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ON A GENERAL INEQUALITY RELATED TO THE GENERALIZED–EULER–CONSTANT FUNCTION

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Abstract. Let $\gamma(z)$ be the generalized-Euler-constant function. In this paper, we establish a general inequality related to $\gamma(z)$, which contains a result due to Chen and Han as a special case. We also obtain an inequality for the generalized Somos recurrence constant, using its relation with the generalized-Euler-constant function.

1. Introduction

Somos' quadratic recurrence constant σ is usually defined by

$$\sigma = \sqrt{1\sqrt{2\sqrt{3\sqrt{4\cdots}}}} = \prod_{k=1}^{\infty} k^{\frac{1}{2^k}} = 1.66168794\cdots \tag{1.1}$$

or

$$\sigma = \exp\left\{-\int_0^1 \frac{1-x}{(2-x)\ln x}dx\right\} = \exp\left\{-\int_0^1\int_0^1 \frac{x}{(2-xy)\ln(xy)}dxdy\right\}. \tag{1.2}$$

See [4, 13, 14]. It arises in the study of the asymptotic behavior of the sequence (see for example [3, p. 446] and [18]):

$$g_n \sim \frac{\sigma^{2^n}}{n}\left(1+\frac{2}{n}-\frac{1}{n^2}+\frac{4}{n^3}-\frac{21}{n^4}+\frac{138}{n^5}-\frac{1091}{n^6}+\frac{10088}{n^7}-\frac{106918}{n^8}\right.$$
$$\left.+\frac{1279220}{n^9}-\frac{17070418}{n^{10}}+\frac{251560472}{n^{11}}-\frac{4059954946}{n^{12}}+\cdots\right)^{-1}, \tag{1.3}$$

where the g_n's are defined recursively by

$$g_0 = 1, \quad g_n = ng_{n-1}^2, \quad n \in \mathbb{N} := \{1,2,3,\ldots\}, \tag{1.4}$$

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with the first few terms being

$$g_0 = 1,\ g_1 = 1,\ g_2 = 2,\ g_3 = 12,\ g_4 = 576,\ g_5 = 1658880, \dots.$$

It is remarkable particularly that Nemes [10] found recurrence relations and an asymptotic approximation for the coefficients of (1.3). Xu [19] extended the Nemes results to the generalized Somos recurrence. The constant σ introduced in (1.1) appears in many important problems in pure and applied analysis and it was investigated by a great number of mathematicians [1, 2, 4, 5, 6, 7, 9, 10, 11, 12, 17, 20].

Sondow and Hadjicostas [17] introduced and studied the generalized-Euler-constant function $\gamma(z)$, defined by the power series

$$\gamma(z) = \sum_{k=1}^{\infty} z^{k-1}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right) \tag{1.5}$$

when $|z| \leqslant 1$. The function was independently introduced by Pilehrood and Pilehrood [11] almost at the same time. Its values include Euler's constant $\gamma = \gamma(1)$ and the "alternating" Euler constant $\log\frac{4}{\pi} = \gamma(-1)$; see for example [15, 16]. In particular, at $z = 1/2$, the function takes the value

$$\gamma\left(\frac{1}{2}\right) = 2\ln\frac{2}{\sigma}, \tag{1.6}$$

which is equivalent to

$$\sigma = 2\exp\left\{-\frac{1}{2}\gamma\left(\frac{1}{2}\right)\right\}. \tag{1.7}$$

Mortici [9] proved that for $n \geqslant 1$,

$$\frac{270(n+1)}{2^n(270n^3+1530n^2+1065n+6293)} < \gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) < \frac{18}{2^n(18n^2+84n-13)}, \tag{1.8}$$

where the partial sum of $\gamma(z)$ is

$$\gamma_n(z) = \sum_{k=1}^{n} z^{k-1}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right), \quad |z| \leqslant 1.$$

He further provided a slightly weaker but simpler version of the above inequality, i.e., for $n \geqslant 8$,

$$\frac{1}{2^n(n+3)^2} < \gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) < \frac{1}{2^n(n+2)^2}. \tag{1.9}$$

Lu and Song [7] improved Mortici's estimate and proved that for $n \geqslant 1$,

$$\frac{690n^2+3524n+145}{6(2^n)(n+1)^2(115n^2+894n+779)} < \gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) < \frac{48n+127}{3(2^n)(16n+85)(n+1)^2}. \tag{1.10}$$

They also provided a simpler version which improved (1.9), namely, for $n \geqslant 6$,

$$\frac{1}{2^n\left(n+\frac{7}{3}\right)^2} < \gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) < \frac{1}{2^n\left(n+2\right)^2}. \tag{1.11}$$

You and Chen [20] improved these inequalities by using continued fractions.

