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SHARP COEFFICIENTS BOUNDS FOR CLASS OF ALMOST STARLIKE MAPPINGS OF ORDER α IN $\mathbb{C}^n$

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Abstract. Let Ω be the bounded starlike circular domain. In this paper, we obtain the sharp bounds for the Fekete-Szegö functional $|A_3 - \mu A_2^2|$ of the class $\mathscr{A}\mathscr{S}_\alpha^*(\Omega)$ of almost starlike mappings of order α in $\mathbb{C}^n$ $(n \geqslant 2)$, where $\mu \in \mathbb{R}$, and A_2, A_3 are the first two coefficients of the homogeneous expansion of mappings $f \in \mathscr{A}\mathscr{S}_\alpha^*(\Omega)$. Our results can be regarded as the extensions of corresponding works from the case in one dimension to the case in higher dimensions.

1. Introduction

Let $\mathscr{S}$ be the family of univalent functions f of the unit disk $\mathbb{U}$, normalized in such a way that

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots. \tag{1}$$

A classical theorem of Fekete-Szegö (see [5]) states that for $f \in \mathscr{S}$, we have

$$\left|a_3 - \mu a_2^2\right| \leqslant \begin{cases} 3 - 4\mu, & if \quad \mu \leqslant 0, \\ 1 + 2\exp(\frac{-2\mu}{1-\mu}), & if \quad 0 \leqslant \mu \leqslant 1, \\ 4\mu - 3, & if \quad \mu \geqslant 1. \end{cases}$$

The inequalities are sharp for each $\mu \in \mathbb{R}$. Let

$$\Phi_\mu(n, f) = |a_{2n-1} - \mu a_n^2|, \; n \geqslant 2, f \in \mathscr{S} \tag{2}$$

denote the generalized Zalcman coefficient functional so that $\Phi_\mu(2, f) = \Phi_\mu(f)$ is the Fekete-Szegö functional (see, e.g. Li-Ponnusamy-Qiao [12], Li-Ponnusamy [13]). Then it is quite natural to discuss the behavior of $\Phi_\mu(f)$ for kinds of subclasses of normalized univalent functions in the unit disk $\mathbb{U}$. This is called Fekete-Szegö problem. It attracts a lot of attentions (see, e.g. [1, 2, 9, 11, 13, 16, 17, 18, 19, 21, 22]).

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Let $\mathscr{S}^*(\alpha)$ and $\mathscr{A}\mathscr{S}^*(\alpha)$ denote classes of starlike and almost starlike univalent functions of order $\alpha\,(0 \leqslant \alpha < 1)$, respectively, i.e.

$$f \in \mathscr{S}^*(\alpha) \Longleftrightarrow f \in \mathscr{S},\ \Re\frac{zf'(z)}{f(z)} > \alpha,\ z \in \mathbb{U} \tag{3}$$

and

$$f \in \mathscr{A}\mathscr{S}^*(\alpha) \Longleftrightarrow f \in \mathscr{S},\ \Re\frac{f(z)}{zf'(z)} > \alpha,\ z \in \mathbb{U}. \tag{4}$$

We can note that $\mathscr{A}\mathscr{S}^*(0) = \mathscr{S}^*(0) = \mathscr{S}^*$ represents standard class of starlike functions.

The following Theorem A was obtained by Kanas-Darwish [10] in Theorem 2.4 with the parameters $n=0,\ b=1-\alpha$ (also, see Orhan-Deniz-Çağlar [18]) and Theorem B was obtained by Xiong-Feng-Zhang [23] in Theorem 3 with the parameters $n=0$, $b=1,\ \lambda=0,\ A_1=2(1-\alpha)$ and $A_2=2(\alpha-1)(2\alpha-1)$.

