

SOME MULTIPLE INTEGRAL INEQUALITIES VIA THE DIVERGENCE THEOREM

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Abstract. In this paper, by the use of the divergence theorem, we establish some inequalities for functions defined on closed and bounded subsets of the Euclidean space \mathbb{R}^n , $n \geq 2$.

1. Introduction

Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives on D . In the recent paper [4], by the use of *Green's identity*, we have shown among others that

$$\left| \int \int_D f(x, y) dx dy - \frac{1}{2} \oint_{\partial D} [(\beta - y)f(x, y) dx + (x - \alpha)f(x, y) dy] \right| \leq \frac{1}{2} \int \int_D \left[|\alpha - x| \left| \frac{\partial f(x, y)}{\partial x} \right| + |\beta - y| \left| \frac{\partial f(x, y)}{\partial y} \right| \right] dx dy =: M(\alpha, \beta; f) \quad (1.1)$$

for all $\alpha, \beta \in \mathbb{C}$ and

$$M(\alpha, \beta; f) \leq \begin{cases} \left\| \frac{\partial f}{\partial x} \right\|_{D, \infty} \iint_D |\alpha - x| dx dy + \left\| \frac{\partial f}{\partial y} \right\|_{D, \infty} \iint_D |\beta - y| dx dy; \\ \left\| \frac{\partial f}{\partial x} \right\|_{D, p} \left(\iint_D |\alpha - x|^q dx dy \right)^{1/q} + \left\| \frac{\partial f}{\partial y} \right\|_{D, p} \left(\iint_D |\beta - y|^q dx dy \right)^{1/q} \\ \text{where } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sup_{(x, y) \in D} |\alpha - x| \left\| \frac{\partial f}{\partial x} \right\|_{D, 1} + \sup_{(x, y) \in D} |\beta - y| \left\| \frac{\partial f}{\partial y} \right\|_{B, 1}, \end{cases} \quad (1.2)$$

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where $\|\cdot\|_{D,p}$ are the usual Lebesgue norms, we recall that

$$\|g\|_{D,p} := \begin{cases} (\iint_D |g(x,y)|^p dx dy)^{1/p}, & p \geq 1; \\ \sup_{(x,y) \in D} |g(x,y)|, & p = \infty. \end{cases}$$

Applications for rectangles and disks were also provided in [4]. For some recent double integral inequalities see [1], [2] and [3].

We also considered similar inequalities for 3-dimensional bodies as follows, see [5]. Let B be a solid in the three dimensional space \mathbb{R}^3 bounded by an orientable closed surface ∂B . If $f : B \rightarrow \mathbb{C}$ is a continuously differentiable function defined on a open set containing B , then by making use of the *Gauss-Ostrogradsky identity*, we have obtained the following inequality

$$\begin{aligned} & \left| \iiint_B f(x,y,z) dx dy dz - \frac{1}{3} \left[\int \int_{\partial B} (x - \alpha) f(x,y,z) dy \wedge dz \right. \right. \\ & \quad \left. \left. + \int \int_{\partial B} (y - \beta) f(x,y,z) dz \wedge dx + \int \int_{\partial B} (z - \gamma) f(x,y,z) dx \wedge dy \right] \right| \\ & \leq \frac{1}{3} \iiint_B \left[|\alpha - x| \left| \frac{\partial f(x,y,z)}{\partial x} \right| + |\beta - y| \left| \frac{\partial f(x,y,z)}{\partial y} \right| + |\gamma - z| \left| \frac{\partial f(x,y,z)}{\partial z} \right| \right] dx dy dz \\ & =: M(\alpha, \beta, \gamma; f) \end{aligned} \tag{1.3}$$

