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EXTENDING A RESULT OF HAYNSWORTH

QIAN LI, QINGWEN WANG AND SHENG DONG*

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Abstract. Haynsworth [4] refined a determinant inequality for two positive definite matrices. We extend Haynsworth's result to more than two positive definite matrices and obtain some inequalities for sum of positive definite matrices. Moreover, we show some generalizations of these to sector matrices.

1. Introduction

Let $\mathbb{M}_n$ be the set of $n \times n$ complex matrices. If X is positive semidefinite, we put $X \geqslant 0$, and $X > 0$ means that X is positive definite. For two Hermitian matrices $X, Y \in \mathbb{M}_n, X \geqslant Y$ means $X - Y$ is positive semidefinite.

If $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ with X_{11} nonsingular, the Schur complement of X_{11} in X is defined as

$$X/X_{11} = X_{22} - X_{21}X_{11}^{-1}X_{12}.$$

And it holds

$$\det X = \det X_{11} \det(X/X_{11}). \tag{1}$$

For $A \in \mathbb{M}_n$, recall the Cartesian decomposition (see, e.g. [6, p.7])

$$A = \Re A + i\Im A,$$

where

$$\Re A = \frac{1}{2}(A + A^*), \quad \Im A = \frac{1}{2i}(A - A^*).$$

The matrix $A \in \mathbb{M}_n$ is accretive-dissipative if $\Re A$ and $\Im A$ are positive definie. Relevant studies on this class of matrices can be found in [2, 7, 11].

The numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax | x \in \mathbb{C}^n, x^*x = 1\}.$$

Also, for $\alpha \in [0, \frac{\pi}{2})$, let S_α be the sector in the complex plane given by

$$S_\alpha = \{z \in \mathbb{C} | \Re z > 0, |\Im z| \leqslant (\Re z)\tan(\alpha)\}.$$

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* Corresponding author.

If $W(A) \subset S_0$, then A is positive definite. As $0 \notin S_\alpha$, if $W(A) \subset S_\alpha$, then A is necessarily nonsingular. For more details on matrices with numerical ranges in a sector, please refer to [9, 1, 10].

Let $A, B \in \mathbb{M}_n$ be positive definite matrices. It is well known that

$$\det(A+B) \geqslant \det A + \det B. \tag{2}$$

A simple generalization of inequality (2) could be

$$\det\left(\sum_{i=1}^{m} A_i\right) \geqslant \sum_{i=1}^{m} \det A_i, \tag{3}$$

where $A_i \in \mathbb{M}_n, i = 1, \ldots, m$ are positive definite matrices.

In [4], Haynsworth proved the following refinement of (2),

$$\det(A+B) \geqslant \left(1 + \sum_{k=1}^{n-1} \frac{\det B^{(k)}}{\det A^{(k)}}\right) \det A + \left(1 + \sum_{k=1}^{n-1} \frac{\det A^{(k)}}{\det B^{(k)}}\right) \det B. \tag{4}$$

where $A^{(k)}, B^{(k)}, k = 1, \ldots, n-1$, denote the k-th leading principal submatrices of A and B, respectively.

For Haynsworth's inequality (4), Hartfiel [3] gave an improvement, Lin [5] extended their results to sector matrices, Hou and Dong [5] gave some results related to three positive definite matrices.

In this paper, by some properties for the Schur complement, we extend Haynsworth's inequality (4) to more than two positive definite matrices, and obtain some refinements of (3). Moreover, we give some generalizations of the results obtained to sector matrices.

2. Main results

In this section, we first give some lemmas which will be used in the proof of our main results.

LEMMA 1. [6, Corollary 7.7.4] *If* $A, B \in \mathbb{M}_n$ *such that* $A \geqslant B > 0$*, then* $\det A \geqslant \det B$.

LEMMA 2. [4, Theorem 2] *Suppose* $A, B \in \mathbb{M}_n$ *are positive definite. Let* $A^{(k)}$ *and* $B^{(k)}, k = 1, \ldots, n-1$*, denote the* k*-th leading principal submatrices of* A *and* B *respectively. Then*

$$(A+B)/(A^{(k)} + B^{(k)}) \geqslant A/A^{(k)} + B/B^{(k)}.$$

LEMMA 3. [6, Theorem 7.8.19] *Let* $A \in \mathbb{M}_n$*. If* $\Re(A) > 0$*, then*

$$\det(\Re A) \leqslant |\det A|.$$

LEMMA 4. [9, Lemma 2.6] *Let* $A \in \mathbb{M}_n$ *with* $W(A) \subset S_\alpha$. *Then*

$$\sec^n(\alpha)\det(\Re A) \geqslant |\det A|.$$

LEMMA 5. [9, Proposition 2.1] *Let* $A \in \mathbb{M}_n$ *be partitioned as* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ *with* A_{11} *square. If* $W(A) \subset S_\alpha$, *then* $W(A_{11}) \subset S_\alpha$.

