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REVERSING BELLMAN OPERATOR INEQUALITY

MOHAMMAD SABABHEH, HAMID REZA MORADI AND SHIGERU FURUICHI

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Abstract. The main aim of the present paper is to obtain several reverses of the operator Bellman inequality. To this end, we employ Mond-Pečarić method to achieve a general inequality treating the arithmetic mean and unital positive linear maps. In particular, we show that, for certain scalars α,β,

$$\alpha(\Phi(I-A\nabla_v B))^{1/p}+\beta I\leqslant\Phi\left((I-A)^{1/p}\nabla_v(I-B)^{1/p}\right)$$

for the positive operators A,B, the normalized positive linear map Φ and $p>1$. As a consequence, we get multiplicative and additive reverses of operator Bellman inequality. Further, we show some inequalities involving concave and convex functions. In the end, we present a simple proof of the scalar Bellman inequality and its reverses.

1. Introduction

Throughout this paper, A and B are positive operators on a Hilbert space $\mathscr{H}$, with identity I. For convenience, we write $A\geqslant 0$ (respectively, $A>0$) if A is a positive (respectively, positive invertible) operator. In the sequel, we use m and M for positive real numbers, and the order between operators is that in which $A\leqslant B$ means $B-A$ is positive. The notation ∇_v will be used for the arithmetic mean, defined for two positive operators A and B by $A\nabla_v B=(1-v)A+vB$. A real valued function $f:J\to(0,\infty)$ is said to be operator concave if $f(A\nabla_v B)\geqslant f(A)\nabla_v f(B)$ for $0\leqslant v\leqslant 1$ and all self adjoint operators A,B whose spectra are contained in the real interval J. A linear map $\Phi:\mathscr{B}(\mathscr{H})\to\mathscr{B}(\mathscr{H})$ is said to be a normalized positive linear map if $\Phi(A)\geqslant 0$ whenever $A\geqslant 0$ and $\Phi(I)=I$. For further details about the notations of this paper, we refer the reader to [4]. In this context, $\mathscr{B}(\mathscr{H})$ is the algebra of all bounded linear operators acting on $\mathscr{H}$.

The following inequality is well known in the literature as the operator Bellman inequality [6]

$$(\Phi(I-A\nabla_v B))^{1/p}\geqslant\Phi\left((I-A)^{1/p}\nabla_v(I-B)^{1/p}\right) \tag{1.1}$$

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for $0 \leqslant v \leqslant 1$, $p > 1$, $0 < A, B \leqslant I$ and a normalized positive linear map Φ. This inequality was proved in [6] as an operator version of the scalar Bellman inequality [2]

$$\left(a^p - \sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} + \left(b^p - \sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}} \leqslant \left((a+b)^p - \sum_{k=1}^{n} (a_k + b_k)^p\right)^{\frac{1}{p}}, \tag{1.2}$$

for the positive numbers a, a_k, b, b_k satisfying $\sum_{k=1}^{n} a_k^p \leqslant a^p$ and $\sum_{k=1}^{n} b_k^p \leqslant b^p$, where $p > 1$.

The proof of (1.1) was based on the operator inequality [6]

$$f(\Phi(A\nabla_v B)) \geqslant \Phi(f(A)\nabla_v f(B)), \tag{1.3}$$

valid for the operator concave function $f : J \subset (0,\infty) \to (0,\infty)$, the normalized positive linear map Φ and the positive operators A, B whose spectra are contained in the interval J.

We refer the reader to [5, 7] for further discussion of (1.1).

In this article, we prove a more elaborated reverse of (1.3), valid for concave functions (not necessarily operator concave). This reverse-type inequality will be used to find a reversed version of (1.1) and a reversed version of (1.2). Further, we present a simple approach that can be used to prove the scalar Bellman inequality and its reverse. The new approach will be useful in obtaining several refinements of these inequalities.

2. Main results

In the sequel, we present a general inequality by applying Mond-Pečarić method. We refer the reader to [4] as a comprehensive reference of this method.

The following notations will be used in Theorem 1, for the positive numbers m, M and the function $f : [m, M] \to \mathbb{R}$.

$$a_f = \frac{f(M) - f(m)}{M - m} \quad \text{and} \quad b_f = \frac{Mf(m) - mf(M)}{M - m}.$$

THEOREM 1. *Let Φ be a normalized positive linear map on $\mathscr{B}(\mathscr{H})$, $A, B \in \mathscr{B}(\mathscr{H})$ be two positive operators such that $mI \leqslant A, B \leqslant MI$ for some scalars $0 < m < M$. If $f, g : [m, M] \to [0, \infty)$ are continuous functions such that f is concave, then for a given $\alpha > 0$,*

$$\alpha g(\Phi(A\nabla_v B)) + \beta I \leqslant \Phi(f(A)\nabla_v f(B)) \tag{2.1}$$

where $\beta = \min\limits_{t \in [m,M]} \left\{ a_f t + b_f - \alpha g(t) \right\}$.

