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MILLOUX INEQUALITY OF NONLINEAR DIFFERENCE MONOMIALS AND ITS APPLICATION

ZHAOJUN WU* AND HONGYAN XU

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Abstract. Let $f(z)$ be a transcendental meromorphic function of finite order and $c_1, c_2, \cdots, c_m$ be complex constants satisfying that at least one of them is non-zero. The authors establish an inequality (Milloux inequality) about the nonlinear difference monomials $f^{d_1}(z+c_1)f^{d_2}(z+c_2)\cdots f^{d_m}(z+c_m)$, where $d_1, d_2, \cdots, d_m \in \mathbb{N}$. As an application of the inequality, the authors investigate the value distribution of $f^{d_1}(z+c_1)f^{d_2}(z+c_2)\cdots f^{d_m}(z+c_m)$. Results obtained partially promote and improve relevant results of Laine, Yang and Chen et al..

1. Some fundamental inequalities in Nevanlinna theory

First of all, we introduce the standard notations of R. Nevanlinna's theory of meromorphic functions which will be used in this paper (see [9, 12, 16]). Let $f(z)$ be a non-constant meromorphic function defined in the complex plane $\mathbb{C}$ and $a \in \mathbb{C}$, Nevanlinna defined the following functions.

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

$$m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta})-a|} d\theta,$$

$$N(r,f) = \int_0^r \frac{n(t,f)-n(0,f)}{t} dt + n(0,f)\log r,$$

$$N\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{n\left(t, \frac{1}{f-a}\right) - n\left(0, \frac{1}{f-a}\right)}{t} dt + n\left(0, \frac{1}{f-a}\right)\log r,$$

$$T(r,f) = m(r,f) + N(r,f),$$

$$T\left(r, \frac{1}{f-a}\right) = m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right),$$

where $T(r,f)$ is called the characteristic function of $f(z)$, $\log^+ x = \max\{\log x, 0\}$ $(x > 0)$, $n(t,f)$ and $n\left(t, \frac{1}{f-a}\right)$ denote the number of poles of $f(z)$ and the number of zeros

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* Corresponding author.

of $f(z)-a$ in the disc $|z|\leqslant t$, counting multiplicities, respectively. The Nevanlinna's deficiency of f with respect to a is defined by

$$\delta(a,f)=\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)}=1-\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)}.$$

If $a=\infty$, then one should replace $N\left(r,\frac{1}{f-a}\right)$ in the above formula by $N(r,f)$.

Let $f(z)$ be a non-constant meromorphic function defined in the complex plane $\mathbb{C}$, we will let $\sigma(f)$ and $\lambda(a,f)$ denote the order of $f(z)$ and the exponent of convergence of zeros of $f(z)-a$, respectively. We use $Q(r,f)$ to denote any quantity of $Q(r,f)=o(T(r,f))(r\to\infty)$, possibly outside a set E with finite linear measure, and use $S(r,f)$ to denote any quantity of $S(r,f)=o(T(r,f))(r\to\infty)$, possibly outside a set E with finite logarithmic measure.

Previously, we introduced some standard notations and definitions of Nevenlinna theory. Below, we will list a few relevant results of Nevanlinna theory. And we also assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna theory.

In 1920s, R. Nevanlinna has established the first fundamental theorem and the second fundamental theorem.

THEOREM A. *(The first fundamental theorem) Suppose that $f(z)$ is a meromorphic function in the complex plane $\mathbb{C}$ and a is any complex number. Then*

$$T\left(r,\frac{1}{f-a}\right)=T(r,f)+O(1).$$

THEOREM B. *(The second fundamental theorem) Suppose that $f(z)$ is a non-constant meromorphic function in the complex plane $\mathbb{C}$ and $a_i(1\leqslant i\leqslant q)$ are $q(q\geqslant 3)$ distinct values in the extended complex plane. Then*

$$(q-2)T(r,f)<\sum_{i=1}^{q}N\left(r,\frac{1}{f-a_i}\right)+Q(r,f).$$

In 1940, H. Milloux has established the following inequality about the $f^{(k)}$.

THEOREM C. *(Milloux inequality) Suppose that $f(z)$ is a non-constant meromorphic function in the complex plane and k is a positive integer. Then*

$$T(r,f)<N(r,f)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}-1}\right)+Q(r,f).$$

In 1989, Yi [13] has proved the following Milloux type inequality about the nonlinear differential monomial $f'f$.