LEMMA 6. *Suppose* $A_i \in \mathbb{M}_n, i = 1, \ldots, m$, *are positive definite. Let* $A_i^{(k)}, k = 1, \ldots, n-1$, *denote the* k*-th leading principal submatrices of* A_i. *Then*

$$\det\left(\left(\sum_{i=1}^m A_i\right) / \left(\sum_{i=1}^m A_i^{(k)}\right)\right) \geqslant \sum_{i=1}^m \frac{\det A_i}{\det A_i^{(k)}}.$$

Proof. For $A_i, i = 1, \ldots, m$, by Lemma 2, we obtain

$$\left(\sum_{i=1}^m A_i\right) / \left(\sum_{i=1}^m A_i^{(k)}\right) \geqslant \sum_{i=1}^m \left(A_i / A_i^{(k)}\right).$$

Taking determinants on both sides, by Lemma 1, we get

$$\det\left(\left(\sum_{i=1}^m A_i\right) / \left(\sum_{i=1}^m A_i^k\right)\right) \geqslant \det\left(\sum_{i=1}^m \left(A_i / A_i^k\right)\right).$$

By (3) and (1), we have

$$\begin{aligned} \det\left(\sum_{i=1}^m \left(A_i / A_i^{(k)}\right)\right) &\geqslant \sum_{i=1}^m \det\left(A_i / A_i^{(k)}\right). \\ &= \sum_{i=1}^m \frac{\det A_i}{\det A_i^{(k)}}. \end{aligned}$$

This completes the proof. □

Now, we give the following extension of Haynsworth's inequality (4).

THEOREM 1. *Suppose* $A_i \in \mathbb{M}_n, i = 1, \ldots, m$, *are positive definite. Let* $A_i^{(k)}, k = 1, \ldots, n-1$, *denote the* k*-th leading principal submatrices of* A_i. *Then*

$$\det\left(\sum_{i=1}^m A_i\right) \geqslant \sum_{i=1}^m \left(1 + \sum_{k=1}^{n-1} \frac{\sum_{j=1, j\neq i}^m \det A_j^{(k)}}{\det A_i^{(k)}}\right) \det A_i. \tag{5}$$

Proof. We prove the theorem by induction on n. For $n = 2$, we get

$$\det\left(\sum_{i=1}^m A_i\right) = \det\left(\sum_{i=1}^m A_i^{(1)}\right) \cdot \det\left(\left(\sum_{i=1}^m A_i\right) / \left(\sum_{i=1}^m A_i^{(1)}\right)\right)$$

$$\begin{aligned}
&\geqslant \det\left(\sum_{i=1}^{m} A_i^{(1)}\right)\cdot\sum_{i=1}^{m}\frac{\det A_i}{\det A_i^{(1)}}\\
&\geqslant \left(\sum_{i=1}^{m}\det A_i^{(1)}\right)\cdot\sum_{i=1}^{m}\frac{\det A_i}{\det A_i^{(1)}}\\
&=\sum_{i=1}^{m}\left(1+\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(1)}}{\det A_i^{(1)}}\right)\det A_i,
\end{aligned}$$

where the first equality above is by (1); the first inequality is due to Lemma 6. This proves (5) for $n=2$.

Suppose the inequality (5) holds for all A_i of order less than or equal to $n-1$. If $A_i, i=1,\dots,m$, are of order n, by (1) and Lemma 6, we have

$$\begin{aligned}
\det\left(\sum_{i=1}^{m} A_i\right) &= \det\left(\sum_{i=1}^{m} A_i^{(n-1)}\right)\cdot\det\left(\left(\sum_{i=1}^{m} A_i\right)\Big/\left(\sum_{i=1}^{m} A_i^{(n-1)}\right)\right)\\
&\geqslant \det\left(\sum_{i=1}^{m} A_i^{(n-1)}\right)\cdot\sum_{i=1}^{m}\frac{\det A_i}{\det A_i^{(n-1)}}.
\end{aligned}$$