The reverse inequality of (2.1) holds when f is a convex function.

Proof. According to the assumptions, we have, for any $t \in [m, M]$,

$$f(t) \geqslant a_f t + b_f.$$

A standard functional calculus argument implies

$$f(A) \geqslant a_f A + b_f I \quad \text{and} \quad f(B) \geqslant a_f B + b_f I.$$

Consequently, we infer for any $v \in [0,1]$,

$$(1-v)f(A) \geqslant (1-v)a_f A + (1-v)b_f I \quad \text{and} \quad vf(B) \geqslant va_f B + vb_f I,$$

and hence

$$f(A)\nabla_v f(B) \geqslant a_f(A\nabla_v B) + b_f I.$$

It follows from the linearity and the normality of Φ that

$$\Phi(f(A)\nabla_v f(B)) \geqslant a_f \Phi(A\nabla_v B) + b_f I.$$

Whence

$$\begin{aligned}\Phi(f(A)\nabla_v f(B)) - \alpha g(\Phi(A\nabla_v B)) &\geqslant a_f \Phi(A\nabla_v B) + b_f I - \alpha g(\Phi(A\nabla_v B)) \\ &\geqslant \min_{t\in[m,M]} \left\{a_f t + b_f - \alpha g(t)\right\} I\end{aligned}$$

which implies the desired inequality (2.1).

A reverse of the operator Bellman inequality (1.1) is obtained by taking $f(t) = g(t) = (1-t)^{1/p}$ on $(0,1)$ with $p > 1$ in Theorem 1.

COROLLARY 2.1. *(Reverse of operator Bellman inequality) Let $A,B \in \mathscr{B}(\mathscr{H})$ be two positive invertible operators such that $0 < mI \leqslant A,B \leqslant MI < I$, and Φ be a normalized positive linear map on $\mathscr{B}(\mathscr{H})$. Then for a given $\alpha > 0$,*

$$\alpha(\Phi(I - A\nabla_v B))^{1/p} + \beta I \leqslant \Phi\left((I-A)^{1/p}\nabla_v (I-B)^{1/p}\right)$$

where $p > 1$, $v \in [0,1]$ and

$$\beta = \min_{t\in[m,M]} \left\{ \frac{(1-M)^{1/p} - (1-m)^{1/p}}{M-m} t + \frac{M(1-m)^{1/p} - m(1-M)^{1/p}}{M-m} - \alpha(1-t)^{1/p} \right\}.$$

We remark that a similar result as in Corollary 2.1 was shown in [1, Corollary 2.8]. However, the advantage of our result is that the inclusion of a free constant α. This allows obtaining a multiplicative reverse, by choosing appropriate α and β in Corollary 2.1. This is our next result.

COROLLARY 2.2. *Let $A,B \in \mathscr{B}(\mathscr{H})$ be two positive invertible operators such that $0 < mI \leqslant A,B \leqslant MI < I$, and Φ be a normalized positive linear map on $\mathscr{B}(\mathscr{H})$. Then*

$$\alpha(\Phi(I - A\nabla_v B))^{1/p} \leqslant \Phi\left((I-A)^{1/p}\nabla_v (I-B)^{1/p}\right)$$

where $p>1$, $v\in[0,1]$ *and*

$$\alpha=\min_{t\in[m,M]}\left\{\frac{1}{(1-t)^{1/p}}\left(\frac{(1-M)^{1/p}-(1-m)^{1/p}}{M-m}t+\frac{M(1-m)^{1/p}-m(1-M)^{1/p}}{M-m}\right)\right\}.$$

Additionally,

$$(\Phi(I-A\nabla_v B))^{1/p}+\beta I\leqslant\Phi\left((I-A)^{1/p}\nabla_v(I-B)^{1/p}\right)$$

where

$$\beta=\min_{t\in[m,M]}\left\{\frac{(1-M)^{1/p}-(1-m)^{1/p}}{M-m}t+\frac{M(1-m)^{1/p}-m(1-M)^{1/p}}{M-m}-(1-t)^{1/p}\right\}.$$