THEOREM A. *(Kanas-Darwish, [10]) Suppose $f \in \mathscr{S}^*(\alpha)$ is given by (1.1). Then for any $\mu \in \mathbb{R}$,*

$$\left|a_3-\mu a_2^2\right| \leqslant \begin{cases} (1-\alpha)[2(1-\alpha)(1-2\mu)+1], & if \quad \mu \leqslant \frac{1}{2}, \\ 1-\alpha, & if \quad \frac{1}{2} \leqslant \mu \leqslant \frac{3-2\alpha}{2(1-\alpha)}, \\ (1-\alpha)[2(1-\alpha)(2\mu-1)-1], & if \quad \mu \geqslant \frac{3-2\alpha}{2(1-\alpha)}. \end{cases}$$

The above estimates are sharp for each real μ.

THEOREM B. *(Xiong-Feng-Zhang, [23]) Suppose $f \in \mathscr{A}\mathscr{S}^*(\alpha)$ is given by (1.1). Then for any $\mu \in \mathbb{R}$,*

$$\left|a_3-\mu a_2^2\right| \leqslant \begin{cases} (1-\alpha)[3-4\alpha-4(1-\alpha)\mu], & if \quad \mu \leqslant \frac{4\alpha-2}{4(\alpha-1)}, \\ 1-\alpha, & if \quad \frac{4\alpha-2}{4(\alpha-1)} \leqslant \mu \leqslant 1, \\ (1-\alpha)[4\alpha+4(1-\alpha)\mu-3], & if \quad \mu \geqslant 1. \end{cases}$$

The above estimates are sharp for each real μ.

QUESTION 1.1. For the class of almost starlike mappings of order α in $\mathbb{C}^n$ $(n \geqslant 2)$, whether or not we can give a higher dimensional version for Theorem B?

The Fekete-Szegö problem is related to the Bieberbach conjecture (see [3]). However, Cartan [4] stated that the Bieberbach conjecture does not hold in several complex variables. Thus, compare with the case in $\mathbb{C}$, it is more difficult to obtain the complete results for the inequalities of homogeneous expansions for subclasses of biholomorphic mappings in $\mathbb{C}^n$ (see, e.g. [6, 7, 8]).

The study of Fekete-Szegö problem in higher dimensions with a subclass of starlike mappings defined on the unit ball in a complex Banach space or on the unit polydisk was firstly done by Xu-Liu [26], and some of sequels to Xu-Liu [26] have appeared in the literature since then (see, e.g., [14, 24, 25, 26]). At present, these results mainly consider the Fekete-Szegö problem for classes of strongly starlike mappings of order α or starlike mappings of order α $(0 \leqslant \alpha < 1)$ in $\mathbb{C}^n$ when $\mu \in \mathbb{C}$.

Although there are some of significant results which cope with the Fekete-Szegö problem for subclasses of starlike mapping or strongly starlike mappings in $\mathbb{C}^n$ $(n \geqslant 2)$, there exists no work directly concerning this problem for subclasses of almost starlike mapping. Moreover, compare Theorem A with Theorem B, we note that the case for class of almost starlike function is quite different from the case for class of starlike function in one dimension. We think that this situation is similar for the class of almost starlike mappings and starlike mappings in $\mathbb{C}^n$. These stimulate us to consider Question 1.1.

In this paper, we shall try to give an affirmative answer to Question 1.1.

2. Preliminaries

Let $\langle z, w\rangle$ stand for the inner product in the complex n-dimensional space $\mathbb{C}^n$ given by

$$\langle z, w\rangle = z_1\overline{w_1} + z_2\overline{w_2} + \ldots + z_n\overline{w_n},$$

where $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ are coordinate representations of z and w in the standard base $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{C}^n$. Norm in $\mathbb{C}^n$ induced by the inner product is denoted by $\|z\| = \sqrt{\langle z, z\rangle}$. The open ball $\mathbb{B}^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$. Denote by $\mathbb{U}$ the unit disk in $\mathbb{C}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f(z) \prec g(z)$, provided there exists an analytic function $w(z)$ defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$.