Recently, Chen and Han [2] obtained the following new inequalities for $\gamma(1/2) - \gamma_n(1/2)$:

$$\begin{aligned}
&\frac{1}{2^n}\left(\frac{1}{(n+1)^2} - \frac{8}{3(n+1)^3} + \frac{23}{2(n+1)^4} - \frac{332}{5(n+1)^5} + \frac{479}{(n+1)^6} - \frac{29024}{7(n+1)^7}\right)\\
<&\gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right)\\
<&\frac{1}{2^n}\left(\frac{1}{(n+1)^2} - \frac{8}{3(n+1)^3} + \frac{23}{2(n+1)^4} - \frac{332}{5(n+1)^5} + \frac{479}{(n+1)^6}\right).
\end{aligned} \tag{1.12}$$

In their paper, Chen and Han pointed out that the lower bound in (1.12) is sharper than the one in (1.10) for $n \geqslant 24$, and the upper bound in (1.12) is sharper than the one in (1.10) for $n \geqslant 18$.

Besides, there were some researches which focus on the estimates for $\gamma(1/3)$ and $\gamma(1/4)$; see the related works in [7, 8, 9, 20]. Motivated by those interesting works, we establish a general inequality for $\gamma(z) - \gamma_n(z)$, which contains (1.12) as a special case. By using its relation with the generalized-Euler-constant function, we further obtain an inequality for the generalized Somos quadratic recurrence constant.

2. Main results

First of all, for $0 < z < 1$, we let

$$\begin{aligned}
c_2(z) &= \frac{1}{2(1-z)}, \quad c_3(z) = -\frac{2z+1}{3(1-z)^2},\\
c_4(z) &= \frac{3z^2+8z+1}{4(1-z)^3}, \quad c_5(z) = -\frac{4z^3+33z^2+22z+1}{5(1-z)^4},\\
c_6(z) &= \frac{5z^4+104z^3+198z^2+52z+1}{6(1-z)^5},\\
c_7(z) &= -\frac{6z^5+285z^4+1208z^3+906z^2+114z+1}{7(1-z)^6}.
\end{aligned}$$

In this section, we obtain a general inequality related to $\gamma(z)$ which generalizes the Chen-Han result. Let $\mathbb{N}$ be a set for all positive integers. We have the following theorem.

THEOREM 2.1. *For $n \in \mathbb{N}$ and $0 < z < 1$, we have*

$$z^n l(n+1;z) < \gamma(z) - \gamma_n(z) < z^n u(n+1;z), \tag{2.1}$$

where

$$u(x;z)=\sum_{k=2}^{6}\frac{c_k(z)}{x^k},\quad l(x;z)=\sum_{k=2}^{7}\frac{c_k(z)}{x^k}. \tag{2.2}$$

REMARK 2.1. Here some special examples of $u(x;z)$ and $l(x;z)$ for $z=1/2$, $z=1/3$ and $z=1/4$, respectively, are presented.

$$\begin{aligned}
u(x;1/2)&=\frac{1}{x^2}-\frac{8}{3x^3}+\frac{23}{2x^4}-\frac{332}{5x^5}+\frac{479}{x^6},\\
l(x;1/2)&=\frac{1}{x^2}-\frac{8}{3x^3}+\frac{23}{2x^4}-\frac{332}{5x^5}+\frac{479}{x^6}-\frac{29024}{7x^7},\\
u(x;1/3)&=\frac{3}{4x^2}-\frac{5}{4x^3}+\frac{27}{8x^4}-\frac{123}{10x^5}+\frac{56}{x^6},\\
l(x;1/3)&=\frac{3}{4x^2}-\frac{5}{4x^3}+\frac{27}{8x^4}-\frac{123}{10x^5}+\frac{56}{x^6}-\frac{17127}{56x^7},\\
u(x;1/4)&=\frac{2}{3x^2}-\frac{8}{9x^3}+\frac{17}{9x^4}-\frac{736}{135x^5}+\frac{1594}{81x^6},\\
l(x;1/4)&=\frac{2}{3x^2}-\frac{8}{9x^3}+\frac{17}{9x^4}-\frac{736}{135x^5}+\frac{1594}{81x^6}-\frac{48296}{567x^7}.
\end{aligned}$$

Thus, taking $z=1/2$ in Theorem 2.1, we obtain the Chen-Han inequality (1.12) again. Additionally, we also obtain the similar inequalities for $z=1/3$ and $z=1/4$.