REMARK 2.1. Here we find the exact value of α appearing in Corollary 2.2. This will help us better understand the operator Bellman inequality.
For simplicity, let

$$a=\frac{(1-M)^{1/p}-(1-m)^{1/p}}{M-m},\ b=\frac{M(1-m)^{1/p}-m(1-M)^{1/p}}{M-m},\ r=\frac{1}{p},$$

and let

$$f(t)=\frac{at+b}{(1-t)^r},\quad 0<m\leqslant t\leqslant M<1.$$

To find α, we find $\min_{m\leqslant t\leqslant M}f(t)$. Notice that

$$f'(t)=\frac{a+br+a(r-1)t}{(1-t)^{r+1}}.$$

Solving $f'(t)=0$, we obtain $t_0=\frac{a+br}{a(1-r)}$. Noting that $a,r-1<0$, it is easily seen that f attains its minimum at t_0, provided that $m\leqslant t_0\leqslant M$; which we show in this remark. We will prove that $m\leqslant t_0$ and leave the similar proof of $t_0\leqslant M$ to the reader. So, define $g(m)=t_0-m$. Simplifying this using the above a,b, we obtain

$$g(m)=\frac{1}{p-1}\left(p(1-m)+\frac{(1-m)^r(m-M)}{(1-m)^r-(1-M)^r}\right).$$

Calculus computations show that

$$g'(m)=\frac{h(M)}{p(p-1)(m-1)((1-m)^r-(1-M)^r)^2},$$

where

$$\begin{aligned}h(M)=&(1-m)((1-m)^r-(1-M)^r)^2(p-1)+(1-M)^r((1-m)(1-M)^r\\&+(1-m)^r(-1+m-mr+Mr)).\end{aligned}$$

Then

$$h'(M) = -(1-M)^{r-1}H(M),$$

where

$$H(M) = 2(1-m)(1-M)^r + (1-m)^r(-2+Mr(1+r)-m(-2+r+r^2)).$$

Further

$$H'(M) = (1-m)r\left[(1+r)(1-m)^{r-1} - 2(1-M)^{r-1}\right].$$

Noting that $r<1$ and $m<M$, it follows that $H'(M) \leqslant 0$. Since $m \leqslant M$, it follows that $H(M) \leqslant H(m) = 0$, and hence $h'(M) \geqslant 0$. Consequently, $h(M) \geqslant h(m) = 0$ and $g'(m) \leqslant 0$. This implies $g(m) \geqslant \lim\limits_{m \to M^-} g(m) = 0$, showing that $g \geqslant 0$ and hence $t_0 \geqslant m$.

Following similar computations, one can show that $t_0 \leqslant M$. We leave these computations to the reader.

Now having shown that $m \leqslant t_0 \leqslant M$, it follows that f attains its minimum on $[m,M]$ at t_0. That is

$$\alpha = f(t_0) = \frac{|a|}{r}\left(\frac{r(a+b)}{|a|(1-r)}\right)^{1-r} = p|a|^{\frac{1}{p}}\left(\frac{a+b}{p-1}\right)^{\frac{p-1}{p}}.$$

REMARK 2.2. To find β appearing in Corollary 2.2, we set

$$f(t) = at+b-(1-t)^r$$

for the same parameters as in Remark 2.1. Direct computations show that f attains its minimum on $[m,M]$ at

$$t_0 = 1-\left(\frac{r}{|a|}\right)^{\frac{1}{1-r}},$$

provided that $t_0 \in [m,M]$. In fact, tedious Calculus computations show that this is always the case. Consequently,

$$\beta = f(t_0) = a+b-a(p|a|)^{\frac{-p}{p-1}} - (p|a|)^{\frac{-1}{p-1}} = a+b-\frac{p-1}{p}(p|a|)^{-\frac{1}{p-1}}.$$

As an application of Corollary 2.2, we have the following scalar Bellman-type inequality.

COROLLARY 2.3. *For $1 \leqslant i \leqslant n$, let a_i, b_i be positive numbers satisfying $0 < m \leqslant a_i, b_i \leqslant M < 1$ for some scalars m, M. Then, for $p>1$ and $q \leqslant 1$,*

$$2^{1-\frac{q}{p}}\alpha\sum_{i=1}^{n}(2^q-(a_i+b_i)^q)^{\frac{1}{p}} \leqslant \sum_{i=1}^{n}\left\{(1-a_i^q)^{\frac{1}{p}} + (1-b_i^q)^{\frac{1}{p}}\right\},$$

where α is as in Corollary 2.2.

