

HERMITE–HADAMARD, FEJER AND SHERMAN TYPE INEQUALITIES FOR GENERALIZATIONS OF SUPERQUADRATIC AND CONVEX FUNCTIONS

SHOSHANA ABRAMOVICH

*In honour of Academician Professor Josip Pečarić
on the occasion of his 70th birthday*

(Communicated by J. Pečarić)

Abstract. In this paper we prove some Hermite-Hadamard, Fejer and Sherman type inequalities for generalizations of superquadratic functions and convex functions. These results, under a monotonicity condition, lead to refinements of the Hermite-Hadamard, Fejer and Sherman inequalities of non-negative convex functions. Also, the obtained inequalities are discussed about and compared with some recent generalizations of weighted Hermite-Hadamard inequalities.

1. Introduction

The Hermite-Hadamard inequality says that for any convex function $f : I \rightarrow \mathbb{R}$, I an interval, and for $a, b \in I$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a)+f(b)}{2}$$

holds, and the Fejer inequality reads

$$f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \leq \int_a^b f(t) p(t) dt \leq \frac{f(a)+f(b)}{2} \int_a^b p(x) dx,$$

when f is convex and $p : [a, b] \rightarrow \mathbb{R}$ is non-negative, integrable and symmetric around the midpoint $x = \frac{a+b}{2}$. The Sherman inequality shows that

$$\sum_{i=1}^n a_i f(x_i) \geq \sum_{j=1}^m b_j f(y_j)$$

Mathematics subject classification (2010): 26D15, 94A17.

Keywords and phrases: Convexity, superquadracity, N -quasisuperquadracity, N -quasiconvexity, Jensen type inequalities, Hermite-Hadamard type inequalities, Fejer type inequalities, Sherman type inequalities, f -divergence.

holds for any convex function f on \mathbb{R}_+ and for \mathbf{x} and $\mathbf{a} \in \mathbb{R}_+^n$, \mathbf{y} and $\mathbf{b} \in \mathbb{R}_+^m$ where $\mathbf{y} = \mathbf{x}S$ and $\mathbf{a} = \mathbf{b}S^T$ for some $n \times m$ column stochastic matrix $S = (s_{ij})$.

Numerous publications deal with convex functions, their properties and applications. In particular we refer to the classical 1964 book "Inequalities" by Hardy, Littlewood and Polya [7], the 1992 book "Convex functions, partial ordering and statistical applications" by Pečarić, Proschan and Tong [17] and to the 2006 book and its 2018 second edition "Convex functions and their applications - a contemporary approach" by Niculescu and Persson [16]. Out of dealing with the classical convex functions evolved many generalizations and refinements of this notion.

In this paper we prove some of Hermite-Hadamard and Sherman type inequalities for superquadratic functions, convex functions and their generalizations. These results, under a monotonicity condition, lead to refinements of the Hermite-Hadamard, Fejer and Sherman inequalities of non-negative convex functions.

The obtained inequalities are also compared with the results dealt with in [10] about weighted generalizations of Hermite-Hadamard and Fejer type inequalities.

We state some definitions and lemmas that we use in the sequel.

DEFINITION 1. A function $\varphi : [0, B) \rightarrow \mathbb{R}$ is superquadratic provided that for all $x \in [0, B)$ there exists a constant $C_\varphi(x) \in \mathbb{R}$ such that the inequality

$$\varphi(y) \geq \varphi(x) + C_\varphi(x)(y-x) + \varphi(|y-x|) \quad (1.1)$$

holds for all $y \in [0, B)$, (see [1, Definition 2.1], there $[0, \infty)$ instead $[0, B)$).

LEMMA 1. [1, inequality 1.2] *The inequality*

$$\int \varphi(f(s)) d\mu(s) \geq \varphi\left(\int f d\mu\right) + \int \varphi\left(\left|f(s) - \int f d\mu\right|\right) d\mu(s)$$

holds for all probability measures μ and all non-negative, μ -integrable functions f if and only if φ is superquadratic.

LEMMA 2. [1, Lemma 2.1] *Let φ be a superquadratic function with $C_\varphi(x)$ as in Definition 1.*

(i) *Then $\varphi(0) \leq 0$.*

(ii) *If $\varphi(0) = \varphi'(0) = 0$, then $C_\varphi(x) = \varphi'(x)$ whenever φ is differentiable on $[0, B)$.*

(iii) *If $\varphi \geq 0$, then φ is convex and $\varphi(0) = \varphi'(0) = 0$.*

LEMMA 3. [1, Lemma 3.1] *Suppose $\varphi : [0, B) \rightarrow \mathbb{R}$ is continuously differentiable and $\varphi(0) \leq 0$. If φ' is superadditive or $\varphi'(x)/x$ is non-decreasing, then φ is superquadratic and $C_\varphi(x) = \varphi'(x)$ with $C_\varphi(x)$ as in Definition 1.*

DEFINITION 2. A function $\phi_N : [0, B) \rightarrow \mathbb{R}$ is N -quasisuperquadratic if $\phi_N(x) = x^N \varphi(x)$ where φ is superquadratic.

In Lemma 4 we get a Jensen type inequality:

LEMMA 4. [4, Lemma 3] Let $\phi_N(x) = x^N \varphi(x)$, where $\varphi(x)$ is superquadratic on $[0, B)$, that is ϕ_N is N -quasisuperquadratic. Then

$$\phi_N(y) - \phi_N(x) = \varphi(y)y^N - \varphi(x)x^N \geq \varphi(x)(y^N - x^N) + C_\varphi(x)y^N(y-x) + y^N\varphi(|y-x|) \tag{1.2}$$

holds for $x \in [0, B)$, $y \in [0, B)$, $C_\varphi(x)$ as in (1.1).

Also, Jensen type inequality for N -quasisuperquadratic functions

$$\begin{aligned} \int_{\Omega} \phi_N(f(s)) d\mu(s) - \phi_N(\bar{x}) &= \int_{\Omega} \varphi(f(s))(f(s))^N d\mu - \varphi(\bar{x})\bar{x}^N \\ &\geq \int_{\Omega} [C_\varphi(\bar{x})f^N(s)(f(s) - \bar{x}) + f^N(s)\varphi(|f(s) - \bar{x}|)] d\mu(s) \end{aligned}$$

holds, where f is any non-negative μ -integrable function on the probability measure space (Ω, μ) and $\bar{x} = \int_{\Omega} f d\mu > 0$.

LEMMA 5. [5, Lemma 1] Let φ be a differentiable function on an interval $I \subset \mathbb{R}$, and let $x, y \in I$. Then, for $N = 0, 1, 2, \dots$,

$$\begin{aligned} \varphi(x)(y^N - x^N) + \varphi'(x)y^N(y-x) &\tag{1.3} \\ &= (x^N \varphi(x))'(y-x) + (y-x)^2 \sum_{k=1}^N y^{k-1} (x^{N-k} \varphi(x))' \\ &= (x^N \varphi(x))'(y-x) + (y-x)^2 \frac{\partial}{\partial x} \left(\frac{x^N - y^N}{x-y} \varphi(x) \right). \end{aligned}$$

In particular, for $N = 1$ we have that

$$\varphi(x)(y-x) + \varphi'(x)y(y-x) = (x\varphi(x))'(y-x) + \varphi'(x)(y-x)^2.$$

DEFINITION 3. [2] A function $\phi_N : [0, B) \rightarrow \mathbb{R}$ is N -quasiconvex, where $N \in \mathbb{R}$, provided that for all $x \in [0, B)$ $\phi_N(x) = x^N \varphi(x)$, where φ is convex on $[0, B)$.