Let $\Omega \subset \mathbb{C}^n$ be a bounded starlike circular domain with $0 \in \Omega$, and its Minkowski functional $\rho(z) \in \mathscr{C}^1$ except for some lower dimensional manifolds in $\mathbb{C}^n$. In particular, if $\Omega = \mathbb{B}^n$, then $\rho(z) = \|z\|$. Let $\overline{\Omega}$ and $\partial\Omega$ represent the closure of Ω and the boundary of Ω, respectively. We denote by $H(\Omega)$ the set of all holomorphic mappings from Ω into $\mathbb{C}^n$. Throughout this paper, we write a point $z \in \mathbb{C}^n$ as a column vector in the following $n \times 1$ matrix form

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

and the symbol $'$ stands for the transpose of vectors or matrices. For $f \in H(\Omega)$, we also write it as $f = (f_1, f_2, \ldots, f_n)'$, where f_j is a holomorphic function from Ω to $\mathbb{C}$, $j = 1, \ldots, n$. The derivative of $f \in H(\Omega)$ at a point $a \in \Omega$ is the complex Jacobian matrix of f given by

$$Df(a) = \left(\frac{\partial f_i}{\partial z_j}(a)\right)_{n\times n}.$$

If $f \in H(\Omega)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I_n$, where I_n is the identity matrix. We say that $f \in H(\Omega)$ is locally biholomorphic on Ω if $Df(z)$ is nonsingular at each $z \in \Omega$. A holomorphic mapping $f : \Omega \to \mathbb{C}^n$ is said to be biholomorphic on Ω if the inverse f^{-1} exists and is holomorphic on the open set $f(\Omega)$. In fact, if $f \in H(\Omega)$, then

$$f(w) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(z)((w-z)^n),$$

for all w in some neighborhood of $z \in \Omega$, where $D^n f(z)$ is the nth-Fr$\acute{e}$chet derivative of f at z, and for $n \geqslant 1$,

$$D^n f(z)((w-z)^n) = D^n f(z) \underbrace{(w-z, \ldots, w-z)}_{n}.$$

Suppose that $\Omega \subset \mathbb{C}^n$ is a bounded circular domain. The m $(m > 2)$-Fr$\acute{e}$chet derivative of a mapping $f \in H(\Omega)$ at point $z \in \Omega$ is written as $D^m f(z)(a^{m-1}, \cdot)$. The matrix representation (see, e.g. Xu-You [24]) is

$$D^m f(z)(a^{m-1}, \cdot) = \left(\sum_{l_1, l_2, \ldots, l_{m-1}=1}^{n} \frac{\partial^m f_p(z)}{\partial z_k \partial z_{l_1} \ldots \partial z_{l_{m-1}}} a_{l_1} \ldots a_{l_{m-1}} \right)_{1 \leqslant p,k \leqslant n},$$

where $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))', a = (a_1, a_2, \ldots, a_n)' \in \mathbb{C}^n$.

DEFINITION 1. (Xu-Liu, [24]) Let $\mathscr{A}\mathscr{S}^*(\Omega)$ denote the class of almost starlike mappings of order α $(0 \leqslant \alpha < 1)$ on bounded starlike circular domain Ω, for $\forall z \in \Omega \backslash \{0\}$,

$$f \in \mathscr{A}\mathscr{S}_\alpha^*(\Omega) \Longleftrightarrow f \in H(\Omega),\ \Re \left\{ \frac{2 \frac{\partial \rho(z)}{\partial z} (Df(z))^{-1} f(z)}{\rho(z)} \right\} > \alpha, \tag{5}$$

where f is normalized, $(Df(z))^{-1}$ is the inverse matrix of $Df(z)$ and

$$\frac{\partial \rho(z)}{\partial z} = \left(\frac{\partial \rho(z)}{\partial z_1}, \frac{\partial \rho(z)}{\partial z_2}, \ldots, \frac{\partial \rho(z)}{\partial z_n} \right).$$