THEOREM D. [13, Theorem 2] *Suppose that $f(z)$ is a non-constant meromorphic function in the complex plane and a is any non-zero complex number. Then*

$$2T(r,f)<N(r,f)+2N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{ff'-a}\right)+Q(r,f).$$

In the present paper, Milloux inequality about the nonlinear difference monomials are established. As an application of the Milloux inequality, we investigate the value distribution of nonlinear difference monomials.

2. Milloux inequality of nonlinear difference monomials

Recently, a number of papers (including [3, 4, 6, 7, 8]) have focused on the difference analogues of the Nevanlinan theory. In this section, we will give a difference counterpart of Milloux inequality. In the course of the proof of Milloux inequality, we need to make use of the following lemmas.

LEMMA 1. [7] *Let $f(z)$ be a transcendental meromorphic function of finite order, then*

$$m\left(r, \frac{f(z+c)}{f}\right) = S(r,f).$$

LEMMA 2. [3, 14] *Let f be a transcendental meromorphic function of finite order. Then*

$$N(r,f(z+c)) = N(r,f)+S(r,f),$$
$$T(r,f(z+c)) = T(r,f)+S(r,f),$$

where $S(r,f) = o(T(r,f))(r \to \infty)$, possibly outside a set E of r with finite logarithmic measure.

LEMMA 3. [6] *Let f be a transcendental meromorphic function of finite order. Then for any positive integer n, we have*

$$m\left(r, \frac{\Delta_c^n f(z)}{f(z)}\right) = S(r,f).$$

LEMMA 4. [11] *Suppose that $f(z)$ is a transcendental meromorphic function in the complex plane and $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, where $a_0 (\not\equiv 0), a_1, \cdots, a_n$ are constants. Then*

$$T(r,P(f)) = nT(r,f)+S(r,f).$$

LEMMA 5. [1, 5] *Let $F(r)$ and $G(r)$ be monotone increasing function such that $F(r) \leqslant G(r)$ outside of exceptional set E that is of finite logarithmic measure. Then for any $\alpha > 0$, there exists $r_0 > 1$ such that $F(r) \leqslant G(\alpha r)$ for all $r > r_0$.*

THEOREM 1. *Let f be a transcendental meromorphic function of finite order and let $c_1, c_2, \cdots, c_m, (m \in \mathbb{N}^+)$ be complex constants satisfying that at least one of them is non-zero. Put*

$$\Phi(z) = f^{d_1}(z+c_1) f^{d_2}(z+c_2) \cdots f^{d_m}(z+c_m), \tag{1}$$

and

$$d = d_1 + d_2 + \cdots + d_m,$$

where $d_1, d_2, \cdots, d_m \in \mathbb{N}$*. Suppose that* $\Phi(z)$ *is a non-constant meromorphic function. Then, for every* $b \in \mathbb{C}\backslash\{0\}$*, we have*

$$dT(r,f) \leqslant dN\left(r,\frac{1}{f}\right) + 2dN(r,f) + N\left(r,\frac{1}{\Phi(z)-b}\right) + S(r,f). \tag{2}$$

Proof. Since $\Phi(z)$ is a non-constant meromorphic function. There is a $\eta \in \mathbb{C}\backslash\{0\}$ such that $\Delta_\eta \Phi(z) = \Delta_\eta(\Phi(z)-b) \not\equiv 0$. Note that

$$\begin{aligned} \frac{1}{f^d} &= \frac{\Phi(z)}{bf^d} - \frac{\Delta_\eta(\Phi(z)-b)}{bf^d}\frac{\Phi(z)-b}{\Delta_\eta(\Phi(z)-b)} \\ &= \frac{\Phi(z)}{bf^d} - \frac{\Delta_\eta \Phi(z)}{bf^d}\frac{\Phi(z)-b}{\Delta_\eta(\Phi(z)-b)} \end{aligned} \tag{3}$$

