^1 (1-t)^s dt = -\int_1^0 u^s du, \text{(doing the change } u = 1-t) = \left[\frac{u^{s+1}}{s+1}\right]_0^1 = \frac{1}{s+1}. \tag{17}$$

Substituting (15)–(17) in (14), we get

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leqslant \frac{f(a)}{s+1} + \frac{mf(mb)}{s+1} = \frac{f(a)+mf(mb)}{s+1}. \tag{18}$$

Performing the same procedure for $x = b$ and $y = a$, we have

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leqslant \frac{f(b)+mf(ma)}{s+1}. \tag{19}$$

Thus, from (18) and (19)

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leqslant \min\left\{\frac{f(a)+mf(mb)}{s+1}, \frac{f(b)+mf(ma)}{s+1}\right\}.$$

Hence we get the inequality (12).

REMARK 2. Note that if $m = 1$ in (7) and (12) it is obtained the Hermite–Hadamard type inequalities for harmonically s–convex function (see [6]).

REMARK 3. Note that if $s = 1$ in (12) it is obtained the following inequality:

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leqslant \min\left\{\frac{f(a)+mf(mb)}{2}, \frac{f(b)+mf(ma)}{2}\right\}. \tag{20}$$

4. Fejér type inequalities

To establish the inequality of type Fejér for differentiable functions, use is made of a previous lemma and the following definition.

DEFINITION 6. ([42]) We say that a function $g : [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is harmonically symmetric with respect to $\frac{2ab}{a+b}$ if

$$g(x) = g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right) \tag{21}$$

holds for all $x \in [a,b]$.

LEMMA 1. [42] *Let* $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be a differentiable function on* I° *and* $a, b \in I^\circ$ *with* $a < b$ *and let* $g : [a,b] \to [0,+\infty)$ *be continuous positive mapping and harmonically symmetric to* $\dfrac{2ab}{a+b}$. *If* $f' \in L[a,b]$, *then the following identity holds*

$$\frac{f(a)+f(b)}{2} \int_a^b \frac{g(x)}{x^2} dx - \int_a^b \frac{g(x)f(x)}{x^2} dx$$
$$= \frac{b-a}{4ab} \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x^2} dx \right) \left[(U(t))^2 f'(U(t)) - (L(t))^2 f'(L(t)) \right] dt,$$

where

$$L(t) = \frac{2ab}{(1-t)a+(1+t)b} \quad \text{and} \quad U(t) = \frac{2ab}{(1+t)a+(1-t)b}.$$

In the following result we show the Fejér inequality for functions differentiable $f : \mathbb{R}_+ \to \mathbb{R}$ and $|f'|^q$ is recíprocally (s,m)–convex.

THEOREM 10. *Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be a differentiable function on* $\mathbb{R}_+$ *and* $a, b \in \mathbb{R}_+$ *with* $a < b$ *and let* $g : [a,b] \to [0,+\infty)$ *be continuous positive mapping and harmonically symmetric to* $\dfrac{2ab}{a+b}$ *such that* $f' \in L[a,b]$. *If* $|f'|^q$ *is reciprocally* (s,m)–*convex for* $q \geqslant 1$, *then the following inequality holds*

$$\left| \frac{f(a)+f(b)}{2} \int_a^b \frac{g(x)}{x^2} dx - \int_a^b \frac{g(x)f(x)}{x^2} dx \right| \tag{22}$$
$$\leqslant \left(\frac{b-a}{2ab} \right)^2 \|g\|_\infty \left[\mu_1^{1-\frac{1}{q}} v^{\frac{1}{q}} + \mu_2^{1-\frac{1}{q}} \eta^{\frac{1}{q}} \right],$$

where

$$\mu_1 = \frac{2ab^2}{b-a} + \left(\frac{2ab}{b-a} \right)^2 \ln \left(\frac{2a}{a+b} \right),$$

$$\mu_2 = \left(\frac{2ab}{b-a} \right)^2 \ln \left(\frac{2b}{a+b} \right) - \frac{2a^2b}{(b-a)^2},$$

$$v = \min \left\{ \frac{1}{2^s} \left[v_1 |f'(b)|^q + m v_2 |f'(ma)|^q \right], \frac{1}{2^s} \left[m v_1 |f'(mb)|^q + v_2 |f'(a)|^q \right] \right\},$$

$$\eta = \min \left\{ \frac{1}{2^s} \left[v_2 |f'(b)|^q + m v_1 |f'(ma)|^q \right], \frac{1}{2^s} \left[m v_2 |f'(mb)|^q + v_1 |f'(a)|^q \right] \right\}$$