COROLLARY 2.1. *For $n\in\mathbb{N}$, we have*

$$\begin{aligned}
&\frac{1}{3^n}\left(\frac{3}{4(n+1)^2}-\frac{5}{4(n+1)^3}+\frac{27}{8(n+1)^4}-\frac{123}{10(n+1)^5}+\frac{56}{(n+1)^6}-\frac{17127}{56(n+1)^7}\right)\\
&<\gamma\left(\frac{1}{3}\right)-\gamma_n\left(\frac{1}{3}\right)\\
&<\frac{1}{3^n}\left(\frac{3}{4(n+1)^2}-\frac{5}{4(n+1)^3}+\frac{27}{8(n+1)^4}-\frac{123}{10(n+1)^5}+\frac{56}{(n+1)^6}\right).
\end{aligned} \tag{2.3}$$

COROLLARY 2.2. *For $n\in\mathbb{N}$, we have*

$$\begin{aligned}
&\frac{1}{4^n}\left(\frac{2}{3(n+1)^2}-\frac{8}{9(n+1)^3}+\frac{17}{9(n+1)^4}-\frac{736}{135(n+1)^5}+\frac{1594}{81(n+1)^6}-\frac{48296}{567(n+1)^7}\right)\\
&<\gamma\left(\frac{1}{4}\right)-\gamma_n\left(\frac{1}{4}\right)\\
&<\frac{1}{4^n}\left(\frac{2}{3(n+1)^2}-\frac{8}{9(n+1)^3}+\frac{17}{9(n+1)^4}-\frac{736}{135(n+1)^5}+\frac{1594}{81(n+1)^6}\right).
\end{aligned} \tag{2.4}$$

Before a proof of Theorem 2.1 is given, we need the following lemma.

LEMMA 2.1. *For* $0 < z < 1$ *and* $x \geqslant 1$, *we have*

$$l(x;z) - zl(x+1;z) < \frac{1}{x} - \ln\left(1+\frac{1}{x}\right) < u(x;z) - zu(x+1;z). \tag{2.5}$$

Proof. For $t \in (-1,1]$ and $m \in \mathbb{N}$, it is clear that

$$\sum_{k=1}^{2m} \frac{(-1)^{k-1}}{k} t^k < \ln(1+t) < \sum_{k=1}^{2m-1} \frac{(-1)^{k-1}}{k} t^k,$$

which leads to the following inequality, namely, for $x \geqslant 1$ and $m \in \mathbb{N}$,

$$\sum_{k=2}^{2m} \frac{(-1)^k}{kx^k} > \frac{1}{x} - \ln\left(1+\frac{1}{x}\right) > \sum_{k=2}^{2m-1} \frac{(-1)^k}{kx^k}. \tag{2.6}$$

It follows from (2.6) that

$$\begin{aligned}
&\frac{1}{x} - \ln\left(1+\frac{1}{x}\right) - l(x;z) + zl(x+1;z)\\
>&\frac{1}{2x^2} - \frac{1}{3x^3} + \frac{1}{4x^4} - \frac{1}{5x^5} + \frac{1}{6x^6} - \frac{1}{7x^7} - l(x;z) + zl(x+1;z)\\
=&\frac{1}{x^7(1+x)^7}\left\{(x+1)^7 \sum_{k=2}^{7}\left[\frac{(-1)^k}{k} - c_k(z)\right]x^{7-k} + zx^7 \sum_{k=2}^{7} c_k(z)(x+1)^{7-k}\right\}\\
=&\frac{1}{x^7(1+x)^7}\left\{\left(\sum_{k=0}^{7}\binom{7}{k}x^k\right)\sum_{k=2}^{7}\left[\frac{(-1)^k}{k} - c_k(z)\right]x^{7-k} + zx^7 \sum_{k=2}^{7} c_k(z) \sum_{j=0}^{7-k}\binom{7-k}{j}x^j\right\}\\
=&\frac{1}{x^7(1+x)^7}\sum_{j=0}^{12} b_j(z)x^j,
\end{aligned}$$