Proof. For the given a_i, b_i, define the $n\times n$ matrices $A=\mathrm{diag}(a_i^q)$ and $B=\mathrm{diag}(b_i^q)$. Apply the first inequality of Corollary 2.2 with $v=\frac{1}{2}$ to get

$$\alpha(I-A\nabla B)^{\frac{1}{p}}\leqslant(I-A)^{\frac{1}{p}}\nabla(I-B)^{\frac{1}{p}},$$

where we have chosen Φ to be the identity mapping. In particular, it follows that

$$\alpha\|(I-A\nabla B)^{\frac{1}{p}}\|\leqslant\|(I-A)^{\frac{1}{p}}\nabla(I-B)^{\frac{1}{p}}\|,$$

for any unitarily invariant norm $\|\ \|$. Selecting the trace norm $\|\ \|_1$, we obtain

$$\alpha\sum_{i=1}^{n}s_i\left((I-A\nabla B)^{\frac{1}{p}}\right)\leqslant\sum_{i=1}^{n}s_i\left((I-A)^{\frac{1}{p}}\nabla(I-B)^{\frac{1}{p}}\right),$$

where s_i is the i^{th} singular value. This implies

$$\alpha\sum_{i=1}^{n}\left(1-a_i^q\nabla b_i^q\right)^{\frac{1}{p}}\leqslant\sum_{i=1}^{n}\left\{(1-a_i^q)^{\frac{1}{p}}\nabla(1-b_i^q)^{\frac{1}{p}}\right\}.$$

That is, noting concavity of the mapping $t\mapsto t^q$,

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}\left\{(1-a_i^q)^{\frac{1}{p}}+(1-b_i^q)^{\frac{1}{p}}\right\}&\geqslant\alpha\sum_{i=1}^{n}\left(1-\frac{a_i^q+b_i^q}{2}\right)^{\frac{1}{p}}\geqslant\alpha\sum_{i=1}^{n}\left(1-\left(\frac{a_i+b_i}{2}\right)^q\right)^{\frac{1}{p}}\\&=\frac{\alpha}{2^{q/p}}\sum_{i=1}^{n}\left(2^q-(a_i+b_i)^q\right)^{\frac{1}{p}},\end{aligned}$$

which completes the proof.

The main observation in [3, Lemma 3.2] can be stated as follows.

COROLLARY 2.4. *Let $A,B\in\mathscr{B}(\mathscr{H})$ be two positive operators such that $mI\leqslant A,B\leqslant MI$ for some scalars $0<m<M$. If $f:[m,M]\to[0,\infty)$ is a concave function and $v\in[0,1]$, then the ratio inequality*

$$\alpha f(A\nabla_v B)\leqslant f(A)\nabla_v f(B)\tag{2.2}$$

holds, where $\alpha=\min\limits_{t\in[m,M]}\left\{\frac{a_f t+b_f}{f(t)}\right\}$. *Additionally, the following difference inequality*

$$f(A\nabla_v B)+\beta I\leqslant f(A)\nabla_v f(B)\tag{2.3}$$

holds, where $\beta=\min\limits_{t\in[m,M]}\left\{a_f t+b_f-f(t)\right\}$.

The reverse inequalities in (2.2) *and* (2.3) *hold when f is a convex function.*

We conclude this paper, by presenting the following simple proof of (1.2) and some reversed versions.

PROPOSITION 2.1. *Let a_k, b_k be positive numbers such that $\sum_{k=1}^{n} a_k^p \leqslant 1$ and $\sum_{k=1}^{n} b_k^p \leqslant 1$, for $p \in \mathbb{R}$. Then, for $0 \leqslant v \leqslant 1$,*

$$\left(1-\sum_{k=1}^{n}(a_k^p \nabla_v b_k^p)\right)^{\frac{1}{p}} \geqslant \left(1-\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \nabla_v \left(1-\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}}, \quad \text{if } p>1 \tag{2.4}$$

and

$$\left(1-\sum_{k=1}^{n}(a_k^p \nabla_v b_k^p)\right)^{\frac{1}{p}} \leqslant \left(1-\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \nabla_v \left(1-\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}}, \quad \text{if } p<1.$$