with

$$v_1 = a^2 \left\{ \beta(s+2,1) \left[2^{s+2} \cdot {}_2F_1 \left(2,s+2;s+3;\frac{b-a}{b} \right) - {}_2F_1 \left(2,s+2;s+3;\frac{b-a}{2b} \right) \right] \right.$$
$$\left. - \beta(s+1,1) \left[2^{s+1} \cdot {}_2F_1 \left(2,s+1;s+2;\frac{b-a}{b} \right) - {}_2F_1 \left(2,s+1;s+2;\frac{b-a}{2b} \right) \right] \right\},$$

$$v_2 = \left(\frac{2ab}{a+b} \right)^2 \beta(2,s+1) \cdot {}_2F_1 \left(2,2;s+3;\frac{b-a}{a+b} \right).$$

Proof. From Lemma 1 and Hölder's inequality, we get

$$\left| \frac{f(a)+f(b)}{2} \int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{g(x)f(x)}{x^2}dx \right|$$

$$= \left| \frac{b-a}{4ab} \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x^2}dx \right) \left[(U(t))^2 f'(U(t)) - (L(t))^2 f'(L(t)) \right] dt \right|$$

$$\leqslant \frac{b-a}{4ab} \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{|g(x)|}{x^2}dx \right) \left| (U(t))^2 f'(U(t)) - (L(t))^2 f'(L(t)) \right| dt$$

$$\leqslant \frac{b-a}{4ab} \|g\|_\infty \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{1}{x^2}dx \right) \left| (U(t))^2 f'(U(t)) - (L(t))^2 f'(L(t)) \right| dt$$

$$\leqslant \left(\frac{b-a}{2ab} \right)^2 \|g\|_\infty \left\{ \int_0^1 \left[t(U(t))^2 \right]^{\frac{1}{p}} \left(\left[t(U(t))^2 \right]^{\frac{1}{q}} |f'(U(t))| \right) dt \right.$$

$$\left. + \int_0^1 \left[t(L(t))^2 \right]^{\frac{1}{p}} \left(\left[t(L(t))^2 \right]^{\frac{1}{q}} |f'(L(t))| \right) dt \right\}$$

$$\left(\text{where } \frac{1}{p} = 1 - \frac{1}{q} \right)$$

$$\leqslant \left(\frac{b-a}{2ab} \right)^2 \|g\|_\infty \left\{ \left[\int_0^1 \left(\left(t(U(t))^2 \right)^{\frac{1}{p}} \right)^p dt \right]^{\frac{1}{p}} \left[\int_0^1 \left(\left[t(U(t))^2 \right]^{\frac{1}{q}} |f'(U(t))| \right)^q dt \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[\int_0^1 \left(\left(t(L(t))^2 \right)^{\frac{1}{p}} \right)^p dt \right]^{\frac{1}{p}} \left[\int_0^1 \left(\left[t(L(t))^2 \right]^{\frac{1}{q}} |f'(L(t))| \right)^q dt \right]^{\frac{1}{q}} \right\}$$

$$= \left(\frac{b-a}{2ab} \right)^2 \|g\|_\infty \left\{ \left[\int_0^1 t(U(t))^2 dt \right]^{\frac{1}{p}} \left[\int_0^1 t(U(t))^2 |f'(U(t))|^q dt \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[\int_0^1 t(L(t))^2 dt \right]^{\frac{1}{p}} \left[\int_0^1 t(L(t))^2 |f'(L(t))|^q dt \right]^{\frac{1}{q}} \right\}$$

$$= \left(\frac{b-a}{2ab} \right)^2 \|g\|_\infty \left\{ \left[\int_0^1 t(U(t))^2 dt \right]^{1-\frac{1}{q}} \left[\int_0^1 t(U(t))^2 |f'(U(t))|^q dt \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[\int_0^1 t(L(t))^2 dt \right]^{1-\frac{1}{q}} \left[\int_0^1 t(L(t))^2 |f'(L(t))|^q dt \right]^{\frac{1}{q}} \right\}. \tag{23}$$

Resolving each of the integrals of (23)

By making the change $x = \dfrac{2ab}{(1+t)a+(1-t)b}$, you have to proceed to

$$\mu_1 = \int_0^1 t(U(t))^2 dt = \frac{2ab^2}{b-a} + \left(\frac{2ab}{b-a} \right)^2 \ln \left(\frac{2a}{a+b} \right). \tag{24}$$