where

$$b_j(z) = \begin{cases} \sum_{k=0}^{j}\binom{7}{k}\left[\frac{(-1)^{7-j+k}}{7-j+k} - c_{7-j+k}(z)\right], & 0 \leqslant j \leqslant 5,\\ \sum_{k=1}^{6}\binom{7}{k}\left[\frac{(-1)^{k+1}}{k+1} - c_{k+1}(z)\right], & j = 6,\\ \sum_{k=j-5}^{7}\binom{7}{k}\left[\frac{(-1)^{7-j+k}}{7-j+k} - c_{7-j+k}(z)\right] + z\sum_{k=2}^{14-j}\binom{7-k}{j-7}c_k(z), & 7 \leqslant j \leqslant 12. \end{cases} \tag{2.7}$$

For $0 < z < 1$, it is easy to verify that

$$b_0(z) = \frac{z\left(120 + 891z + 1228z^2 + 270z^3 + 12z^4 - z^5\right)}{7(1-z)^6} > 0,$$

$$b_1(z) = \frac{z\left(663 + 5215z + 7442z^2 + 1734z^3 + 71z^4 - 5z^5\right)}{6(1-z)^6} > 0,$$

$$b_2(z)=\frac{z\left(8961+75455z+112990z^2+28350z^3+1105z^4-61z^5\right)}{30(1-z)^6}>0,$$
$$b_3(z)=\frac{z\left(8683+79395z+127250z^2+35330z^3+1395z^4-53z^5\right)}{20(1-z)^6}>0,$$
$$b_4(z)=\frac{z\left(21527+218275z+386250z^2+124450z^3+5615z^4-117z^5\right)}{60(1-z)^6}>0,$$
$$b_5(z)=\frac{z\left(1927+22439z+46290z^2+18866z^3+1207z^4-9z^5\right)}{12(1-z)^6}>0,$$
$$b_6(z)=\frac{3z\left(41+593z+1614z^2+990z^3+121z^4+z^5\right)}{4(1-z)^6}>0,$$

and

$$b_7(z)=b_8(z)=b_9(z)=b_{10}(z)=b_{11}(z)=b_{12}(z)=0.$$

Therefore,

$$\frac{1}{x}-\ln\left(1+\frac{1}{x}\right)-l(x;z)+zl(x+1;z)>0.$$

Similarly, we have

$$\begin{aligned}
&\frac{1}{x}-\ln\left(1+\frac{1}{x}\right)-u(x;z)+zu(x+1;z)\\
<&\frac{1}{2x^2}-\frac{1}{3x^3}+\frac{1}{4x^4}-\frac{1}{5x^5}+\frac{1}{6x^6}-u(x;z)+zu(x+1;z)\\
=&\frac{1}{x^6(1+x)^6}\left\{(x+1)^6\sum_{k=2}^{6}\left[\frac{(-1)^k}{k}-c_k(z)\right]x^{6-k}+zx^6\sum_{k=2}^{6}c_k(z)(x+1)^{6-k}\right\}\\
=&\frac{1}{x^6(1+x)^6}\left\{\left(\sum_{k=0}^{6}\binom{6}{k}x^k\right)\sum_{k=2}^{6}\left[\frac{(-1)^k}{k}-c_k(z)\right]x^{6-k}+zx^6\sum_{k=2}^{6}c_k(z)\sum_{j=0}^{6-k}\binom{6-k}{j}x^j\right\}\\
=&\frac{1}{x^6(1+x)^6}\sum_{j=0}^{10}\hat{b}_j(z)x^j,
\end{aligned}$$

where

$$\hat{b}_j(z)=\begin{cases}\sum_{k=0}^{j}\binom{6}{k}\left[\frac{(-1)^{6-j+k}}{6-j+k}-c_{6-j+k}(z)\right], & 0\leqslant j\leqslant 4,\\ \sum_{k=1}^{5}\binom{6}{k}\left[\frac{(-1)^{k+1}}{k+1}-c_{k+1}(z)\right], & j=5,\\ \sum_{k=j-4}^{6}\binom{6}{k}\left[\frac{(-1)^{6-j+k}}{6-j+k}-c_{6-j+k}(z)\right]+z\sum_{k=2}^{12-j}\binom{6-k}{j-6}c_k(z), & 6\leqslant j\leqslant 10.\end{cases} \tag{2.8}$$