If $\Omega = \mathbb{B}^n$ in (5), then we denote the class of almost starlike mappings of order α on $\mathbb{B}^n$ by $\mathscr{A}\mathscr{S}_\alpha^*(\mathbb{B}^n)$. It is easy to see that

$$f \in \mathscr{A}\mathscr{S}_\alpha^*(\mathbb{B}^n) \Longleftrightarrow f \in H(\mathbb{B}^n),\ \Re \frac{\|z\|^2}{\langle (Df(z))^{-1} f(z), z \rangle} > \alpha, \forall z \in \mathbb{B}^n \backslash \{0\}.$$

If $n = 1, \Omega = \mathbb{U}$, then $\mathscr{A}\mathscr{S}_\alpha^*(\mathbb{U}) = \mathscr{A}\mathscr{S}^*(\alpha)$.

The following Lemma is needed in the proof of Theorems 1 and 2.

LEMMA 1. (Liu-Ren, [15]) *$\Omega \in \mathbb{C}^n$ is a bounded starlike circular domain if and only if there exists a unique real continuous function $\rho : \mathbb{C}^n \to \mathbb{R}$, called the Minkowski functional of Ω, such that (i) $\rho(z) \geqslant 0$, $z \in \mathbb{C}^n$; $\rho(z) = 0 \Leftrightarrow z = 0$; (ii) $\rho(tz) = |t| \rho(z)$,*

$t \in \mathbb{C}$, $z \in \mathbb{C}^n$; *(iii)* $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 1\}$. *Furthermore, the function* $\rho(z)$ *has the following properties:*

$$2\frac{\partial \rho(z)}{\partial z} z = \rho(z),\ z \in \mathbb{C}^n,$$
$$2\frac{\partial \rho(z_0)}{\partial z} z_0 = 1,\ z_0 \in \partial\Omega,$$
$$\frac{\partial \rho(\lambda z)}{\partial z} = \frac{\partial \rho(z)}{\partial z},\ \lambda \in (0,+\infty),$$
$$\frac{\partial \rho(e^{i\theta} z)}{\partial z} = e^{-i\theta}\frac{\partial \rho(z)}{\partial z},\ \theta \in \mathbb{R}.$$

3. Main theorems and some corollaries

THEOREM 1. *Suppose that mappings* $f \in \mathscr{A}\mathscr{S}^*_\alpha(\Omega)$ *and satisfy the following condition*

$$\frac{\partial \rho(z)}{\partial z} D^2 f(0)\Big(z, \frac{D^2 f(0)(z^2)}{2!}\Big)\rho(z) = \Big(\frac{\partial \rho(z)}{\partial z} D^2 f(0)(z^2)\Big)^2,\ z \in \Omega. \tag{6}$$

Then for $z \in \Omega \backslash \{0\}$, *we have*

$$\left|A_3 - \mu A_2^2\right| \leqslant \begin{cases} (1-\alpha)[3-4\alpha-4(1-\alpha)\mu], & if \quad \mu \leqslant \frac{4\alpha-2}{4(\alpha-1)}, \\ 1-\alpha, & if \quad \frac{4\alpha-2}{4(\alpha-1)} \leqslant \mu \leqslant 1, \\ (1-\alpha)[4\alpha+4(1-\alpha)\mu-3], & if \quad \mu \geqslant 1, \end{cases}$$

where

$$A_3 = 2\frac{\partial \rho(z)}{\partial z}\frac{D^3 f(0)(z^3)}{3!\rho^3(z)}, \quad A_2 = 2\frac{\partial \rho(z)}{\partial z}\frac{D^2 f(0)(z^2)}{2!\rho^2(z)}.$$

The above estimates are sharp for each real μ.