Proof. For $0 \leqslant v \leqslant 1$, let

$$f(v) = \left(1-\sum_{k=1}^{n}(a_k^p \nabla_v b_k^p)\right)^{\frac{1}{p}}.$$

Since the summannds are linear in v, it is readily seen that f is concave if $p>1$ and is convex if $p<1$. Then both inequalities follow from concavity/convexity of f. Notice that when $p>1$, the function $x \mapsto x^p, x>0$ is convex. Therefore, $a_k^p \nabla_v b_k^p \geqslant (a_k \nabla_v b_k)^p$. This observation together with (2.4) imply

$$\left(1-\sum_{k=1}^{n}(a_k \nabla_v b_k)^p\right)^{\frac{1}{p}} \geqslant \left(1-\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \nabla_v \left(1-\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}}, \quad \text{if } p>1.$$

An elaborated proof of this inequality was given in [6] as an application of (1.1). Further, in [6], it was shown that this last inequality is equivalent to (1.2).

Notice that convexity of the mapping $x \mapsto x^p, p>1$ allowed the passage from $a_k^p \nabla_v b_k^p$ to $(a_k \nabla_v b_k)^p$. Unfortunately, the same logic does not apply for $p<1$. However, the following is a more elaborated convexity result. The proof follows immediately upon finding the second derivative of the given function.

PROPOSITION 2.2. *For the positive numbers a_k, b_k satisfying $\sum_{k=1}^{n} a_k^p, \sum_{k=1}^{n} b_k^p \leqslant 1$, where $p \in \mathbb{R}$, define the function*

$$f(v) = \left(1-\sum_{k=1}^{n}(a_k \nabla_v b_k)^p\right)^{\frac{1}{p}}, \quad 0 \leqslant v \leqslant 1.$$

Then f is concave if $p>1$, while it is convex if $p<0$.

From this, we have

$$\left(1-\sum_{k=1}^{n}(a_k \nabla_v b_k)^p\right)^{\frac{1}{p}} \leqslant \left(1-\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \nabla_v \left(1-\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}}, \quad \text{if } p<0.$$

Using this inequality and following the proof of [6, Theorem 2.5] imply the following reverse of (1.2).

COROLLARY 2.5. *Let* a, a_k, b, b_k *be positive scalars satisfying* $\sum_{k=1}^{n} a_k^p \leqslant a^p$ *and* $\sum_{k=1}^{n} b_k^p \leqslant b^p$, *where* $p < 0$. *Then the following reverse of* (1.2) *holds*

$$\left(a^p - \sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} + \left(b^p - \sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}} \geqslant \left((a+b)^p - \sum_{k=1}^{n} (a_k+b_k)^p\right)^{\frac{1}{p}}.$$

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REFERENCES

[1] M. BAKHERAD AND A. MORASSAEI, *Some operator Bellman type inequalities*, Indag. Math., **26** (2015), 646–659.

[2] R. BELLMAN, *On an inequality concerning an indefinite form*, Amer. Math. Monthly., **63** (1956), 108–109.

[3] M. FUJII, J. MIĆIĆ HOT, J, PEČARIĆ AND Y. SEO, *Reverse inequalities on chaotically geometric mean via Specht ratio, II*, J. Inequal. Pure and Appl. Math., **4**(2) (2003), Article 40.

[4] T. FURUTA, J. MIĆIĆ, J. PEČARIĆ AND Y. SEO, *Mond–Pečarić method in operator inequalities*, Element, Zagreb, 2005.

[5] F. MIRZAPOUR, A. MORASSAEI AND M.S. MOSLEHIAN, *More on operator Bellman inequality*, Quaest. Math., **37** (2014), 9–17.

[6] A. MORASSAEI, F. MIRZAPOUR AND M.S. MOSLEHIAN, *Bellman inequality for Hilbert space operators*, Linear Algebra Appl., **438** (2013), 3776–3780.

[7] S. SHEYBANI, M.E. OMIDVAR AND H.R. MORADI, *New inequalities for operator concave functions involving positive linear maps*, Math. Inequal. Appl., **21**(4) (2018), 1167–1174.

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Mohammad Sababheh
Dept. of Basic Sciences
Princess Sumaya Univ. for Tech.
Amman, Jordan
e-mail: sababheh@psut.edu.jo

Hamid Reza Moradi
Department of Mathematics
Payame Noor University (PNU)
P.O. Box 19395-4697, Tehran, Iran
e-mail: hrmoradi@mshdiau.ac.ir

Shigeru Furuichi
Department of Information Science
College of Humanities and Sciences, Nihon University
3-25-40, Sakurajyousui, Setagaya-ku, Tokyo, 156-8550, Japan
e-mail: furuichi@chs.nihon-u.ac.jp