where

$$\begin{aligned} \frac{\Delta_\eta \Phi(z)}{bf^d} &= \frac{f^{d_1}(z+c_1+\eta)f^{d_2}(z+c_2+\eta)\cdots f^{d_m}(z+c_m+\eta)}{bf^d} - \frac{\Phi(z)}{bf^d} \\ &= \frac{1}{b}\left(\frac{f(z+c_1+\eta)}{f}\right)^{d_1}\cdot\left(\frac{f(z+c_2+\eta)}{f}\right)^{d_2}\cdots\cdots\left(\frac{f(z+c_m+\eta)}{f}\right)^{d_m} - \frac{\Phi(z)}{bf^d}, \\ \frac{\Phi(z)}{bf^d} &= \frac{f^{d_1}(z+c_1)f^{d_2}(z+c_2)\cdots f^{d_m}(z+c_m)}{bf^d} \\ &= \frac{1}{b}\left(\frac{f(z+c_1)}{f}\right)^{d_1}\cdot\left(\frac{f(z+c_2)}{f}\right)^{d_2}\cdots\cdots\left(\frac{f(z+c_m)}{f}\right)^{d_m}. \end{aligned}$$

It follows from Lemma 1 that

$$m\left(r,\frac{\Phi(z)}{bf^d}\right) = S(r,f), \tag{4}$$

$$m\left(r,\frac{\Delta_\eta\Phi(z)}{bf^d}\right) = S(r,f). \tag{5}$$

From (3)-(5), we get

$$m\left(r,\frac{1}{f^d}\right) \leqslant m\left(r,\frac{\Phi(z)-b}{\Delta_\eta(\Phi(z)-b)}\right) + S(r,f).$$

Therefore

$$\begin{aligned} T\left(r,\tfrac{1}{f^d}\right) &\leqslant N\left(r,\tfrac{1}{f^d}\right) + m\left(r,\tfrac{\Phi(z)-b}{\Delta_\eta(\Phi(z)-b)}\right) + S(r,f) \\ &\leqslant dN\left(r,\tfrac{1}{f}\right) + m\left(r,\tfrac{\Phi(z)-b}{\Delta_\eta(\Phi(z)-b)}\right) + S(r,f). \end{aligned} \tag{6}$$

Since

$$T(r,\Phi(z)) \leqslant \sum_{i=1}^{m} d_i T(r,f(z+c_i)) + O(1). \tag{7}$$

Using Lemma 2, we can derive from (7) that

$$T(r,\Phi(z)) \leqslant dT(r,f)+S(r,f). \tag{8}$$

Hence $\sigma(\Phi(z)) \leqslant \sigma(f)$ and

$$S(r,\Phi(z)) = S(r,f). \tag{9}$$

From the first fundamental theorem of Nevanlinna theory, we have

$$m\left(r,\frac{\Phi(z)-b}{\Delta_\eta(\Phi(z)-b)}\right) \leqslant m\left(r,\frac{\Delta_\eta(\Phi(z)-b)}{\Phi(z)-b}\right)+N\left(r,\frac{\Delta_\eta(\Phi(z)-b)}{\Phi(z)-b}\right)+O(1). \tag{10}$$

It follows from Lemma 3 and (9) that

$$m\left(r,\frac{\Delta_\eta(\Phi(z)-b)}{\Phi(z)-b}\right) = S(r,\Phi(z)) = S(r,f). \tag{11}$$

It follows from Lemma 2 that

$$N\left(r,\frac{\Delta_\eta(\Phi(z)-b)}{\Phi(z)-b}\right) \leqslant N\left(r,\frac{1}{\Phi(z)-b}\right)+2dN(r,f)+S(r,f). \tag{12}$$

From (6), (10)-(12) and Lemma 4, we have

$$\begin{aligned} dT(r,f) &= T\left(r,\frac{1}{f^d}\right)+S(r,f) \\ &\leqslant dN\left(r,\frac{1}{f}\right)+2dN(r,f)+N\left(r,\frac{1}{\Phi(z)-b}\right)+S(r,f) \end{aligned} \tag{13}$$

From Theorem 1, we can get the following Corollaries.

COROLLARY 1. *Let f be a transcendental meromorphic function of finite order and let c be a non-zero complex constant. Then, for every $b \in \mathbb{C}\backslash\{0\}$, we have*

$$T(r,f) \leqslant N\left(r,\frac{1}{f}\right)+2N(r,f)+N\left(r,\frac{1}{f(z+c)-b}\right)+S(r,f).$$

Corollary 1 is a difference counterpart of Theorem C.