Proof. Fix $z \in \Omega \setminus \{0\}$, and set $z_0 = \frac{z}{h(z)}$. We define a function $\mathscr{P} : \mathbb{U} \to \mathbb{C}$ by

$$\mathscr{P}(\zeta) = \begin{cases} \dfrac{\zeta}{2\frac{\partial \rho(z_0)}{\partial z}(Df(\zeta z_0))^{-1} f(\zeta z_0)}, & \zeta \neq 0, \\ 1, & \zeta = 0. \end{cases} \tag{7}$$

Then $\mathscr{P} \in H(\mathbb{U})$. Furthermore, using (7), Lemma 1 and $f \in \mathscr{A}\mathscr{S}^*_\alpha(\Omega)$, it gives that

$$\begin{aligned} \Re(\mathscr{P}(\zeta)) &= \Re\frac{\zeta}{2\frac{\partial \rho(z_0)}{\partial z}(Df(\zeta z_0))^{-1} f(\zeta z_0)} \\ &= \Re\frac{\rho(\zeta z_0)}{2\frac{\partial \rho(\xi z_0)}{\partial z}(Df(\zeta z_0))^{-1} f(\zeta z_0)} < \frac{1}{\alpha},\ \zeta \in \mathbb{U}. \end{aligned} \tag{8}$$

Since $\mathscr{P}(0)=\psi(0)=1$, following (8), it is easy to see that $\mathscr{P}(\zeta)\prec\psi(\zeta)$, where

$$\psi(\zeta)=\frac{1+\zeta}{1+(2\alpha-1)\zeta},\ \zeta\in\mathbb{U}.$$

Thus, there is a function $w(z)$, such that $\mathscr{P}(\zeta)=\psi(w(\zeta)),\ \zeta\in\mathbb{U}$. Take a function

$$p(z)=\frac{1+w(z)}{1-w(z)}=1+r_1z+r_2z^2+\ldots\prec\frac{1+z}{1-z},\ z\in\mathbb{U}. \tag{9}$$

We note that $p(0)=1$ and p is a function with positive real part. By (8) and (9), it is easy to obtain

$$\begin{aligned}\mathscr{P}(\zeta)=&\psi(w(\zeta))=1+(1-\alpha)r_1\zeta\\&+\left((1-\alpha)\left(r_2-\frac{r_1^2}{2}\right)+\frac{1}{2}(\alpha-1)(2\alpha-1)r_1^2\right)\zeta^2+\ldots,\zeta\in\mathbb{U}.\end{aligned}\tag{10}$$

Here, for later convenience, we set $g(z)=(Df(z))^{-1}f(z), z\in\Omega$. Taking into account the relations (7) and (10), we deduce that

$$\begin{aligned}&\left[1+(1-\alpha)r_1\zeta+\left((1-\alpha)\left(r_2-\frac{r_1^2}{2}\right)+\frac{1}{2}(\alpha-1)(2\alpha-1)r_1^2\right)\zeta^2+\ldots\right]\\&\times\left(\zeta+2\frac{\partial h(z)}{\partial z}\frac{D^2g(0)(z_0)^2}{2!}\zeta^2+2\frac{\partial h(z)}{\partial z}\frac{D^3g(0)(z_0)^3}{3!}\zeta^3+\ldots\right)=\zeta,\end{aligned}\tag{11}$$

Comparing with the coefficient of two sides of the (11) in ξ^2 and ξ^3, we obtain

$$(1-\alpha)r_1=-2\frac{\partial\rho(z)}{\partial z}\frac{D^2g(0)(z_0)^2)}{2!}\tag{12}$$

and

$$\begin{aligned}&(1-\alpha)\left(r_2-\frac{r_1^2}{2}\right)+\frac{1}{2}(\alpha-1)(2\alpha-1)r_1^2\\=&\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2g(0)(z_0)^2}{2!}\right)^2-2\frac{\partial\rho(z)}{\partial z}\frac{D^3g(0)(z_0)^3}{3!}.\end{aligned}\tag{13}$$

By (12), (13) and Lemma 2 in Xiong-Feng-Zhang [23] (also, see Pommerenke [20]), we can get

$$\begin{aligned}\left|r_2-\frac{1}{2}r_1^2\right|=&\left|\frac{1}{1-\alpha}\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2g(0)(z_0)^2}{2!}\right)^2-\frac{2}{1-\alpha}\frac{\partial\rho(z)}{\partial z}\frac{D^3g(0)(z_0)^3}{3!}\right.\\&\left.+\frac{2\alpha-1}{2(1-\alpha)^2}\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2g(0)(z_0)^2}{2!}\right)^2\right|\\\leqslant&2-\frac{1}{2}\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2g(0)(z_0)^2}{2!}\right)^2\frac{1}{(1-\alpha)^2}.\end{aligned}\tag{14}$$