By the induction hypothesis

$$\det\left(\sum_{i=1}^{m} A_i^{(n-1)}\right)\geqslant\sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-2}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\det A_i^{(n-1)},$$

we get

$$\begin{aligned}
\det\left(\sum_{i=1}^{m} A_i\right) &\geqslant \sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-2}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\det A_i^{(n-1)}\cdot\sum_{h=1}^{m}\frac{\det A_h}{\det A_h^{(n-1)}}\\
&= \sum_{i=1,h=1,i=h}^{m}\left(1+\sum_{k=1}^{n-2}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\det A_i^{(n-1)}\cdot\frac{\det A_h}{\det A_h^{(n-1)}}\\
&\quad+\sum_{i=1,h=1,i\neq h}^{m}\left(1+\sum_{k=1}^{n-2}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\det A_i^{(n-1)}\cdot\frac{\det A_h}{\det A_h^{(n-1)}}\\
&= \sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-2}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\cdot\det A_i\\
&\quad+\sum_{i=1,h=1,i\neq h}^{m}\left(1+\sum_{k=1}^{n-2}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\det A_i^{(n-1)}\cdot\frac{\det A_h}{\det A_h^{(n-1)}}\\
&\geqslant \sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-2}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\cdot\det A_i
\end{aligned}$$

$$+\sum_{i=1,h=1,i\neq h}^{m}\det A_i^{(n-1)}\cdot\frac{\det A_h}{\det A_h^{(n-1)}}$$

$$=\sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-1}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\cdot\det A_i.$$

This completes the proof of Theorem 1. □

The following two inequalities are refinements of (3).

THEOREM 2. *Suppose $A_i\in\mathbb{M}_n, i=1,\ldots,m$, are positive definite. Let $A_i^{(k)}, k=1,\ldots,n-1$, denote the k-th leading principal submatrices of A_i. Then*

$$\det\left(\sum_{i=1}^{m}A_i\right)\geqslant\sum_{i=1}^{m}\det A_i+2(n-1)\sum_{i,j=1,i\neq j}^{m}\sqrt{\det A_iA_j}. \tag{6}$$

Proof. By (5), we get

$$\begin{aligned}\det\left(\sum_{i=1}^{m}A_i\right)&\geqslant\sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-1}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\det A_i\\&=\sum_{i=1}^{m}\det A_i+\sum_{i=1}^{m}\left(\sum_{k=1}^{n-1}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\cdot\det A_i\right)\\&=\sum_{i=1}^{m}\det A_i+\sum_{k=1}^{n-1}\left(\sum_{i=1}^{m}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\cdot\det A_i\right)\\&=\sum_{i=1}^{m}\det A_i+\sum_{k=1}^{n-1}\left(\sum_{i,j=1,i\neq j}^{m}\left(\frac{\det A_j^{(k)}}{\det A_i^{(k)}}\cdot\det A_i+\frac{\det A_i^{(k)}}{\det A_j^{(k)}}\cdot\det A_j\right)\right)\\&\geqslant\sum_{i=1}^{m}\det A_i+\sum_{k=1}^{n-1}\sum_{i,j=1,i\neq j}^{m}2\sqrt{\det A_i\cdot\det A_j}\\&=\sum_{i=1}^{m}\det A_i+2(n-1)\sum_{i,j=1,i\neq j}^{m}\sqrt{\det A_iA_j},\end{aligned}$$

the proof is completed. □

THEOREM 3. *Suppose $A_i\in\mathbb{M}_n, i=1,\ldots,m$, are positive definite. Let $A_i^{(k)}, k=1,\ldots,n-1$, denote the k-th leading principal submatrices of A_i. Then*

$$\det\left(\sum_{i=1}^{m}A_i\right)\geqslant\sum_{i=1}^{m}\det A_i+m(m-1)(n-1)\left(\prod_{i=1}^{m}\det A_i\right)^{\frac{1}{m}}.$$

Proof. This follows from inequality (6) and the arithmetic-geometric inequality. $\square$

Next, we extend Theorem 1, Theorem 2 and Theorem 3 to sector matrices. The generalization of Theorem 1 is as follows.

THEOREM 4. *Suppose $A_i \in \mathbb{M}_n$, $W(A_i) \subset S_\alpha, i = 1,\ldots,m, \alpha \in [0,\frac{\pi}{2})$. Let $A_i^{(k)}, k = 1,\ldots,n-1$, denote the k-th leading principal submatrices of A_i. Then*

$$\sec^n(\alpha)\left|\det\left(\sum_{i=1}^m A_i\right)\right| \geqslant \sum_{i=1}^m \left(1+\sum_{k=1}^{n-1}\cos^k(\alpha)\frac{\sum_{j=1,j\neq i}^m \left|\det A_j^{(k)}\right|}{\left|\det A_i^{(k)}\right|}\right)|\det A_i|.$$