COROLLARY 2. *Let f be a transcendental meromorphic function of finite order and let c be a non-zero complex constant. Then, for every $b \in \mathbb{C}\backslash\{0\}$, we have*

$$2T(r,f) \leqslant 2N\left(r,\frac{1}{f}\right)+4N(r,f)+N\left(r,\frac{1}{f(z)f(z+c)-b}\right)+S(r,f).$$

Corollary 2 is a difference counterpart of Theorem D.

COROLLARY 3. *Let f be a transcendental meromorphic function of finite order and let c be a non-zero complex constant. Then, for every $b \in \mathbb{C}\backslash\{0\}$ and any $n \in \mathbb{N}^+$, we have*

$$(n+1)T(r,f) \leqslant (n+1)N\left(r,\frac{1}{f}\right)+(2n+2)N(r,f) +N\left(r,\frac{1}{f^n(z)f(z+c)-b}\right)+S(r,f),$$

$$(n+1)T(r,f) \leqslant (n+1)N\left(r,\frac{1}{f}\right)+(2n+2)N(r,f) +N\left(r,\frac{1}{f(z)f^n(z+c)-b}\right)+S(r,f).$$

3. Value distribution of nonlinear difference monomials

For the value distribution of nonlinear difference monomial, Laine and Yang [10] has proved the following theorem.

THEOREM E. [10] *Let f be a transcendental entire function of finite order, and c be a non-zero complex constant. Then for $n \geqslant 2$, $f(z+c)(f(z))^n$ assumes every non-zero value $b \in \mathbb{C}$ infinitely often.*

In [2], Chen et al. have proved the following Theorem which can be seen as a supplement to the case of Theorem E in $n=1$.

THEOREM F. [2] *Let f be a transcendental entire function of finite order, and let $c \in \mathbb{C}\backslash\{0\}$. If $f(z)$ has infinitely many multi-order zeros, then $f(z+c)f(z)$ assumes every non-zero value $b \in \mathbb{C}$ infinitely often.*

The following result is an improvement of Theorems E and F under some other conditions.

THEOREM 2. *Let $f(z)$ be a transcendental meromorphic function of finite order, and assume that $\delta(\infty,f)=1$. Suppose that $\Phi(z)$ is a nonlinear difference monomial of the form (1). Then,*

(i) for $\delta(0,f)>0$, $\Phi(z)$ assumes every non-zero value b infinitely often and $\lambda(b,\Phi(z))=\sigma(f)$;

(ii) for $\delta(0,f)=1$, $\Phi(z)$ assumes every non-zero value b infinitely often and

$$T(r,\Phi(z)) \sim dT(r,f) \sim N\left(r,\frac{1}{\Phi(z)-b}\right)$$

as $r \notin E, r \to \infty$, where E is a possible exception set of r with finite logarithmic measure.

Proof. Since $\delta(0,f)>0$, $\Phi(z)$ is a transcendental meromorphic function. Suppose that $\Phi(z)$ is not a transcendental meromorphic function. Then there is a rational function $P(z)$ such that $p(z)\Phi(z)\equiv 1$, i.e.

$$\begin{aligned}\frac{1}{f^d} &\equiv p(z)\frac{\Phi(z)}{f^d}\\ &\equiv p(z)\left(\frac{f(z+c_1)}{f}\right)^{d_1}\cdot\left(\frac{f(z+c_2)}{f}\right)^{d_2}\cdot\cdots\cdot\left(\frac{f(z+c_m)}{f}\right)^{d_m}.\end{aligned}$$

Applying Lemma 1 and noting that $f(z)$ is transcendental, we can get

$$m\left(r,\frac{1}{f^d}\right)=S(r,f).$$

Therefore

$$\begin{aligned}m\left(r,\frac{1}{f^d}\right)+N\left(r,\frac{1}{f^d}\right) &\leqslant N\left(r,\frac{1}{f^d}\right)+S(r,f)\\ &\leqslant dN\left(r,\frac{1}{f}\right)+S(r,f).\end{aligned}$$

Apply Lemma 4 and the first fundamental theorem of Nevanlinna theory, we can get

$$dT(r,f)\leqslant dN\left(r,\frac{1}{f}\right)+S(r,f).$$

This contradicts with $\delta(0,f)>0$. Thus $\Phi(z)$ is a function transcendental and meromorphic.