Let z_0 be replaced by $\frac{z}{\rho(z)}$ in (14). Then

$$\Big|\Big(2\frac{\partial\rho(z)}{\partial z}\frac{D^2g(0)(z^2)}{2!}\Big)^2-2\frac{\partial\rho(z)}{\partial z}\frac{D^3g(0)(z^3)\rho(z)}{3!}+\frac{2\alpha-1}{2(1-\alpha)}\Big(2\frac{\partial\rho(z)}{\partial z}\frac{D^2g(0)(z^2)}{2!}\Big)^2\Big|$$
$$\leqslant 2(1-\alpha)\rho^4(z)-\frac{1}{2(1-\alpha)}\Big|2\frac{\partial\rho(z)}{\partial z}\frac{D^2g(0)(z^2)}{2!}\Big|^2. \tag{15}$$

On the other hand, since $g(z)=(Df(z))^{-1}f(z)$, we have

$$\begin{aligned}&z+\frac{D^2f(0)(z^2)}{2!}+\frac{D^3f(0)(z^3)}{3!}+\dots\\ =&\Big(I+D^2f(0)(z,.)+\frac{D^3f(0)(z^2,.)}{2!}+\dots\Big)\times\\ &\Big(Dg(0)z+\frac{D^2g(0)(z^2)}{2!}+\frac{D^3g(0)(z^3)}{3!}+\dots\Big).\end{aligned} \tag{16}$$

Comparing with the homogeneous expansion of two sides of the (16), then we have

$$J_g(0)z=z,\ \frac{D^2g(0)(z^2)}{2!}=-\frac{D^2f(0)(z^2)}{2!} \tag{17}$$

and

$$\frac{D^3f(0)(z^3)}{3!}=\frac{D^3g(0)(z^3)}{3!}+\frac{D^3f(0)(z^3)}{2!}-D^2f(0)\Big(z,\frac{D^2f(0)(z^2)}{2!}\Big). \tag{18}$$

In view of (18), and using the conditions (6) and (17), we can give

$$\begin{aligned}&\Big|2\frac{\partial\rho(z)}{\partial z}\frac{D^3f(0)(z^3)\rho(z)}{3!}-\mu\Big(2\frac{\partial\rho(z)}{\partial z}\frac{D^2f(0)(z^2)}{2!}\Big)^2\Big|\\
=&\Big|-\frac{1}{2}2\frac{\partial\rho(z)}{\partial z}\frac{D^3g(0)(z^3)\rho(z)}{3!}+\frac{1}{2}2\frac{\partial\rho(z)}{\partial z}\Big(D^2f(0)\Big(z,\frac{D^2f(0)(z^2)}{2!}\Big)\Big)\rho(z)\\
&-\mu\Big(2\frac{\partial\rho(z)}{\partial z}\frac{D^2f(0)(z^2)}{2!}\Big)^2\Big|\\
=&\frac{1}{2}\Big|-2\frac{\partial\rho(z)}{\partial z}\frac{D^3g(0)(z^3)\rho(z)}{3!}+(2-2\mu)\Big(2\frac{\partial\rho(z)}{\partial z}\frac{D^2f(0)(z^2)}{2!}\Big)^2\Big|\\
=&\frac{1}{2}\Big|-2\frac{\partial\rho(z)}{\partial z}\frac{D^3g(0)(z^3)\rho(z)}{3!}+\Big(2\frac{\partial\rho(z)}{\partial z}\frac{D^2g(0)(z^2)}{2!}\Big)^2\\
&+\frac{2\alpha-1}{2(1-\alpha)}\Big(2\frac{\partial\rho(z)}{\partial z}\frac{D^2g(0)(z^2)}{2!}\Big)^2+\Big(\frac{4\alpha-3}{2(\alpha-1)}-2\mu\Big)\Big(2\frac{\partial\rho(z)}{\partial z}\frac{D^2g(0)(z^2)}{2!}\Big)^2\Big|\\
\leqslant&\frac{1}{2}\Big(2(1-\alpha)\rho^4(z)-\frac{1}{2(1-\alpha)}\Big|2\frac{\partial\rho(z)}{\partial z}\frac{D^2g(0)(z^2)}{2!}\Big|^2\\
&+\Big|\frac{4\alpha-3}{2(\alpha-1)}-2\mu\Big|\Big|2\frac{\partial\rho(z)}{\partial z}\frac{D^2g(0)(z^2)}{2!}\Big|^2\Big).\end{aligned} \tag{19}$$