REMARK 1. From (1.2) in Lemma 4 and (1.3) in Lemma 5 in the case that φ is superquadratic differentiable and satisfies $C_\varphi(x) = \varphi'(x)$ the inequality

$$\phi_N(y) - \phi_N(x) \geq \phi'_N(x)(y-x) + (y-x)^2 \frac{\partial}{\partial x} \left(\frac{x^N - y^N}{x-y} \varphi(x) \right) + y^N\varphi(|y-x|) \tag{1.4}$$

holds, where $\phi_N(x) = x^N \varphi(x)$ and N is an integer.

Also it is obvious that differentiable N -quasiconvex function satisfies

$$\phi_N(y) - \phi_N(x) \geq \phi'_N(x)(y-x) + (y-x)^2 \frac{\partial}{\partial x} \left(\frac{x^N - y^N}{x-y} \varphi(x) \right), \tag{1.5}$$

where $\phi_N(x) = x^N \varphi(x)$, φ is convex and N is an integer (see [2, Lemma 1]).

THEOREM 1. [6, Theorem 8] *Let $\varphi : [0, B] \rightarrow \mathbb{R}$ be a superquadratic function and let $0 \leq a < b < B$, then*

$$\begin{aligned} & \varphi\left(\frac{a+b}{2}\right) + \frac{1}{b-a} \int_a^b \varphi\left(\left|x - \frac{a+b}{2}\right|\right) dx \leq \frac{1}{b-a} \int_a^b \varphi(x) dx \\ & \leq \frac{\varphi(a) + \varphi(b)}{2} - \frac{1}{(b-a)^2} \int_a^b ((b-x)\varphi(x-a) + (x-a)\varphi(b-x)) dx. \end{aligned} \tag{1.6}$$

In Section 2 we generalize the above Theorem 1 and obtain Hermite-Hadamard type inequalities for N -quasisuperquadratic functions.

In Section 3 we generalize for N -quasisuperquadratic and N -quasiconvex functions the following Sherman inequality and Sherman type inequality.

The Sherman inequality says:

THEOREM 2. [18] *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be convex on \mathbb{R}_+ , $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}_+^n$, $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}_+^m$, $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}_+^n$, and $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{R}_+^m$.*

Assume that $\mathbf{y} = \mathbf{xS}$ and $\mathbf{a} = \mathbf{bS}^T$ for some $n \times m$ column stochastic matrix $S = (s_{ij})$. Then,

$$\sum_{i=1}^n a_i f(x_i) \geq \sum_{j=1}^m b_j f(y_j).$$

Sherman type inequality for superquadratic functions is:

THEOREM 3. [14] *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be superquadratic function and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}_+^n$, $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}_+^m$, $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}_+^n$, and $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{R}_+^m$.*

Assume that $\mathbf{y} = \mathbf{xS}$ and $\mathbf{a} = \mathbf{bS}^T$ for some $n \times m$ column stochastic matrix $S = (s_{ij})$. Then,

$$\sum_{j=1}^m b_j f(y_j) + \sum_{i=1}^n \sum_{j=1}^m b_j s_{ij} f(|x_i - y_j|) \leq \sum_{i=1}^n a_i f(x_i)$$

holds.

In the sequel we use the standard notation $\langle \cdot, \cdot \rangle$ for the inner product.

From Theorem 3 the following Sherman type inequality proved in [11, Theorem 1] reads:

THEOREM 4. *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a superquadratic function. Let*

$$\begin{aligned} \mathbf{p} &= (p_1, \dots, p_m) \in \mathbb{R}_{++}^m, \quad \mathbf{q} = (q_1, \dots, q_m) \in \mathbb{R}_+^m, \quad \mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}_+^m \\ \mathbf{r}_j &= (r_{1j}, \dots, r_{mj}) \in \mathbb{R}_+^m, \quad j = 1, \dots, n, \quad \mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}_+^n. \end{aligned}$$

Then, for $a_i = \sum_{j=1}^n b_j \frac{r_{ij} p_i}{\langle \mathbf{r}_j, \mathbf{p} \rangle}$, $i = 1, \dots, m$:

$$\sum_{i=1}^m a_i \varphi\left(\frac{q_i}{p_i}\right) - \sum_{j=1}^n b_j \varphi\left(\frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle}\right) \geq \sum_{j=1}^n \sum_{i=1}^m b_j \frac{p_i r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \varphi\left(\left|\frac{q_i}{p_i} - \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle}\right|\right). \tag{1.7}$$

Inequality (1.7) is called also the Csiszár-Körner inequality for φ -divergence when φ is a superquadratic function (see [15]).

The papers [8], [9], [11], [12], [13] and [15], published lately, deal with various extensions of the Sherman inequality applied to some types of f -divergences such as Kullback-Leibler, Csiszár, Tsallis and other types of f -divergences, and derived new estimations of the Shannon and Rényi entropies (see also the referenses of these papers for definitions and other results).

In Section 4 we compare different types of Hermite-Hadamard and Fejer inequalities for monotonic weight functions which appear in [10] resulting from functions φ and g such that $\frac{\varphi'}{g}$ is non-decreasing.

2. N -quasisuperquadracity, N -quasiconvexity and new Hermite-Hadamard and Fejer type inequalities

Theorems 5 and 6 in this section deal with Hermite-Hadamard and Fejer type inequalities. Theorem 5 generalizes Theorem 1 for N -quasisuperquadratic functions ψ_N where $\psi_N(x) = x^N \varphi(x)$. The proof uses Lemma 4 for $f(x) = x$ and $d\mu = \frac{1}{b-a} dx$ on the interval $0 \leq a < b < B$. It combines the techniques used in Theorem 1 proved in [6, Theorem 8] which deals with superquadratic functions with the technique employed to prove Remark 2 quoted in the end of this section (that appears in [2, Theorem 1]) which deals with N -quasiconvex functions. It reads:

THEOREM 5. *Let $\psi_N : [0, B) \rightarrow \mathbb{R}$, $0 < B \leq \infty$ be a differentiable N -quasisuperquadratic function that satisfies $\psi_N(x) = x^N \varphi(x)$, $N = 0, 1, 2, \dots$, where φ is superquadratic, such that for $0 \leq a < b < B$, for $C_\varphi(x) = \varphi'(x)$ (C_φ as in (1.1)), and for non-negative integrable and symmetric function p on $[a, b]$, the inequalities*

$$\int_a^b \psi_N(x) p(x) dx \tag{2.1}$$

$$\geq \psi_N\left(\frac{a+b}{2}\right) \int_a^b p(x) dx + \int_a^b \left(x - \frac{a+b}{2}\right)^2 \left(\frac{\partial}{\partial \bar{x}} \left(\frac{x^N - \bar{x}^N}{x - \bar{x}} \varphi(\bar{x})\right) \Big|_{\bar{x}=\frac{a+b}{2}}\right) p(x) dx$$

$$+ \int_a^b x^N \varphi\left(\left|x - \frac{a+b}{2}\right|\right) p(x) dx,$$

and

$$\int_a^b \psi_N(x) p(x) dx \tag{2.2}$$

$$\leq \frac{\psi_N(a) + \psi_N(b)}{2} \int_a^b p(x) dx - \frac{1}{(b-a)} \int_a^b \left[(x-a)(b-x)^2 \frac{\partial}{\partial x} \left(\frac{b^N - x^N}{b-x} \varphi(x)\right) \right.$$

$$\left. + (x-a)^2 (b-x) \frac{\partial}{\partial x} \left(\frac{x^N - a^N}{x-a} \varphi(x)\right) \right] p(x) dx$$

$$- \frac{a^N + b^N}{2(b-a)} \int_a^b ((b-x) \varphi(x-a) + (x-a) \varphi(b-x)) p(x) dx$$

hold.

In the special case when $N = 1$ and $p(x) = 1$ on $[a, b]$, we get that

$$\begin{aligned} & \psi_1\left(\frac{a+b}{2}\right) + \varphi'\left(\frac{a+b}{2}\right) \frac{(b-a)^2}{12} + \frac{1}{b-a} \int_a^b x\varphi\left(\left|x - \frac{a+b}{2}\right|\right) dx \quad (2.3) \\ & \leq \frac{1}{b-a} \int_a^b \psi_1(x) dx \\ & \leq \frac{\psi_1(a) + \psi_1(b)}{2} - \frac{1}{b-a} \int_a^b \varphi'(x)(x-a)(b-x) dx \\ & \quad - \frac{a+b}{2} \frac{1}{(b-a)^2} \int_a^b ((b-x)\varphi(x-a) + (x-a)\varphi(b-x)) dx. \end{aligned}$$

In the special case that $N = 0$ we get inequality (1.6).

Proof. Inequality (2.1) follows from inequality (1.4) in Remark 1 satisfied by N -quasisuperquadratic functions where by replacing in (1.4) y by x , x by $\bar{x} = \frac{a+b}{2}$, and $d\mu = \frac{1}{b-a} dx$ and multiplying (1.4) with the non-negative integrable symmetric p and then integrating on the interval $[a, b]$.

To prove (2.2) we use again the basic inequality (1.4). By multiplying inequality (1.4) first by α and then by β where $0 \leq \alpha \leq 1$, $\alpha + \beta = 1$, we get by choosing first $y = y_1$ and then $y = y_2$, that $\bar{x} = \alpha y_1 + \beta y_2$ and when φ is superquadratic on $x \in [a, b]$,

$$\begin{aligned} & \alpha\psi_N(y_1) + \beta\psi_N(y_2) - \psi_N(\bar{x}) \\ & \geq \alpha\beta^2(y_1 - y_2)^2 \frac{\partial}{\partial x} \left(\frac{x^N - y_1^N}{x - y_1} \varphi(x) \right) \Big|_{x=\bar{x}} + \beta\alpha^2(y_1 - y_2)^2 \frac{\partial}{\partial x} \left(\frac{x^N - y_2^N}{x - y_2} \varphi(x) \right) \Big|_{x=\bar{x}} \\ & \quad + \alpha y_1^N \varphi(\beta|y_1 - y_2|) + \beta y_2^N \varphi(\alpha|y_1 - y_2|) \end{aligned}$$

holds.

By a first choice

$$\alpha = \frac{b-x}{b-a}, \quad \beta = \frac{x-a}{b-a}, \quad 0 \leq a \leq x \leq b, \quad y_1 = a, \quad y_2 = b,$$

we get that

$$\bar{x} = \alpha y_1 + \beta y_2 = \frac{b-x}{b-a} a + \frac{x-a}{b-a} b = x$$

and

$$\begin{aligned} & \frac{b-x}{b-a} \psi_N(a) + \frac{x-a}{b-a} \psi_N(b) - \psi_N(x) \quad (2.4) \\ & \geq \frac{(x-a)^2(b-x)}{b-a} \frac{\partial}{\partial x} \left(\frac{x^N - a^N}{x-a} \varphi(x) \right) + \frac{(x-a)(b-x)^2}{b-a} \frac{\partial}{\partial x} \left(\frac{x^N - b^N}{x-b} \varphi(x) \right) \\ & \quad + \frac{b-x}{b-a} a^N \varphi(x-a) + \frac{x-a}{b-a} b^N \varphi(b-x), \end{aligned}$$

and by a second choice which is

$$\alpha = \frac{x-a}{b-a}, \quad \beta = \frac{b-x}{b-a}, \quad 0 \leq a \leq x \leq b, \quad y_1 = a, \quad y_2 = b$$

we get that

$$\bar{x} = \alpha y_1 + \beta y_2 = \frac{x-a}{b-a}a + \frac{b-x}{b-a}b = a + b - x$$

and therefore the inequality

$$\begin{aligned} & \frac{x-a}{b-a} \psi_N(a) + \frac{b-x}{b-a} \psi_N(b) - \psi_N(a+b-x) \\ & \geq \frac{(x-a)(b-x)^2}{b-a} \frac{\partial}{\partial \bar{x}} \left(\frac{\bar{x}^N - a^N}{\bar{x}-a} \varphi(\bar{x}) \right) \Big|_{\bar{x}=a+b-x} \\ & \quad + \frac{(x-a)^2(b-x)}{b-a} \frac{\partial}{\partial \bar{x}} \left(\frac{\bar{x}^N - b^N}{\bar{x}-b} \varphi(\bar{x}) \right) \Big|_{\bar{x}=a+b-x} \\ & \quad + \frac{x-a}{b-a} a^N \varphi(b-x) + \frac{b-x}{b-a} b^N \varphi(x-a) \end{aligned} \tag{2.5}$$

holds.

Adding (2.4) and (2.5) we get that

$$\begin{aligned} & \psi_N(a) + \psi_N(b) \\ & \geq \psi_N(x) + \psi_N(a+b-x) + \frac{(x-a)^2(b-x)}{b-a} \frac{\partial}{\partial \bar{x}} \left(\frac{\bar{x}^N - a^N}{\bar{x}-a} \varphi(\bar{x}) \right) \Big|_{\bar{x}=x} \\ & \quad + \frac{(x-a)(b-x)^2}{b-a} \frac{\partial}{\partial \bar{x}} \left(\frac{\bar{x}^N - b^N}{\bar{x}-b} \varphi(\bar{x}) \right) \Big|_{\bar{x}=x} \\ & \quad + \frac{(x-a)^2(b-x)}{b-a} \frac{\partial}{\partial \bar{x}} \left(\frac{\bar{x}^N - b^N}{\bar{x}-b} \varphi(\bar{x}) \right) \Big|_{\bar{x}=a+b-x} \\ & \quad + \frac{(x-a)(b-x)^2}{b-a} \frac{\partial}{\partial \bar{x}} \left(\frac{\bar{x}^N - a^N}{\bar{x}-a} \varphi(\bar{x}) \right) \Big|_{\bar{x}=a+b-x} \\ & \quad + \frac{a^N + b^N}{b-a} ((b-x) \varphi(x-a) + (x-a) \varphi(b-x)) \end{aligned} \tag{2.6}$$

holds.