for all α, β, γ complex numbers. Moreover, we have the bounds

$$\begin{aligned} & M(\alpha, \beta, \gamma; f) \\ & \leq \frac{1}{3} \left\{ \begin{aligned} & \left\| \frac{\partial f}{\partial x} \right\|_{B,\infty} \iiint_B |\alpha - x| dx dy dz + \left\| \frac{\partial f}{\partial y} \right\|_{B,\infty} \iiint_B |\beta - y| dx dy dz \\ & + \left\| \frac{\partial f}{\partial z} \right\|_{B,\infty} \iiint_B |\gamma - z| dx dy dz; \\ & \left\| \frac{\partial f}{\partial x} \right\|_{B,p} (\iiint_B |\alpha - x|^q dx dy dz)^{1/q} + \left\| \frac{\partial f}{\partial y} \right\|_{B,p} (\iiint_B |\beta - y|^q dx dy dz)^{1/q} \\ & + \left\| \frac{\partial f}{\partial z} \right\|_{B,p} (\iiint_B |\gamma - z|^q dx dy dz)^{1/q}, \quad p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ & \sup_{(x,y,z) \in B} |\alpha - x| \left\| \frac{\partial f}{\partial x} \right\|_{B,1} + \sup_{(x,y,z) \in B} |\beta - y| \left\| \frac{\partial f}{\partial y} \right\|_{B,1} \\ & + \sup_{(x,y,z) \in B} |\gamma - z| \left\| \frac{\partial f}{\partial z} \right\|_{B,1}. \end{aligned} \right. \end{aligned} \tag{1.4}$$

Applications for 3-dimensional balls were also given in [5]. For some triple integral inequalities see [6] and [9].

Motivated by the above results, in this paper we establish several similar inequalities for multiple integrals for functions defined on bonded subsets of \mathbb{R}^n ($n \geq 2$) with smooth (or piecewise smooth) boundary ∂B . To achieve this goal we make use of the well known divergence theorem for multiple integrals as summarized below.

2. Some preliminary facts

Let B be a bounded open subset of \mathbb{R}^n ($n \geq 2$) with smooth (or piecewise smooth) boundary ∂B . Let $F = (F_1, \dots, F_n)$ be a smooth vector field defined in \mathbb{R}^n , or at least in $B \cup \partial B$. Let \mathbf{n} be the unit outward-pointing normal of ∂B . Then the *Divergence Theorem* states, see for instance [8]:

$$\int_B \operatorname{div} F dV = \int_{\partial B} F \cdot n dA, \tag{2.1}$$

where

$$\operatorname{div} F = \nabla \cdot F = \sum_{k=1}^n \frac{\partial F_k}{\partial x_k},$$

dV is the element of volume in \mathbb{R}^n and dA is the element of surface area on ∂B .

If $\mathbf{n} = (\mathbf{n}_1, \dots, \mathbf{n}_n)$, $x = (x_1, \dots, x_n) \in B$ and use the notation dx for dV we can write (2.1) more explicitly as

$$\sum_{k=1}^n \int_B \frac{\partial F_k(x)}{\partial x_k} dx = \sum_{k=1}^n \int_{\partial B} F_k(x) n_k(x) dA. \tag{2.2}$$

By taking the real and imaginary part, we can extend the above equality for complex valued functions F_k , $k \in \{1, \dots, n\}$ defined on B .

If $n = 2$, the normal is obtained by rotating the tangent vector through 90° (in the correct direction so that it points out). The quantity $t ds$ can be written (dx_1, dx_2) along the surface, so that

$$n dA := n ds = (dx_2, -dx_1)$$

Here t is the tangent vector along the boundary curve and ds is the element of arc-length.

From (2.2) we get for $B \subset \mathbb{R}^2$ that

$$\int_B \frac{\partial F_1(x_1, x_2)}{\partial x_1} dx_1 dx_2 + \int_B \frac{\partial F_2(x_1, x_2)}{\partial x_2} dx_1 dx_2 = \int_{\partial B} F_1(x_1, x_2) dx_2 - \int_{\partial B} F_2(x_1, x_2) dx_1, \tag{2.3}$$

which is *Green's theorem* in plane.

If $n = 3$ and if ∂B is described as a level-set of a function of 3 variables i.e. $\partial B = \{x_1, x_2, x_3 \in \mathbb{R}^3 \mid G(x_1, x_2, x_3) = 0\}$, then a vector pointing in the direction of \mathbf{n} is $\operatorname{grad} G$. We shall use the case where $G(x_1, x_2, x_3) = x_3 - g(x_1, x_2)$, $(x_1, x_2) \in D$, a domain in \mathbb{R}^2 for some differentiable function g on D and B corresponds to the inequality $x_3 < g(x_1, x_2)$, namely

$$B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 < g(x_1, x_2)\}.$$

Then

$$\mathbf{n} = \frac{(-g_{x_1}, -g_{x_2}, 1)}{(1 + g_{x_1}^2 + g_{x_2}^2)^{1/2}}, \quad dA = (1 + g_{x_1}^2 + g_{x_2}^2)^{1/2} dx_1 dx_2$$