Proof. From $W(A_i) \subset S_\alpha$, $i = 1,\ldots,m$, we get $W(\sum_{i=1}^m A_i) \subset S_\alpha$ and $W(A_i^{(k)}) \subset S_\alpha, k = 1,\ldots,n-1$. Then

$$\begin{aligned}
\left|\det\left(\sum_{i=1}^m A_i\right)\right| &\geqslant \det\left(\Re\left(\sum_{i=1}^m A_i\right)\right)\\
&= \det\left(\sum_{i=1}^m \Re A_i\right)\\
&\geqslant \sum_{i=1}^m \left(1+\sum_{k=1}^{n-1}\frac{\sum_{j=1,j\neq i}^m \det\left(\Re A_j^{(k)}\right)}{\det\left(\Re A_i^{(k)}\right)}\right)\det(\Re A_i)\\
&\geqslant \sum_{i=1}^m \left(1+\sum_{k=1}^{n-1}\cos^k(\alpha)\frac{\sum_{j=1,j\neq i}^m \left|\det A_j^{(k)}\right|}{\left|\det A_i^{(k)}\right|}\right)\cos^n(\alpha)|\det A_i|,
\end{aligned}$$

where the first inequality above is due to Lemma 3; the second is by inequality (5) and the last inequality holds by Lemma 3 and Lemma 4.

Then, multiplying both sides by $\sec^n(\alpha)$ yields the desired inequality, and this completes the proof of Theorem 4. $\square$

Apparently, Theorem 4 reduces to Theorem 1 when $\alpha = 0$. Note that if A is accretive-dissipative, then $W(e^{-i\pi/4}A) \subset S_{\pi/4}$. Thus, we get the following corollary.

COROLLARY 1. *Suppose $A_i \in \mathbb{M}_n, i = 1,\ldots,m$, are accretive-dissipative. Let $A_i^{(k)}, k = 1,\ldots,n-1$, denote the k-th leading principal submatrices of A_i. Then*

$$2^{\frac{n}{2}}\left|\det\left(\sum_{i=1}^m A_i\right)\right| \geqslant \sum_{i=1}^m \left(1+\sum_{k=1}^{n-1}\frac{1}{2^{k/2}}\frac{\sum_{j=1,j\neq i}^m \left|\det A_j^{(k)}\right|}{\left|\det A_i^{(k)}\right|}\right)|\det A_i|.$$

The generalizations of Theorem 2 and Theorem 3 to sector matrices are as follows.

THEOREM 5. *Let $A_i \in \mathbb{M}_n, W(A) \subset S_\alpha, \alpha \in [0, \frac{\pi}{2}), i = 1, \dots, m$. Then*

$$\sec^n(\alpha) \left|\det\left(\sum_{i=1}^m A_i\right)\right| \geqslant \sum_{i=1}^m |\det A_i| + 2(n-1) \sum_{i,j=1, i\neq j}^m \sqrt{|\det A_i A_j|}.$$

COROLLARY 2. *Let $A_i \in \mathbb{M}_n, i = 1, \dots, m$, be accretive-dissipative. Then*

$$2^{\frac{n}{2}} \left|\det\left(\sum_{i=1}^m A_i\right)\right| \geqslant \sum_{i=1}^m |\det A_i| + 2(n-1) \sum_{i,j=1, i\neq j}^m \sqrt{|\det A_i A_j|}.$$

THEOREM 6. *Let $A_i \in \mathbb{M}_n, W(A) \subset S_\alpha, \alpha \in [0, \frac{\pi}{2}), i = 1, \dots, m$. Then*

$$\sec^n(\alpha) \left|\det\left(\sum_{i=1}^m A_i\right)\right| \geqslant \sum_{i=1}^m |\det A_i| + m(m-1)(n-1) \left(\prod_{i=1}^m |\det A_i|\right)^{\frac{1}{m}}.$$

COROLLARY 3. *Let $A_i \in \mathbb{M}_n, i = 1, \dots, m$, be accretive-dissipative. Then*

$$2^{\frac{n}{2}} \left|\det\left(\sum_{i=1}^m A_i\right)\right| \geqslant \sum_{i=1}^m |\det A_i| + m(m-1)(n-1) \left(\prod_{i=1}^m |\det A_i|\right)^{\frac{1}{m}}.$$

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Qian Li
College of Mathematics
Zhengzhou University of Aeronautics
Zhengzhou 450046, China
e-mail: liq689@163.com

QingWen Wang
Department of Mathematics
Shanghai University
Shanghai 200444, China
e-mail: wqw@t.shu.edu.cn

Sheng Dong
College of Mathematics
Zhengzhou University of Aeronautics
Zhengzhou 450046, China
and Department of Mathematics
Shanghai University
Shanghai 200444, China
e-mail: dongsheng7088@163.com