Now to solve the integral $\int_0^1 t\,(L(t))^2\,dt$, we make the change $x = \dfrac{2ab}{(1-t)a+(1+t)b}$, we get

$$\mu_2 = \int_0^1 t\,(L(t))^2\,dt = \left(\frac{2ab}{b-a}\right)^2 \ln\left(\frac{2b}{a+b}\right) - \frac{2a^2b}{(b-a)^2}. \tag{25}$$

On the other hand by the reciprocal (s,m)–convexity of $|f'|^q$ on $\mathbb{R}^+$ for $q \geqslant 1$, we have

$$\int_0^1 t\,[U(t)]^2\,|f'(U(t))|^q\,dt \tag{26}$$

$$= \int_0^1 t\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2 \left|f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\right|^q\,dt$$

$$= \begin{cases} \displaystyle\int_0^1 t\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2 \left|f'\left(\frac{mab}{\frac{1}{2}(1+t)ma+\frac{1}{2}m(1-t)b}\right)\right|^q\,dt \\[4ex] \displaystyle\int_0^1 t\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2 \left|f'\left(\frac{amb}{\frac{1}{2}m(1+t)a+\frac{1}{2}(1-t)mb}\right)\right|^q\,dt \end{cases}$$

$$\leqslant \begin{cases} \displaystyle\int_0^1 t\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2 \left[\frac{1}{2^s}(1+t)^s|f'(b)|^q + \frac{1}{2^s}m(1-t)^s|f'(ma)|^q\right]\,dt \\[4ex] \displaystyle\int_0^1 t\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2 \left[\frac{1}{2^s}m(1+t)^s|f'(mb)|^q + \frac{1}{2^s}(1-t)^s|f'(a)|^q\right]\,dt. \end{cases}$$

Resolving each of the integrals above,

$$v_1 = \int_0^1 t(1+t)^s \left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2\,dt \tag{27}$$

$$= a^2 \left\{\beta(s+2,1)\left[2^{s+2}\cdot {}_2F_1\left(2,s+2;s+3;\frac{b-a}{b}\right) - {}_2F_1\left(2,s+2;s+3;\frac{b-a}{2b}\right)\right]\right.$$

$$\left. - \beta(s+1,1)\left[2^{s+1}\cdot {}_2F_1\left(2,s+1;s+2;\frac{b-a}{b}\right) - {}_2F_1\left(2,s+1;s+2;\frac{b-a}{2b}\right)\right]\right\},$$

$$v_2 = \int_0^1 t(1-t)^s \left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2\,dt \tag{28}$$

$$= \left(\frac{2ab}{a+b}\right)^2 \beta(2,s+1)\cdot {}_2F_1\left(2,2;s+3;\frac{b-a}{a+b}\right).$$

Substituting (27) and (28) in (26), we get

$$\int_0^1 t\,[U(t)]^2\,|f'(U(t))|^q\,dt \tag{29}$$

$$\leqslant \begin{cases} \dfrac{1}{2^s}\left[v_1\left|f'(b)\right|^q + mv_2\left|f'(ma)\right|^q\right] \\[3mm] \dfrac{1}{2^s}\left[mv_1\left|f'(mb)\right|^q + v_2\left|f'(a)\right|^q\right] \end{cases}$$

$$\leqslant \min\left\{\dfrac{1}{2^s}\left[v_1\left|f'(b)\right|^q + mv_2\left|f'(ma)\right|^q\right], \dfrac{1}{2^s}\left[mv_1\left|f'(mb)\right|^q + v_2\left|f'(a)\right|^q\right]\right\} = v.$$

Again by the reciprocal (s,m)–convexity of $|f'|^q$ on $\mathbb{R}^+$ for $q \geqslant 1$, we have

$$\int_0^1 t\left[L(t)\right]^2\left|f'(L(t))\right|^q dt \tag{30}$$

$$= \int_0^1 t\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2\left|f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right|^q dt$$

$$= \begin{cases} \int_0^1 t\left[\dfrac{2ab}{(1-t)a+(1+t)b}\right]^2\left|f'\left(\dfrac{mab}{\frac{1}{2}(1-t)ma+\frac{1}{2}m(1+t)b}\right)\right|^q dt \\[6mm] \int_0^1 t\left[\dfrac{2ab}{(1-t)a+(1+t)b}\right]^2\left|f'\left(\dfrac{amb}{\frac{1}{2}m(1-t)a+\frac{1}{2}(1+t)mb}\right)\right|^q dt \end{cases}$$