Multiplying (2.6) with the non-negative symmetric function p and integrating we get that

$$\begin{aligned} & \int_a^b \psi_N(x) p(x) dx \\ & = \int_a^{\frac{a+b}{2}} \psi_N(x) p(x) dx + \int_{\frac{a+b}{2}}^b \psi_N(x) p(x) dx \\ & = \int_a^{\frac{a+b}{2}} (\psi_N(x) p(x) + \psi_N(a+b-x) p(a+b-x)) dx \end{aligned}$$

$$\begin{aligned}
 &\leq (\psi_N(a) + \psi_N(b)) \int_a^{\frac{a+b}{2}} p(x) dx \\
 &\quad - \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (b-x)(x-a)^2 \frac{\partial}{\partial x} \left(\frac{x^N - a^N}{x-a} \varphi(x) \right) p(x) dx \\
 &\quad - \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (x-a)(b-x)^2 \frac{\partial}{\partial x} \left(\frac{x^N - b^N}{x-b} \varphi(x) \right) p(x) dx \\
 &\quad - \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (x-a)(b-x)^2 \frac{\partial}{\partial \bar{x}} \left(\frac{\bar{x}^N - a^N}{\bar{x}-a} \varphi(\bar{x}) \right) \Big|_{\bar{x}=a+b-x} p(x) dx \\
 &\quad - \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (b-x)(x-a)^2 \frac{\partial}{\partial \bar{x}} \left(\frac{\bar{x}^N - b^N}{\bar{x}-b} \varphi(\bar{x}) \right) \Big|_{\bar{x}=a+b-x} p(x) dx. \\
 &\quad - \frac{a^N + b^N}{b-a} \int_a^{\frac{a+b}{2}} ((b-x)\varphi(x-a)p(x) + (x-a)\varphi(b-x))p(a+b-x)dx.
 \end{aligned}$$

Therefore by using the symmetry of p around $\frac{a+b}{2}$, (2.2) holds. The proof is complete. In the special case of $N = 1$ and $p(x) = 1$, we can improve (2.3) and get:

THEOREM 6. *Let $\phi : [0, B) \rightarrow \mathbb{R}$ be a differentiable 1-quasisuperquadratic function that satisfies $\phi(x) = x\varphi(x)$, where φ is superquadratic. Then for $0 \leq a < b < B$ and for $C_\varphi(x) = \varphi'(x)$, C_φ as in (1.2), the inequality*

$$\begin{aligned}
 &\phi\left(\frac{a+b}{2}\right) + \varphi'\left(\frac{a+b}{2}\right) \frac{(b-a)^2}{12} + \frac{1}{b-a} \int_a^b x\varphi\left(\left|x - \frac{a+b}{2}\right|\right) dx \quad (2.7) \\
 &\leq \frac{1}{b-a} \int_a^b \phi(x) dx \\
 &\leq \frac{\phi(a) + \phi(b)}{6} + \frac{a+b}{3} \frac{1}{b-a} \int_a^b \varphi(x) dx \\
 &\quad - \frac{a+b}{6} \frac{1}{(b-a)^2} \int_a^b ((b-x)\varphi(x-a) + (x-a)\varphi(b-x)) dx,
 \end{aligned}$$

holds.

Proof. From the identity

$$- \int_a^b \varphi'(x)(x-a)(b-x) dx = (a+b) \int_a^b \varphi(x) dx - 2 \int_a^b \phi(x) dx$$

we get together with (2.3), the right hand-side of the inequality (2.7) holds. This completes the proof of the theorem.

REMARK 2. Comparing the inequalities (1.4) with (1.5) in Remark 1 we get for differentiable convex functions φ when N is an integer, similarly to (2.1), (2.2) and (2.7), that for N -quasiconvex functions ψ_N (see [2, Theorem 1]):

$$\int_a^b \psi_N(x) p(x) dx \tag{2.8}$$

$$\geq \psi_N \left(\frac{a+b}{2} \right) \int_a^b p(x) dx + \int_a^b \left(x - \frac{a+b}{2} \right)^2 \left(\frac{\partial}{\partial \bar{x}} \left(\frac{x^N - \bar{x}^N}{x - \bar{x}} \varphi(\bar{x}) \right) \Big|_{\bar{x} = \frac{a+b}{2}} \right) p(x) dx$$

and

$$\begin{aligned} & \int_a^b \psi_N(x) p(x) dx \\ \leq & \frac{\psi_N(a) + \psi_N(b)}{2} \int_a^b p(x) dx - \frac{1}{(b-a)} \int_a^b \left[(x-a)(b-x)^2 \frac{\partial}{\partial x} \left(\frac{b^N - x^N}{b-x} \varphi(x) \right) \right. \\ & \left. + (x-a)^2 (b-x) \frac{\partial}{\partial x} \left(\frac{x^N - a^N}{x-a} \varphi(x) \right) \right] p(x) dx. \end{aligned}$$

hold. For 1-quasiconvex functions ϕ and $p(x) = 1$ the inequalities

$$\begin{aligned} & \phi \left(\frac{a+b}{2} \right) + \phi' \left(\frac{a+b}{2} \right) \frac{(b-a)^2}{12} \\ & \leq \frac{1}{b-a} \int_a^b \phi(x) dx \leq \frac{\phi(a) + \phi(b)}{6} + \frac{a+b}{3} \frac{1}{b-a} \int_a^b \varphi(x) dx \end{aligned} \tag{2.9}$$

hold.

3. Sherman type inequalities obtained from Jensen type inequalities for N -quasisuperquadratic and N -quasiconvex functions

In this section we state and prove Sherman type inequalities for N -quasisuperquadratic functions. As an immediate result we get Sherman type inequalities for superquadratic and analogous result for convex and N -quasiconvex functions. These results generalize results proved in [11], [14], [15] and [18] using the technique of the proof in [11].