and

$$\mathbf{n}dA = (-g_{x_1}, -g_{x_2}, 1) dx_1 dx_2.$$

From (2.2) we get

$$\begin{aligned} & \int_B \left(\frac{\partial F_1(x_1, x_2, x_3)}{\partial x_1} + \frac{\partial F_2(x_1, x_2, x_3)}{\partial x_2} + \frac{\partial F_3(x_1, x_2, x_3)}{\partial x_3} \right) dx_1 dx_2 dx_3 \\ &= - \int_D F_1(x_1, x_2, g(x_1, x_2)) g_{x_1}(x_1, x_2) dx_1 dx_2 - \int_D F_2(x_1, x_2, g(x_1, x_2)) g_{x_2}(x_1, x_2) dx_1 dx_2 \\ & \quad + \int_D F_3(x_1, x_2, g(x_1, x_2)) dx_1 dx_2 \end{aligned} \tag{2.4}$$

which is the *Gauss-Ostrogradsky theorem* in space.

3. Identities of interest

We have the following identity of interest:

THEOREM 1. *Let B be a bounded open subset of \mathbb{R}^n ($n \geq 2$) with smooth (or piecewise smooth) boundary ∂B . Let f be a continuously differentiable function defined in \mathbb{R}^n , or at least in $B \cup \partial B$ and with complex values. If $\alpha_k, \beta_k \in \mathbb{C}$ for $k \in \{1, \dots, n\}$ with $\sum_{k=1}^n \alpha_k = 1$, then*

$$\int_B f(x) dx = \sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx + \sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) dA. \tag{3.1}$$

We also have

$$\int_B f(x) dx = \frac{1}{n} \sum_{k=1}^n \int_B (\gamma_k - x_k) \frac{\partial f(x)}{\partial x_k} dx + \frac{1}{n} \sum_{k=1}^n \int_{\partial B} (x_k - \gamma_k) f(x) n_k(x) dA \tag{3.2}$$

for all $\gamma_k \in \mathbb{C}$, where $k \in \{1, \dots, n\}$.

Proof. Let $x = (x_1, \dots, x_n) \in B$. We consider

$$F_k(x) = (\alpha_k x_k - \beta_k) f(x), \quad k \in \{1, \dots, n\}$$

and take the partial derivatives $\frac{\partial F_k(x)}{\partial x_k}$ to get

$$\frac{\partial F_k(x)}{\partial x_k} = \alpha_k f(x) + (\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k}, \quad k \in \{1, \dots, n\}.$$

If we sum this equality over k from 1 to n we get

$$\sum_{k=1}^n \frac{\partial F_k(x)}{\partial x_k} = \sum_{k=1}^n \alpha_k f(x) + \sum_{k=1}^n (\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} = f(x) + \sum_{k=1}^n (\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} \tag{3.3}$$

for all $x = (x_1, \dots, x_n) \in B$.

Now, if we take the integral in the equality (3.3) over $(x_1, \dots, x_n) \in B$ we get

$$\int_B \left(\sum_{k=1}^n \frac{\partial F_k(x)}{\partial x_k} \right) dx = \int_B f(x) dx + \sum_{k=1}^n \int_B \left[(\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} \right] dx. \tag{3.4}$$

By the Divergence Theorem (2.2) we also have

$$\int_B \left(\sum_{k=1}^n \frac{\partial F_k(x)}{\partial x_k} \right) dx = \sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) dA \tag{3.5}$$

and by making use of (3.4) and (3.5) we get

$$\int_B f(x) dx + \sum_{k=1}^n \int_B \left[(\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} \right] dx = \sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) dA$$

which gives the desired representation (3.1).

The identity (3.2) follows by (3.1) for $\alpha_k = \frac{1}{n}$ and $\beta_k = \frac{1}{n} \gamma_k$, $k \in \{1, \dots, n\}$.

For the body B we consider the coordinates for the *centre of gravity*

$$G(\overline{x_{B,1}}, \dots, \overline{x_{B,n}})$$

defined by

$$\overline{x_{B,k}} := \frac{1}{V(B)} \int_B x_k dx, \quad k \in \{1, \dots, n\},$$

where

$$V(B) := \int_B x dx$$

is the volume of B .