$$\leqslant \begin{cases} \int_0^1 t\left[\dfrac{2ab}{(1-t)a+(1+t)b}\right]^2\left[\dfrac{1}{2^s}(1-t)^s\left|f'(b)\right|^q + \dfrac{1}{2^s}m(1+t)^s\left|f'(ma)\right|^q\right] dt \\[6mm] \int_0^1 t\left[\dfrac{2ab}{(1-t)a+(1+t)b}\right]^2\left[\dfrac{1}{2^s}m(1-t)^s\left|f'(mb)\right|^q + \dfrac{1}{2^s}(1+t)^s\left|f'(a)\right|^q\right] dt \end{cases}$$

$$= \begin{cases} \dfrac{1}{2^s}\left[v_2\left|f'(b)\right|^q + mv_1\left|f'(ma)\right|^q\right] \\[3mm] \dfrac{1}{2^s}\left[mv_2\left|f'(mb)\right|^q + v_1\left|f'(a)\right|^q\right] \end{cases}$$

$$\leqslant \min\left\{\dfrac{1}{2^s}\left[v_2\left|f'(b)\right|^q + mv_1\left|f'(ma)\right|^q\right], \dfrac{1}{2^s}\left[mv_2\left|f'(mb)\right|^q + v_1\left|f'(a)\right|^q\right]\right\} = \eta.$$

Substituting (24), (25), (29) and (30) in (23), we get

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx - \int_a^b\frac{g(x)f(x)}{x^2}dx\right| \leqslant \left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\left[\mu_1^{1-\frac{1}{q}}v^{\frac{1}{q}} + \mu_2^{1-\frac{1}{q}}\eta^{\frac{1}{q}}\right].$$

Thus the demonstration is completed.

5. Applications for special means

Let us recall the following special means of two numbers $a,b \in \mathbb{R}$ (see [6]):

1. The arithmetic mean
$$A(a,b) := \frac{a+b}{2}.$$

2. The weight arithmetic mean
$$A_\lambda(a,b) := \lambda a + (1-\lambda)b, \text{ with } \lambda \in [0,1].$$

3. The geometric mean
$$G(a,b) := \sqrt{ab}.$$

4. The harmonic mean
$$H(a,b) := \frac{2ab}{a+b}.$$

5. The p–logarithmic mean with $p \in \mathbb{R} \setminus \{0\}$ and $a < b$.

The following theorems are results in which we present the relationship between the means defined above.

THEOREM 11. *Let $0 < a < b$. Then we have the following inequality*

$$2^{s-1}G^2(a^s,b^s)H^{-s}(a,b) \leqslant L_s^s(a,b) \leqslant A_{(s+1)^{-1}}\left(a^s, \frac{b^s}{s}\right), \tag{31}$$

with $s \in (0,1)$.

Proof. By example 2, we have $f : \mathbb{R}_+ \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x^s}$, for $s \in (0,1)$ is reciprocally $(s,1)$–convex and using Remark 2, we obtain

$$2^{s-1}f\left(\frac{2ab}{a+b}\right) \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leqslant \frac{f(a)+f(b)}{s+1}. \tag{32}$$

Solving each of expressions present in the above inequalities,

$$2^{s-1}f\left(\frac{2ab}{a+b}\right) = 2^{s-1}\left(\frac{2ab}{a+b}\right)^{-s} = 2^{s-1}H^{-s}(a,b). \tag{33}$$

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx = \frac{ab}{-(b-a)(s+1)}\left[b^{-(s+1)} - a^{-(s+1)}\right] \tag{34}$$

$$= \frac{ab}{(b-a)(s+1)}\left[\frac{1}{a^{s+1}} - \frac{1}{b^{s+1}}\right] = \frac{ab}{(b-a)(s+1)}\frac{b^{s+1} - a^{s+1}}{a^{s+1}b^{s+1}}$$

$$= \frac{1}{a^s b^s}\frac{b^{s+1} - a^{s+1}}{(s+1)(b-a)} = \frac{1}{G^2(a^s,b^s)}\left\{\left[\frac{b^{s+1} - a^{s+1}}{(s+1)(b-a)}\right]^{\frac{1}{s}}\right\}^s$$

$$= \frac{1}{G^2(a^s,b^s)}L_s^s(a,b),$$

and

$$\frac{f(a)+f(b)}{s+1} = \frac{a^s+b^s}{(s+1)a^sb^s} \tag{35}$$

$$= \frac{1}{a^sb^s}\frac{a^s+(s+1-1)\dfrac{b^s}{s}}{s+1} = \frac{1}{G^2(a^s,b^s)}\left[\frac{1}{s+1}a^s+\left(1-\frac{1}{s+1}\right)\frac{b^s}{s}\right]$$

$$= \frac{1}{G^2(a^s,b^s)}\left[(s+1)^{-1}a^s+\left(1-(s+1)^{-1}\right)\frac{b^s}{s}\right]$$

$$= \frac{1}{G^2(a^s,b^s)}\left[(s+1)^{-1}a^s+\left(1-(s+1)^{-1}\right)\frac{b^s}{s}\right]$$

$$= \frac{1}{G^2(a^s,b^s)}A_{(s+1)^{-1}}\left(a^s,\frac{b^s}{s}\right).$$

Substituting (33)–(35) in (32), we get

$$2^{s-1}H^{-s}(a,b) \leqslant \frac{1}{G^2(a^s,b^s)}L_s^s(a,b) \leqslant \frac{1}{G^2(a^s,b^s)}A_{(s+1)^{-1}}\left(a^s,\frac{b^s}{s}\right)$$