THEOREM 7. *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable superquadratic function on $[0, B)$ for which $C_\varphi(x) = \varphi'(x)$, $C_\varphi(x)$ as in (1.1). Let*

$$\begin{aligned} \mathbf{p} &= (p_1, \dots, p_m) \in \mathbb{R}_{++}^m, & \mathbf{q} &= (q_1, \dots, q_m) \in \mathbb{R}_+^m, \\ \mathbf{a} &= (a_1, \dots, a_m) \in \mathbb{R}_+^m, & \mathbf{b} &= (b_1, \dots, b_n) \in \mathbb{R}_+^n \\ \mathbf{r}_j &= (r_{1j}, \dots, r_{mj}) \in \mathbb{R}_+^m, & j &= 1, \dots, n, & \frac{q_i}{p_i} &\in [0, B), \quad i = 1, \dots, m. \end{aligned}$$

Denote

$$\langle \mathbf{q}, \mathbf{r}_j \rangle = \sum_{i=1}^m q_i r_{ij}, \quad \langle \mathbf{p}, \mathbf{r}_j \rangle = \sum_{i=1}^m p_i r_{ij}, \quad j = 1, \dots, n.$$

Then, for the N -quasisuperquadratic function ϕ , where $\phi(x) = x^N \varphi(x)$, $N = 0, 1, 2, \dots$, and for $a_i = \sum_{j=1}^n b_j \frac{r_{i,j} p_i}{\langle \mathbf{r}_j, \mathbf{p} \rangle}$, $i = 1, \dots, m$, the inequality

$$\sum_{i=1}^m a_i \phi \left(\frac{q_i}{p_i} \right) - \sum_{j=1}^n b_j \phi \left(\frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right) \tag{3.1}$$

$$\begin{aligned} &\geq \sum_{j=1}^n \sum_{i=1}^m b_j \frac{p_i r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \left(\frac{q_i}{p_i} - \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right)^2 \frac{\partial}{\partial \bar{x}_j} \left(\frac{\left(\frac{q_i}{p_i} \right)^N - \bar{x}_j^N}{\frac{q_i}{p_i} - \bar{x}_j} \varphi(\bar{x}_j) \right) \Big|_{\bar{x}_j = \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle}} \\ &\quad + \sum_{j=1}^n \sum_{i=1}^m b_j \frac{p_i^{1-N} q_i^N r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \varphi \left(\left| \frac{q_i}{p_i} - \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right| \right) \end{aligned}$$

holds.

Proof. From the inequality (1.4) satisfied by N -quasisuperquadratic function ϕ , it follows that:

$$\begin{aligned} &\sum_{i=1}^m \alpha_i \phi(x_i) - \phi(\bar{x}) \\ &\geq \sum_{i=1}^m \alpha_i (x_i - \bar{x})^2 \frac{\partial}{\partial \bar{x}} \left(\frac{x_i^N - \bar{x}^N}{x_i - \bar{x}} \varphi(\bar{x}) \right) + \sum_{i=1}^m \alpha_i x_i^N \varphi(|x_i - \bar{x}|), \end{aligned}$$

where $\alpha_i \geq 0, i = 1, \dots, m, \sum_{i=1}^m \alpha_i = 1, \sum_{i=1}^m \alpha_i x_i = \bar{x}$.

Replacing α_i by $\frac{p_i r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle}$ and x_i by $\frac{q_i}{p_i}, i = 1, \dots, m$ we get that \bar{x} is replaced by

$$\bar{x}_j = \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle}, j = 1, \dots, n \text{ and}$$

$$\begin{aligned} &\sum_{i=1}^m \frac{p_i r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \phi \left(\frac{q_i}{p_i} \right) - \phi \left(\frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right) \\ &\geq \sum_{i=1}^m \frac{p_i r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \left(\frac{q_i}{p_i} - \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right)^2 \frac{\partial}{\partial \bar{x}} \left(\frac{x_i^N - \bar{x}^N}{x_i - \bar{x}} \varphi(\bar{x}) \right) \Big|_{\bar{x} = \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle}} \\ &\quad + \sum_{i=1}^m \frac{p_i r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \left(\frac{q_i}{p_i} \right)^N \varphi \left(\left| \frac{q_i}{p_i} - \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right| \right). \end{aligned}$$

Multiplying by b_j and summing on $j = 1, \dots, n$ we get by changing the order of summation that

$$\begin{aligned} &\sum_{i=1}^m \sum_{j=1}^n b_j \frac{p_i r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \phi \left(\frac{q_i}{p_i} \right) - \sum_{j=1}^n b_j \phi \left(\frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right) \\ &\geq \sum_{j=1}^n \sum_{i=1}^m b_j \frac{p_i r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \left(\frac{q_i}{p_i} - \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right)^2 \frac{\partial}{\partial \bar{x}} \left(\frac{x_i^N - \bar{x}^N}{x_i - \bar{x}} \varphi(\bar{x}) \right) \Big|_{\bar{x} = \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle}} \\ &\quad + \sum_{j=1}^n \sum_{i=1}^m b_j \frac{p_i r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \left(\frac{q_i}{p_i} \right)^N \varphi \left(\left| \frac{q_i}{p_i} - \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right| \right), \end{aligned}$$

and as it is given that $a_i = \sum_{j=1}^n b_j \frac{r_{ij} p_i}{\langle \mathbf{r}_j, \mathbf{p} \rangle}, i = 1, \dots, m$, we get that (3.1) holds.

The proof of the theorem is complete.

REMARK 3. In the special case that $N=0$ we get Theorem 4, ([11, Theorem 1]) which is Sherman type inequality for superquadratic functions, also called the Csizsár-Körner inequality for φ -divergence when φ is a superquadratic function.

In the case that $N = 1$ we get that

$$\begin{aligned} & \sum_{i=1}^m a_i \phi \left(\frac{q_i}{p_i} \right) - \sum_{j=1}^n b_j \phi \left(\frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right) \\ & \geq \sum_{j=1}^n \sum_{i=1}^m b_j \frac{p_i r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \left(\frac{q_i}{p_i} - \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right)^2 \phi' \left(\frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right) \\ & \quad + \sum_{j=1}^n \sum_{i=1}^m b_j \frac{q_i r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \phi \left(\left| \frac{q_i}{p_i} - \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right| \right), \end{aligned} \tag{3.2}$$

which is a Sherman type inequality for 1-quasisuperquadratic function.

If $\varphi \geq 0$ on $[0, B)$ then inequalities (3.1), and (3.2) are refinements of the Sherman inequality for convex increasing functions on $[0, B)$.

Moreover, if $\varphi \geq 0$ on $[0, B)$, then the superquadratic function φ is also convex and increasing and therefore in this case we get in (3.1), and (3.2) refinements of the Sherman inequality for convex increasing functions on $[0, B)$.

By replacing in Theorem 7 $\frac{p_i r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle}$ by p_{ij} and $\frac{q_i}{p_i}$ by x_i , $i = 1, \dots, m$, $j = 1, \dots, n$ we get that:

COROLLARY 1. Let $\psi : [0, B) \rightarrow \mathbb{R}$, be a N -quasisuperquadratic function that is $\psi(x) = x^N \varphi(x)$, where φ is superquadratic on $[0, B)$.

Let $\mathbf{x} = (x_1, \dots, x_m)$, $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n)$, $x_i, y_j \in [0, B)$, $i = 1, \dots, m, j = 1, \dots, n$, $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}_+^m$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}_+^n$.