(i) Since $\delta(0,f)>0$ and $\delta(\infty,f)=1$, there is a positive number $\theta<1$ such that

$$N\left(r,\frac{1}{f}\right)<\theta T(r,f), \tag{14}$$

$$N(r,f)=o(1)T(r,f). \tag{15}$$

By Theorem 1, we have

$$dT(r,f)\leqslant 2dN(r,f)+dN\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{\Phi(z)-b}\right)+S(r,f). \tag{16}$$

Combining (14)-(16) we can get

$$d(1-o(1)-\theta)T(r,f)\leqslant N\left(r,\frac{1}{\Phi(z)-b}\right),r\notin E,r\to\infty, \tag{17}$$

where E is a possible exceptional set with finite logarithmic measure. Noticing f is transcendental, applying Lemma 5 and (17), we can get that $\Phi(z)$ assumes every non-zero value b infinitely often and $\lambda(b,\Psi(z))=\sigma(f)$.

(ii) Since $\delta(0,f)=1$ and $\delta(\infty,f)=1$,

$$N\left(r,\frac{1}{f}\right)=S(r,f), \tag{18}$$

$$N(r,f)=S(r,f). \tag{19}$$

From (8), (16), (18), (19), we have

$$
\begin{aligned}
dT(r,f) &\leqslant N\left(r, \tfrac{1}{\Phi(z)-b}\right) + S(r,f) \\
&\leqslant T\left(r, \Phi(z)\right) + S(r,f) \\
&\leqslant dT(r,f) + S(r,f).
\end{aligned}
\tag{20}
$$

Since f is transcendental, (20) means that $\Phi(z)$ assumes every non-zero value b infinitely often and

$$
T(r,\Phi) \sim dT(r,f) \sim N\left(r, \frac{1}{\Phi - b}\right)
$$

as $r \notin E, r \to \infty$, where E is a possible exception set of r with finite logarithmic measure.

EXAMPLE 1. $f(z) = e^z$, $c_1 = \log 1, c_2 = \log 2, \cdots, c_m = \log m, (m \in \mathbb{N}^+)$. Then

$$
\Phi(z) = f(z+c_1)f(z+c_2)\cdots f(z+c_m) = e^{z+\log 1} \cdot e^{z+\log 2} \cdot \cdots \cdot e^{z+\log m} = m! e^{mz}.
$$

Obviously, we can get $\delta(0,f) = \delta(\infty,f) = 1$ and $\Phi(z)$ assumes every non-zero value b infinitely often and $\lambda(b,\Phi(z)) = \sigma(f)$,

$$
T(r,\Phi(z)) \sim mT(r,f) \sim N\left(r, \frac{1}{\Phi(z) - b}\right)
$$

as $r \to \infty$. And above all, $\Phi(z) = m! e^{mz} \neq 0$. Therefore, the condition $b \neq 0$ in Theorem 2 is necessary.

From Theorem 2, we can get the following Corollary.

COROLLARY 4. *Let $f(z)$ be a transcendental meromorphic function of finite order and let c be a non-zero complex constant. Assume that $\delta(\infty,f) = 1, m,n \in \mathbb{N}$ are not all zero. Then:*

(i) for $\delta(0,f) > 0$, $f^m(z)f^n(z+c)$ assumes every non-zero value b infinitely often and $\lambda(b, f^m(z)f^n(z+c)) = \sigma(f)$;

(ii) for $\delta(0,f) = 1$, $f^m(z)f^n(z+c)$ assumes every non-zero value b infinitely often and

$$
T(r, f^m(z)f^n(z+c)) \sim (m+n)T(r,f) \sim N\left(r, \frac{1}{f^m(z)f^n(z+c) - b}\right),
$$

as $r \notin E, r \to \infty$, where E is a possible exception set of r with finite logarithmic measure.

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Zhaojun Wu
School of Mathematics and Statistics
Hubei University of Science and Technology
Xianning, 437100, China
e-mail: wuzj52@hotmail.com

Hongyan Xu
School of mathematics and computer science
Shangrao Normal University
Shangrao, 334001, China
e-mail: 305180@sru.edu.cn