So we get the inequality (31).

THEOREM 12. *Let $0 < a < b$. Then we have the following inequality*

$$2^{s-1}G^2(a^p,b^p)H^{-p}(a,b) \leqslant L_p^p(a,b) \leqslant A_{(s+1)^{-1}}\left(a^p,\frac{b^p}{s}\right), \tag{36}$$

with $s \in (0,1)$.

Proof. By example 1, we have $f : \mathbb{R}_+ \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x^p}$, for $p \in [1,+\infty)$ is reciprocally $(s,1)$–convex for $s \in (0,1]$ and using Remark 2, we obtain

$$2^{s-1}f\left(\frac{2ab}{a+b}\right) \leqslant \frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx \leqslant \frac{f(a)+f(b)}{s+1}. \tag{37}$$

Solving each of expressions present in the above inequalities,

$$2^{s-1}f\left(\frac{2ab}{a+b}\right) = 2^{s-1}\left(\frac{2ab}{a+b}\right)^{-p} = 2^{s-1}H^{-p}(a,b). \tag{38}$$

$$\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx = \frac{ab}{b-a}\int_a^b\frac{1}{x^{p+2}}dx \tag{39}$$

$$= \frac{ab}{-(b-a)(p+1)}\left[b^{-(p+1)}-a^{-(p+1)}\right]$$

$$= \frac{ab}{(b-a)(p+1)}\left[\frac{1}{a^{p+1}}-\frac{1}{b^{p+1}}\right] = \frac{ab}{(b-a)(p+1)}\frac{b^{p+1}-a^{p+1}}{a^{p+1}b^{p+1}}$$

$$= \frac{1}{a^p b^p} \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} = \frac{1}{G^2(a^p, b^p)} \left\{ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} \right\}^p$$

$$= \frac{1}{G^2(a^p, b^p)} L_p^p(a,b).$$

$$\frac{f(a) + f(b)}{s+1} = \frac{a^p + b^p}{(s+1)a^p b^p} \tag{40}$$

$$= \frac{1}{a^p b^p} \frac{a^p + (s+1-1)\dfrac{b^p}{s}}{s+1}$$

$$= \frac{1}{G^2(a^p, b^p)} \left[(s+1)^{-1} a^p + \left(1 - (s+1)^{-1} \right) \frac{b^p}{s} \right]$$

$$= \frac{1}{G^2(a^p, b^p)} \left[(s+1)^{-1} a^p + \left(1 - (s+1)^{-1} \right) \frac{b^p}{s} \right]$$

$$= \frac{1}{G^2(a^p, b^p)} A_{(s+1)^{-1}} \left(a^p, \frac{b^p}{s} \right).$$

Substituting (38)–(40) in (37), we get

$$2^{s-1} H^{-p}(a,b) \leqslant \frac{1}{G^2(a^p, b^p)} L_p^p(a,b) \leqslant \frac{1}{G^2(a^p, b^p)} A_{(s+1)^{-1}} \left(a^p, \frac{b^p}{s} \right).$$

So we get the inequality (36).

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Mireya R. Bracamonte
ESPOL Polytechnic University, Escuela Superior Politécnica del Litoral
ESPOL, Facultad de Ciencias Naturales y Matemática
Campus Gustavo Galindo Km. 30.5 Vía Perimetral, P.O. Box
09-01-5863, Guayaquil, Ecuador
e-mail: mirebrac@gmail.com, mrbracam@espol.edu.ec

Jesús Medina-Viloria
Departamento de Matemática
Universidad Centroccidental Lisandro Alvarado
Barquisimeto, Venezuela
e-mail: jesus.medina@ucla.edu.ve

Miguel Vivas-Cortez
Pontificia Universidad Católica del Ecuador
Facultad de Ciencias Exactas y Naturales, Escuela de
Ciencias Físicas y Matemática
Av. 12 de octubre 1076, Quito, Ecuador
e-mail: mjvivas@puce.edu.ec

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