If $\bar{\mathbf{x}} = \mathbf{x}P, \mathbf{a} = \mathbf{b}P^T$ for some $m \times n$ column stochastic matrix $P = (p_{ij})$, then

$$\begin{aligned} & \sum_{i=1}^m a_i \psi(x_i) - \sum_{j=1}^n b_j \psi(\bar{x}_j) \\ & \geq \sum_{j=1}^n b_j \sum_{i=1}^m p_{ij} (x_i - \bar{x}_j)^2 \frac{\partial}{\partial \bar{x}_j} \left(\frac{\bar{x}_j^N - x_i^N}{\bar{x}_j - x_i} \varphi(\bar{x}_j) \right) + \sum_{j=1}^n b_j \sum_{i=1}^m p_{ij} x_i^N \varphi(|x_i - \bar{x}_j|). \end{aligned} \tag{3.3}$$

If φ is also non-negative then, as φ is also increasing

$$\begin{aligned} & \sum_{i=1}^m a_i \psi(x_i) - \sum_{j=1}^n b_j \psi(\bar{x}_j) \\ & \geq \sum_{j=1}^n b_j \sum_{i=1}^m p_{ij} (x_i - \bar{x}_j)^2 \frac{\partial}{\partial \bar{x}_j} \left(\frac{\bar{x}_j^N - x_i^N}{\bar{x}_j - x_i} \varphi(\bar{x}_j) \right) + \sum_{j=1}^n b_j \sum_{i=1}^m p_{ij} x_i^N \varphi(|x_i - \bar{x}_j|) \geq 0 \end{aligned} \tag{3.4}$$

holds.

REMARK 4. Inequality (3.4) is a refinement of the Sherman inequality for differentiable, non-negative increasing convex function ψ on $[0, B)$.

As a special case of Corollary 1 for 1-quasisuperquadratic functions we get the next result:

COROLLARY 2. Let $\psi : [0, B) \rightarrow \mathbb{R}$, $0 < B \leq \infty$ be a 1-quasisuperquadratic function, that is $\psi(x) = x\phi(x)$, where ϕ is superquadratic on $[0, B)$, then

$$\begin{aligned} & \sum_{i=1}^m a_i \psi(x_i) - \sum_{j=1}^n b_j \psi(\bar{x}_j) \\ & \geq \sum_{j=1}^n b_j \sum_{i=1}^m p_{ij} \phi'(\bar{x}_j) (x_i - \bar{x}_j)^2 + \sum_{j=1}^n b_j \sum_{i=1}^m p_{ij} x_i \phi(|x_i - \bar{x}_j|). \end{aligned} \tag{3.5}$$

If ϕ is also increasing then

$$\begin{aligned} & \sum_{i=1}^m a_i \psi(x_i) - \sum_{j=1}^n b_j \psi(\bar{x}_j) \\ & \geq \sum_{j=1}^n b_j \sum_{i=1}^m p_{ij} \phi'(\bar{x}_j) (x_i - \bar{x}_j)^2 + \sum_{j=1}^n b_j \sum_{i=1}^m p_{ij} x_i \phi(|x_i - \bar{x}_j|) \geq 0. \end{aligned} \tag{3.6}$$

REMARK 5. By (3.6) we get a refinement of the Sherman inequality for the differentiable, non-negative, increasing convex function ψ on $[0, B)$.

EXAMPLE 1. By choosing in Theorem 7, $b_j := \langle \mathbf{p}, \mathbf{r}_j \rangle$, $j = 1, \dots, n$, we get that $a_i = \sum_{j=1}^n p_i r_{ij} = p_i R_i$, $i = 1, \dots, m$ where $R_i = \sum_{j=1}^n r_{ij}$, $i = 1, \dots, m$. Denote $R = (r_{ij})$, $i = 1, \dots, m$, $j = 1, \dots, n$.

As $\sum_{j=1}^n r_{ij} p_i \phi\left(\frac{q_i}{p_i}\right) = p_i \phi\left(\frac{q_i}{p_i}\right) R_i$ therefore under the same condition as in Theorem 7, (3.1) under the new notation is

$$\begin{aligned} & \sum_{i=1}^m p_i R_i \phi\left(\frac{q_i}{p_i}\right) - \sum_{j=1}^n \langle \mathbf{p}, \mathbf{r}_j \rangle \phi\left(\frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle}\right) \\ & \geq \sum_{j=1}^n \sum_{i=1}^m p_i r_{ij} \left(\frac{q_i}{p_i} - \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle}\right)^2 \frac{\partial}{\partial \bar{x}_j} \left(\frac{\left(\frac{q_i}{p_i}\right)^N - \bar{x}_j^N}{\frac{q_i}{p_i} - \bar{x}_j} \phi(\bar{x}_j) \right)_{\bar{x}_j = \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle}} \\ & \quad + \sum_{j=1}^n \sum_{i=1}^m p_i^{1-N} q_i^N r_{ij} \phi\left(\left| \frac{q_i}{p_i} - \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right|\right). \end{aligned}$$

Now we state analog to the results in the former theorem, this time for N -quasiconvex functions $\psi(x) = x^N \phi(x)$ where N is an integer. The proof is omitted because it is the same proof as the proof of Theorem 7 using (1.5) and Corollary 1 by excluding the last term in each of (3.1)-(3.6).

THEOREM 8. Let $\psi : [0, B) \rightarrow \mathbb{R}$, be a N -quasiconvex function, that is, $\psi(x) = x^N \varphi(x)$, where φ is convex on $[0, B)$.

Let $\mathbf{x} = (x_1, \dots, x_m), \bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n), x_i, \bar{x}_j \in [0, B), i = 1, \dots, m, j = 1, \dots, n, \mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}_+^m, \mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}_+^n$.

If $\bar{\mathbf{x}} = \mathbf{xP}, \mathbf{a} = \mathbf{bP}^T$ for some $m \times n$ column stochastic matrix $P = (p_{ij})$, then

$$\sum_{i=1}^m a_i \psi(x_i) - \sum_{j=1}^n b_j \psi(\bar{x}_j) \geq \sum_{j=1}^n b_j \sum_{i=1}^m p_{ij} (x_i - \bar{x}_j)^2 \frac{\partial}{\partial \bar{x}_j} \left(\frac{\bar{x}_j^N - x_i^N}{\bar{x}_j - x_i} \varphi(\bar{x}_j) \right) \quad (3.7)$$

holds.

If φ is also non-negative and increasing then the inequalities

$$\sum_{i=1}^m a_i \psi(x_i) - \sum_{j=1}^n b_j \psi(\bar{x}_j) \geq \sum_{j=1}^n b_j \sum_{i=1}^m p_{ij} (x_i - \bar{x}_j)^2 \frac{\partial}{\partial \bar{x}_j} \left(\frac{\bar{x}_j^N - x_i^N}{\bar{x}_j - x_i} \varphi(\bar{x}_j) \right) \geq 0 \quad (3.8)$$

hold.

REMARK 6. By (3.8) we get a refinement of the Sherman inequality for a differentiable non-negative and convex function ψ on $[0, B)$.