COROLLARY 1. *With the assumptions of Theorem 1 we have*

$$\int_B f(x) dx = \sum_{k=1}^n \int_B \alpha_k (\overline{x_{B,k}} - x_k) \frac{\partial f(x)}{\partial x_k} dx + \sum_{k=1}^n \int_{\partial B} \alpha_k (x_k - \overline{x_{B,k}}) f(x) n_k(x) dA \tag{3.6}$$

and, in particular,

$$\int_B f(x) dx = \frac{1}{n} \sum_{k=1}^n \int_B (\overline{x_{B,k}} - x_k) \frac{\partial f(x)}{\partial x_k} dx + \frac{1}{n} \sum_{k=1}^n \int_{\partial B} (x_k - \overline{x_{B,k}}) f(x) n_k(x) dA. \tag{3.7}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k \overline{x_{B,k}}$, $k \in \{1, \dots, n\}$.

For a function f as in Theorem 1 above, we define the points

$$x_{B,\partial f,k} := \frac{\int_B x_k \frac{\partial f(x)}{\partial x_k} dx}{\int_B \frac{\partial f(x)}{\partial x_k} dx}, \quad k \in \{1, \dots, n\},$$

provided that all denominators are not zero.

COROLLARY 2. *With the assumptions of Theorem 1 we have*

$$\int_B f(x) dx = \sum_{k=1}^n \int_{\partial B} \alpha_k (x_k - x_{B,\partial f,k}) f(x) n_k(x) dA \tag{3.8}$$

and, in particular,

$$\int_B f(x) dx = \frac{1}{n} \sum_{k=1}^n \int_{\partial B} (x_k - x_{B,\partial f,k}) f(x) n_k(x) dA. \tag{3.9}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k x_{B,\partial f,k}$, $k \in \{1, \dots, n\}$ and observing that

$$\sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx = \sum_{k=1}^n \alpha_k \int_B (x_{B,\partial f,k} - x_k) \frac{\partial f(x)}{\partial x_k} dx = 0.$$

For a function f as in Theorem 1 above, we define the points

$$x_{\partial B,f,k} := \frac{\int_{\partial B} x_k f(x) n_k(x) dA}{\int_{\partial B} f(x) n_k(x) dA}, \quad k \in \{1, \dots, n\}$$

provided that all denominators are not zero.

COROLLARY 3. *With the assumptions of Theorem 1 we have*

$$\int_B f(x) dx = \sum_{k=1}^n \int_B \alpha_k (x_{\partial B,f,k} - x_k) \frac{\partial f(x)}{\partial x_k} dx \tag{3.10}$$

and, in particular,

$$\int_B f(x) dx = \frac{1}{n} \sum_{k=1}^n \int_B (x_{\partial B,f,k} - x_k) \frac{\partial f(x)}{\partial x_k} dx. \tag{3.11}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k x_{\partial B,f,k}$, $k \in \{1, \dots, n\}$ and observing that

$$\sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) dA = 0.$$

4. Some integral inequalities

We have the following result generalizing the inequalities from the introduction:

THEOREM 2. *Let B be a bounded open subset of \mathbb{R}^n ($n \geq 2$) with smooth (or piecewise smooth) boundary ∂B . Let f be a continuously differentiable function defined in \mathbb{R}^n , or at least in $B \cup \partial B$ and with complex values. If $\alpha_k, \beta_k \in \mathbb{C}$ for*

$k \in \{1, \dots, n\}$ with $\sum_{k=1}^n \alpha_k = 1$, then

$$\left| \int_B f(x) dx - \sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) dA \right| \leq \sum_{k=1}^n \int_B |\beta_k - \alpha_k x_k| \left| \frac{\partial f(x)}{\partial x_k} \right| dx \leq \begin{cases} \sum_{k=1}^n \int_B |\beta_k - \alpha_k x_k| dx \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B, \infty} \\ \sum_{k=1}^n \left(\int_B |\beta_k - \alpha_k x_k|^q dx \right)^{1/q} \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B, p} \\ \text{where } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{k=1}^n \sup_{x \in B} |\beta_k - \alpha_k x_k| \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B, 1} \end{cases} \quad (4.1)$$

We also have

$$\left| \int_B f(x) dx - \frac{1}{n} \sum_{k=1}^n \int_{\partial B} (x_k - \gamma_k) f(x) n_k(x) dA \right| \leq \frac{1}{n} \sum_{k=1}^n \int_B |\gamma_k - x_k| \left| \frac{\partial f(x)}{\partial x_k} \right| dx \leq \frac{1}{n} \begin{cases} \sum_{k=1}^n \int_B |\gamma_k - x_k| dx \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B, \infty} \\ \sum_{k=1}^n \left(\int_B |\gamma_k - x_k|^q dx \right)^{1/q} \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B, p} \\ \text{where } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{k=1}^n \sup_{x \in B} |\gamma_k - x_k| \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B, 1} \end{cases} \quad (4.2)$$

for all $\gamma_k \in \mathbb{C}$, where $k \in \{1, \dots, n\}$.