It can be further checked that for $0<z<1$, we have

$$\hat{b}_0(z)=-\frac{z\left(57+188z+114z^2+z^4\right)}{6(1-z)^5}<0,$$

$$\hat{b}_1(z) = -\frac{z\left(259+939z+589z^2+9z^3+4z^4\right)}{5(1-z)^5} < 0,$$
$$\hat{b}_2(z) = -\frac{z\left(2281+9266z+6216z^2+206z^3+31z^4\right)}{20(1-z)^5} < 0,$$
$$\hat{b}_3(z) = -\frac{z\left(763+3570z+2700z^2+158z^3+9z^4\right)}{6(1-z)^5} < 0,$$
$$\hat{b}_4(z) = -\frac{z\left(289+1630z+1516z^2+162z^3+3z^4\right)}{4(1-z)^5} < 0,$$
$$\hat{b}_5(z) = -\frac{z\left(17+128z+174z^2+40z^3+z^4\right)}{(1-z)^5} < 0,$$

and

$$\hat{b}_6(z) = \hat{b}_7(z) = \hat{b}_8(z) = \hat{b}_9(z) = \hat{b}_{10}(z) = 0.$$

Thus, the desired result has been derived.

Proof of Theorem 2.1. By (2.5), we have

$$z^{k-1}l(k;z) - z^k l(k+1;z) < z^{k-1}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right) < z^{k-1}u(k;z) - z^k u(k+1;z). \tag{2.9}$$

Adding the above inequalities, from $k = n+1$ to $k = \infty$, yields

$$z^n l(n+1;z) < \sum_{k=n+1}^{\infty} z^{k-1}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right) < z^n u(n+1;z), \tag{2.10}$$

which implies that

$$z^n l(n+1;z) < \gamma(z) - \gamma_n(z) < z^n u(n+1;z), \tag{2.11}$$

and the proof is complete.

Sondow and Hadjicostas [17] generalized Somons' quadratic recurrence constant as

$$\sigma_t = \sqrt[t]{1\sqrt[t]{2\sqrt[t]{3\sqrt[t]{4\cdots}}}} = \prod_{k=1}^{\infty} k^{\frac{1}{t^k}}, \tag{2.12}$$

and established the following relation between the generalized Somos quadratic recurrence constant σ_t and the generalized Euler-constant function $\gamma\left(\frac{1}{t}\right)$:

$$\gamma\left(\frac{1}{t}\right) = t\ln\frac{t}{(t-1)\sigma_t^{t-1}}, \quad t > 1. \tag{2.13}$$

Thus, together with Theorem 2.1 we obtain the following estimates for the generalized Somos recurrence constant σ_t.

THEOREM 2.2. *For $n \in \mathbb{N}$ and $t > 1$, we have*

$$\left\{\frac{t}{t-1}\exp\left\{-\frac{1}{t}\gamma_n\left(\frac{1}{t}\right)-\frac{u(n+1;1/t)}{t^{n+1}}\right\}\right\}^{\frac{1}{t-1}} < \sigma_t < \left\{\frac{t}{t-1}\exp\left\{-\frac{1}{t}\gamma_n\left(\frac{1}{t}\right)-\frac{l(n+1;1/t)}{t^{n+1}}\right\}\right\}^{\frac{1}{t-1}}, \tag{2.14}$$

where $l(x;z)$ and $u(x;z)$ are given in Theorem 2.1.

In particular, we immediately have the following inequalities for the Somos quadratic recurrence constant σ and the Somos cubic recurrence constant σ_3 [17], respectively.

COROLLARY 2.3. *For $n \in \mathbb{N}$, we have*

$$2\exp\left\{-\frac{1}{2}\gamma_n\left(\frac{1}{2}\right)-\frac{u(n+1;1/2)}{2^{n+1}}\right\} < \sigma < 2\exp\left\{-\frac{1}{2}\gamma_n\left(\frac{1}{2}\right)-\frac{l(n+1;1/2)}{2^{n+1}}\right\}. \tag{2.15}$$

COROLLARY 2.4. *For $n \in \mathbb{N}$, we have*

$$\sqrt{\frac{3}{2}\exp\left\{-\frac{1}{3}\gamma_n\left(\frac{1}{3}\right)-\frac{u(n+1;1/3)}{3^{n+1}}\right\}} < \sigma_3 < \sqrt{\frac{3}{2}\exp\left\{-\frac{1}{3}\gamma_n\left(\frac{1}{3}\right)-\frac{l(n+1;1/3)}{3^{n+1}}\right\}}. \tag{2.16}$$

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