As a special case of Theorem 8 we get for 1-quasiconvex function the next result:

COROLLARY 3. Let $\psi : [0, B) \rightarrow \mathbb{R}, 0 < B \leq \infty$ be a 1-quasiconvex function that is $\psi(x) = x\varphi(x)$, where φ is convex on $[0, B)$, then

$$\sum_{i=1}^m a_i \psi(x_i) - \sum_{j=1}^n b_j \psi(\bar{x}_j) \geq \sum_{j=1}^n b_j \sum_{i=1}^m p_{ij} \varphi'(\bar{x}_j) (x_i - \bar{x}_j)^2. \quad (3.9)$$

If φ is also increasing then

$$\sum_{i=1}^m a_i \psi(x_i) - \sum_{j=1}^n b_j \psi(\bar{x}_j) \geq \sum_{j=1}^n b_j \sum_{i=1}^m p_{ij} \varphi'(\bar{x}_j) (x_i - \bar{x}_j)^2 \geq 0. \quad (3.10)$$

REMARK 7. For the differentiable increasing and convex function ψ , (3.10) is a refinement of the Sherman inequality.

COROLLARY 4. For $N = 0$ inequality (3.7) is the Sherman inequality.

Moreover, when $\mathbf{p}, \mathbf{q}, \mathbf{a}, \mathbf{b}$, and \mathbf{r}_j are as in Theorem 7, then for the N -quasiconvex function $\phi(x) = x^N \varphi(x)$ where φ is convex

$$\begin{aligned} & \sum_{i=1}^m a_i \phi \left(\frac{q_i}{p_i} \right) - \sum_{j=1}^n b_j \phi \left(\frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right) \\ & \geq \sum_{j=1}^n \sum_{i=1}^m b_j \frac{p_i r_{ij}}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \left(\frac{q_i}{p_i} - \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right)^2 \frac{\partial}{\partial \bar{x}_j} \left(\frac{\left(\frac{q_i}{p_i} \right)^N - \bar{x}_j^N}{\frac{q_i}{p_i} - \bar{x}_j} \varphi(\bar{x}_j) \right) \Big|_{\bar{x}_j = \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle}} \end{aligned}$$

holds.

For $N = 0$ we get the Csiszár-Körner's inequality for φ -divergence, (see [15]).

Similar to Example 1, and with the same notation there, we get from Theorem 8:

EXAMPLE 2.

$$\begin{aligned} & \sum_{i=1}^m p_i R_i \phi \left(\frac{q_i}{p_i} \right) - \sum_{j=1}^n \langle \mathbf{p}, \mathbf{r}_j \rangle \phi \left(\frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right) \\ & \geq \sum_{j=1}^n \sum_{i=1}^m p_i r_{ij} \left(\frac{q_i}{p_i} - \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle} \right)^2 \frac{\partial}{\partial \bar{x}_j} \left(\frac{\left(\frac{q_i}{p_i} \right)^N - \bar{x}_j^N}{\frac{q_i}{p_i} - \bar{x}_j} \varphi(\bar{x}_j) \right) \Big|_{\bar{x}_j = \frac{\langle \mathbf{q}, \mathbf{r}_j \rangle}{\langle \mathbf{p}, \mathbf{r}_j \rangle}} \end{aligned}$$

4. Comments on different Hermite-Hadamard and Fejer type inequalities

In [10] new generalizations for weighted Hermite-Hadamard inequalities are proved. These generalizations leads us to compare these results with other results of theorems we stated or proved here. In the three examples shown at the end of this section we get conclusive results when comparing inequality (2.9) in Remark 2 with Theorem 10, Theorem 12 with Theorem 10 and Theorem 12 with 11. However, comparing inequalities derived from Theorem 9 with other inequalities we proved or stated here do not give conclusive results.

First we quote Hermite-Hadamard and Fejer type inequalities resulting from functions φ and g such that $\frac{\varphi'}{g}$ is non-decreasing that appear in [10] where the following are proved:

THEOREM 9. [10, Corollary 2.3] *Let $\varphi : [a, b] \rightarrow \mathbb{R}$ be a differentiable function and let $g : [a, b] \rightarrow (0, \infty)$ be an integrable function. If $\frac{\varphi'}{g}$ is increasing then*

$$\frac{1}{\int_a^b g(x) dx} \int_a^b \varphi(x) g(x) dx \leq \frac{\varphi(a) + \varphi(b)}{2}. \tag{4.1}$$

THEOREM 10. [10, Theorem 2.6] *Let $\varphi : [a, b] \rightarrow \mathbb{R}$ be a convex function and let $g : [a, b] \rightarrow (0, \infty)$ be an integrable function. If φ and g are monotonic in the same direction then*

$$\int_a^b \varphi(x) g(x) dx \geq \varphi \left(\frac{a+b}{2} \right) \int_a^b g(x) dx. \tag{4.2}$$

THEOREM 11. [10, Theorem 2.11] *Let $\varphi : [a, b] \rightarrow \mathbb{R}$ be a convex function and let $g : [a, b] \rightarrow (0, \infty)$ be an continuous function. If the function φ and g are monotonic in the opposite directions then*

$$\frac{1}{\int_a^b g(x) dx} \int_a^b \varphi(x) g(x) dx \leq \frac{\varphi(a) + \varphi(b)}{2} - \left(\frac{1}{2} - \frac{G}{b-a} \right) \delta_\varphi$$

where

$$G = \frac{1}{\int_a^b g(x) dx} \int_a^b \left| x - \frac{a+b}{2} \right| g(x) dx,$$

and

$$\delta_\varphi = \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right).$$

In the following three corollaries we choose g to be $g(x) = x$ that appears in the three Theorems 9, 10, and 11. As $g(x) \geq 0$ in these theorems, we deal now with $x \geq 0$.

COROLLARY 5. *Let $\varphi : [a, b] \rightarrow \mathbb{R}$, $0 \leq a < b$ be a differentiable function for which $\frac{\varphi'(x)}{x}$ is increasing. Then*

$$\frac{1}{b-a} \int_a^b \phi(x) dx = \frac{1}{b-a} \int_a^b \varphi(x) x dx \leq \frac{\varphi(a) + \varphi(b)}{2} \frac{a+b}{2}, \tag{4.3}$$

where $\phi(x) = x\varphi(x)$.