Proof. By the identity (3.1) we have

$$\begin{aligned} & \left| \int_B f(x) dx - \sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) dA \right| \\ &= \left| \sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \leq \sum_{k=1}^n \left| \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \\ &\leq \sum_{k=1}^n \int_B \left| (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} \right| dx, \end{aligned}$$

which proves the first inequality in (4.1).

By Hölder’s integral inequality for multiple integrals we have

$$\int_B \left| (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} \right| dx \leq \begin{cases} \sup_{x \in B} \left| \frac{\partial f(x)}{\partial x_k} \right| \int_B |\beta_k - \alpha_k x_k| dx \\ \left(\int_B \left| \frac{\partial f(x)}{\partial x_k} \right|^p \right)^{1/p} \left(\int_B |\beta_k - \alpha_k x_k|^q dx \right)^{1/q} \\ \text{where } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sup_{x \in B} |\beta_k - \alpha_k x_k| \int_B \left| \frac{\partial f(x)}{\partial x_k} \right| dx \end{cases} = \begin{cases} \int_B |\beta_k - \alpha_k x_k| dx \left\| \frac{\partial f}{\partial x_k} \right\|_{B, \infty} \\ \left(\int_B |\beta_k - \alpha_k x_k|^q dx \right)^{1/q} \left\| \frac{\partial f}{\partial x_k} \right\|_{B, p} \\ \text{where } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sup_{x \in B} |\beta_k - \alpha_k x_k| \left\| \frac{\partial f}{\partial x_k} \right\|_{B, 1} \end{cases},$$

which proves the last part of (4.1).

COROLLARY 4. *With the assumptions of Theorem 2 we have*

$$\left| \int_B f(x) dx - \frac{1}{n} \sum_{k=1}^n \int_{\partial B} (x_k - \overline{x_{B,k}}) f(x) n_k(x) dA \right| \leq \frac{1}{n} \sum_{k=1}^n \int_B |\overline{x_{B,k}} - x_k| \left| \frac{\partial f(x)}{\partial x_k} \right| dx \leq \frac{1}{n} \begin{cases} \sum_{k=1}^n \int_B |\overline{x_{B,k}} - x_k| dx \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B, \infty} \\ \sum_{k=1}^n \left(\int_B |\overline{x_{B,k}} - x_k|^q dx \right)^{1/q} \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B, p} \\ \text{where } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{k=1}^n \sup_{x \in B} |\overline{x_{B,k}} - x_k| \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B, 1} \end{cases} \quad (4.3)$$

and

$$\left| \int_B f(x) dx \right| \leq \frac{1}{n} \sum_{k=1}^n \int_B |x_{\partial B, f, k} - x_k| \left| \frac{\partial f(x)}{\partial x_k} \right| dx \leq \frac{1}{n} \begin{cases} \sum_{k=1}^n \int_B |x_{\partial B, f, k} - x_k| dx \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B, \infty} \\ \sum_{k=1}^n \left(\int_B |x_{\partial B, f, k} - x_k|^q dx \right)^{1/q} \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B, p} \\ \text{where } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{k=1}^n \sup_{x \in B} |x_{\partial B, f, k} - x_k| \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B, 1}. \end{cases} \quad (4.4)$$

We also have the dual result:

THEOREM 3. *With the assumption of Theorem 2 we have*

$$\left| \int_B f(x) dx - \sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \leq \sum_{k=1}^n \int_{\partial B} |\alpha_k x_k - \beta_k| |n_k(x)| |f(x)| dA \leq \begin{cases} \|f\|_{\partial B, \infty} \sum_{k=1}^n \int_{\partial B} |\alpha_k x_k - \beta_k| |n_k(x)| dA; \\ \|f\|_{\partial B, p} \sum_{k=1}^n (\int_{\partial B} |\alpha_k x_k - \beta_k|^q |n_k(x)|^q dA)^{1/q} \\ \text{where } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|f\|_{\partial B, 1} \sum_{k=1}^n \sup_{x \in \partial B} |\alpha_k x_k - \beta_k| |n_k(x)|, \end{cases} \tag{4.5}$$

where

$$\|f\|_{\partial B, p} := \begin{cases} (\int_{\partial B} |f(x)|^p dA)^{1/p}, & p \geq 1; \\ \sup_{x \in \partial B} |f(x)|, & p = \infty. \end{cases}$$