From Theorem 10 we get when $g(x) = x \geq 0$:

COROLLARY 6. *Let $\varphi : [a, b] \rightarrow \mathbb{R}$, $0 \leq a \leq x \leq b$ be a convex increasing function. Then, for $\phi(x) = x\varphi(x)$*

$$\frac{1}{b-a} \int_a^b \phi(x) dx = \frac{1}{b-a} \int_a^b \varphi(x) x dx \geq \varphi\left(\frac{a+b}{2}\right) \left(\frac{a+b}{2}\right) = \phi\left(\frac{a+b}{2}\right). \tag{4.4}$$

Corollary 7 follows from Theorem 11 for $g(x) = x \geq 0$ and it says:

COROLLARY 7. *Let $\varphi : [a, b] \rightarrow \mathbb{R}$, $0 \leq a < b$, be a convex decreasing function for which $\phi(x) = x\varphi(x)$. Then*

$$\begin{aligned} \frac{1}{b-a} \int_a^b \phi(x) dx &= \frac{1}{b-a} \int_a^b \varphi(x) x dx \\ &\leq \left(\frac{\varphi(a) + \varphi(b)}{4} + \frac{1}{2} \varphi\left(\frac{a+b}{2}\right) \right) \frac{a+b}{2}. \end{aligned} \tag{4.5}$$

Now we quote a theorem that deals with Fejer type inequality where the weight function p in monotonic:

THEOREM 12. [3, Theorem 5] *Let $\varphi : [a, b] \rightarrow \mathbb{R}$ be a differentiable and convex function. Let $p : [a, b] \rightarrow \mathbb{R}$ be a non-negative, integrable and monotone function.*

a) *Let $p'(x) \leq 0$, $a \leq x \leq b$ and $\varphi(a) \leq \varphi(b)$. Then*

$$\int_a^b \varphi(t) p(t) dt \leq \frac{\varphi(a) + \varphi(b)}{2} \int_a^b p(x) dx. \tag{4.6}$$

b) Let $p'(x) \geq 0$, $a \leq x \leq b$ and $\varphi(a) \leq \varphi\left(\frac{a+b}{2}\right)$. Then

$$\varphi\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \leq \int_a^b \varphi(t) p(t) dt. \quad (4.7)$$

c) If $p'(x) \geq 0$, $a \leq x \leq b$ and $\varphi(a) \geq \varphi(b)$, then (4.6) holds.

d) If $p'(x) \leq 0$, $a \leq x \leq b$ and $\varphi(a) \geq \varphi\left(\frac{a+b}{2}\right)$, then (4.7) holds.

The three examples below give conclusive results when comparing the results from [10] with other results in our paper.

EXAMPLE 3. The inequality (4.5) in Corollary 7 gives a better result than Theorem 12(c) for the special case where φ besides being convex is also non-increasing and $\phi(x) = x\varphi(x)$. Indeed

$$\begin{aligned} \frac{1}{b-a} \int_a^b \phi(x) dx &= \frac{1}{b-a} \int_a^b \varphi(x) x dx \leq \left(\frac{\varphi(a) + \varphi(b)}{4} + \frac{1}{2} \varphi\left(\frac{a+b}{2}\right) \right) \frac{a+b}{2} \\ &\leq \left(\frac{\varphi(a) + \varphi(b)}{2} \right) \frac{a+b}{2}. \end{aligned}$$

EXAMPLE 4. It is obvious that the left hand-side of the inequality (2.9) in Remark 2 for differentiable convex increasing φ where $\phi(x) = x\varphi(x)$ is better than (4.4) in Corollary 6 (Theorem 10 for $g(x) = x$) because

$$\begin{aligned} \int_a^b \phi(x) dx &= \frac{1}{b-a} \int_a^b \varphi(x) x dx \geq \phi\left(\frac{a+b}{2}\right) + \varphi'\left(\frac{a+b}{2}\right) \frac{(b-a)^2}{12} \geq \phi\left(\frac{a+b}{2}\right) \\ &= \varphi\left(\frac{a+b}{2}\right) \left(\frac{a+b}{2}\right). \end{aligned}$$

EXAMPLE 5. Comparing (4.2) in Theorem 10 with (4.7) in Theorem 12 cases b) and d) we see that we get the same inequalities for convex functions φ . However Theorem 10 deals only with the functions φ and p which are monotonic in the same direction and in Theorem 12 cases b) and d) φ and p are not necessarily monotonic in the same direction.

Acknowledgement. The author thanks the referee for the important remarks and suggestions that greatly improved the accessibility of the paper.

REFERENCES

- [1] S. ABRAMOVICH, G. JAMESON, AND G. SINNAMON, *Refining Jensen's inequality*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 47(95) (2004), no. 1-2, 3–14.
- [2] S. ABRAMOVICH, L-E. PERSSON, *Fejer and Hermite-Hadamard Inequalities for N -quasiconvex functions*, Math. Notes, 102(5), (2017), 644–656.
- [3] S. ABRAMOVICH, L-E. PERSSON, *Extensions and refinements of Fejer and Hermite-Hadamard Inequalities*, Math. Inequal. Applic., 21(3), (2018), 759–772.

- [4] S. ABRAMOVICH, L-E. PERSSON, *Some new scales of refined Hardy type inequalities via functions related to superquadracity*, Math. Inequal. Appl., 16(3), (2013), 679–695.
- [5] S. ABRAMOVICH, L-E. PERSSON, *Some new estimates of the 'Jensen gap'*, J. Inequal. Applic., (2016), 2016:39, 9 pages.
- [6] S. BANIĆ, J. PEČARIĆ AND S. VAROŠANEC, *Superquadratic functions and refinements of some classical inequalities*, J. Korean Math. Soc., 45(2008), 513–525.
- [7] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge University press, 1964.
- [8] S. IVELIĆ BRADANOVIĆ, N. LATIF AND J. PEČARIĆ, *On an upper bound for Sherman's inequality*, J. Inequal. Applic., (2016) 2016:165.
- [9] S. IVELIĆ BRADANOVIĆ, N. LATIF, Đ. PEČARIĆ AND J. PEČARIĆ, *Sherman and related inequalities with applications in information theory*, J. Inequal. Applic., (2018) 2018:98.
- [10] R. JAKSIĆ, L. KVESIĆ AND J. PEČARIĆ, *On Weighted Generalization of the Hermite-Hadamard Inequality*, Math. Inequal. Applic., 18(2), (2015) 649–665.
- [11] P. A. KLUZA AND M. NIEZGODA, *On Csiszár and Tsallis type f -divergences induced by superquadratic and convex functions*, Math. Inequal. Applic., 21(2), (2018), 455–467.
- [12] N. LATIF, Đ. PEČARIĆ AND J. PEČARIĆ, *Majorization, Csiszár divergence and Zipf-Mandelbrot law*, J. Inequal. Applic., (2017) 2017:197.
- [13] N. LOVRIČEVIĆ, Đ. PEČARIĆ AND J. PEČARIĆ, *Zipf-Mandelbrot law, f -divergences and Jensen-type interpolating inequalities*, J. Inequal. Applic., (2018) 2018:36.
- [14] M. NIEZGODA, *Inequalities for H -invex functions with applications for uniformly convex and superquadratic functions*, J. Filomat, 31(15), (2017), 4781–4794.
- [15] M. NIEZGODA, *Vector joint majorization and generalization of Csiszár-Körner's inequality for f -divergence*, Discrete Appl. Math. 198 (2016), 195–205.
- [16] C. NICULESCU AND L-E. PERSSON, *Convex functions and their applications, a contemporary approach*, Canad. Math. Series Books in Mathematics, Second Edition, Springer 2018.
- [17] J. PEČARIĆ, F. PROSCHAN, Y. L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, New York, 1992.
- [18] S. SHERMAN, *On a theorem of Hardy, Littlewood, Pólya and Blackwell*, Proc. Acad. Sci. USA, 37 (1957), 826–831.

(Received September 23, 2018)

Shoshana Abramovich
Department of Mathematics
University of Haifa
Haifa, Israel
e-mail: abramos@math.haifa.ac.il