In particular,

$$\left| \int_B f(x) dx - \frac{1}{n} \sum_{k=1}^n \int_B (\gamma_k - x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \leq \frac{1}{n} \sum_{k=1}^n \int_{\partial B} |\gamma_k - x_k| |n_k(x)| |f(x)| dA \leq \frac{1}{n} \begin{cases} \|f\|_{\partial B, \infty} \sum_{k=1}^n \int_{\partial B} |\gamma_k - x_k| |n_k(x)| dA; \\ \|f\|_{\partial B, p} \sum_{k=1}^n (\int_{\partial B} |\gamma_k - x_k|^q |n_k(x)|^q dA)^{1/q} \\ \text{where } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|f\|_{\partial B, 1} \sum_{k=1}^n \sup_{x \in \partial B} [|\gamma_k - x_k| |n_k(x)|]. \end{cases} \tag{4.6}$$

Proof. From the identity (3.1) we have

$$\begin{aligned} & \left| \int_B f(x) dx - \sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \\ &= \left| \sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) dA \right| \leq \sum_{k=1}^n \left| \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) dA \right| \\ &\leq \sum_{k=1}^n \int_{\partial B} |(\alpha_k x_k - \beta_k) f(x) n_k(x)| dA, \end{aligned}$$

which proves the first inequality in (4.5).

By Hölder's inequality for functions defined on ∂B we have

$$\int_{\partial B} |\alpha_k x_k - \beta_k| |n_k(x)| |f(x)| dA \leq \begin{cases} \int_{\partial B} |\alpha_k x_k - \beta_k| |n_k(x)| dA \|f\|_{\partial B, \infty}; \\ (\int_{\partial B} |\alpha_k x_k - \beta_k|^q |n_k(x)|^q dA)^{1/q} \|f\|_{\partial B, p} \\ \text{where } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sup_{x \in \partial B} |\alpha_k x_k - \beta_k| |n_k(x)| \|f\|_{\partial B, 1}, \end{cases}$$

which proves the second part of the inequality (4.5).

We also have:

COROLLARY 5. *With the assumptions of Theorem 2 we have*

$$\begin{aligned} & \left| \int_B f(x) dx - \frac{1}{n} \sum_{k=1}^n \int_B (\overline{x_{B,k}} - x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \\ & \leq \frac{1}{n} \sum_{k=1}^n \int_{\partial B} |\overline{x_{B,k}} - x_k| |n_k(x)| |f(x)| dA \\ & \leq \begin{cases} \|f\|_{\partial B, \infty} \sum_{k=1}^n \int_{\partial B} |\overline{x_{B,k}} - x_k| |n_k(x)| dA; \\ \frac{1}{n} \|f\|_{\partial B, p} \sum_{k=1}^n \left(\int_{\partial B} |\overline{x_{B,k}} - x_k|^q |n_k(x)|^q dA \right)^{1/q} \\ \text{where } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|f\|_{\partial B, 1} \sum_{k=1}^n \sup_{x \in \partial B} [|\overline{x_{B,k}} - x_k| |n_k(x)|] \end{cases} \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} & \left| \int_B f(x) dx \right| \leq \frac{1}{n} \sum_{k=1}^n \int_{\partial B} |x_{B, \partial f, k} - x_k| |n_k(x)| |f(x)| dA \\ & \leq \begin{cases} \|f\|_{\partial B, \infty} \sum_{k=1}^n \int_{\partial B} |x_{B, \partial f, k} - x_k| |n_k(x)| dA; \\ \frac{1}{n} \|f\|_{\partial B, p} \sum_{k=1}^n \left(\int_{\partial B} |x_{B, \partial f, k} - x_k|^q |n_k(x)|^q dA \right)^{1/q} \\ \text{where } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|f\|_{\partial B, 1} \sum_{k=1}^n \sup_{x \in \partial B} [|x_{B, \partial f, k} - x_k| |n_k(x)|]. \end{cases} \end{aligned} \quad (4.8)$$

If we take $n = 2$ in Theorem 3, then we get other results from [4], while for $n = 3$ we recapture some results from [5].

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