According to the above equality (19), we consider the following four cases by using Lemma 6 in Xu-Liu [24].

Case 1: If $\mu \leqslant \frac{4\alpha-2}{4(\alpha-1)}$, we have

$$\begin{aligned}
&\Big|2\frac{\partial\rho(z)}{\partial z}\frac{D^3 f(0)(z^3)\rho(z)}{3!}-\mu\Big(2\frac{\partial\rho(z)}{\partial z}\frac{D^2 f(0)(z^2)}{2!}\Big)^2\Big| \\
\leqslant&\frac{1}{2}\Big(2(1-\alpha)\rho^4(z)+\Big(\frac{4\alpha-3}{2(\alpha-1)}-2\mu-\frac{1}{2(1-\alpha)}\Big)\Big|2\frac{\partial\rho(z)}{\partial z}\frac{D^2 g(0)(z^2)}{2!}\Big|^2\Big) \\
\leqslant&\frac{1}{2}\Big(2(1-\alpha)\rho^4(z)+4(1-\alpha)^2\Big(\frac{4\alpha-3}{2(\alpha-1)}-2\mu-\frac{1}{2(1-\alpha)}\Big)\rho^4(z)\Big) \\
=&(1-\alpha)\rho^4(z)[3-4\alpha-4(1-\alpha)\mu].
\end{aligned} \tag{20}$$

Case 2: If $\frac{4\alpha-2}{4(\alpha-1)} \leqslant \mu \leqslant \frac{4\alpha-3}{4(\alpha-1)}$, then we have

$$\begin{aligned}
&\Big|2\frac{\partial\rho(z)}{\partial z}\frac{D^3 f(0)(z^3)\rho(z)}{3!}-\mu\Big(2\frac{\partial\rho(z)}{\partial z}\frac{D^2 f(0)(z^2)}{2!}\Big)^2\Big| \\
\leqslant&\frac{1}{2}\Big(2(1-\alpha)\rho^4(z)+4(1-\alpha)^2\Big(\frac{4\alpha-3}{2(\alpha-1)}-2\mu-\frac{1}{2(1-\alpha)}\Big)\rho^4(z)\Big) \\
=&(1-\alpha)\rho^4(z).
\end{aligned} \tag{21}$$

Case 3: If $\frac{4\alpha-3}{4(\alpha-1)} \leqslant \mu \leqslant 1$, then we have

$$\begin{aligned}
&\Big|2\frac{\partial\rho(z)}{\partial z}\frac{D^3 f(0)(z^3)\rho(z)}{3!}-\mu\Big(2\frac{\partial\rho(z)}{\partial z}\frac{D^2 f(0)(z^2)}{2!}\Big)^2\Big| \\
\leqslant&\frac{1}{2}\Big(2(1-\alpha)\rho^4(z)+4(1-\alpha)^2\Big(2\mu-\frac{4\alpha-3}{2(\alpha-1)}-\frac{1}{2(1-\alpha)}\Big)\rho^4(z)\Big) \\
=&(1-\alpha)\rho^4(z).
\end{aligned} \tag{22}$$

Case 4: If $\mu \geqslant 1$, then we have

$$\begin{aligned}
&\Big|2\frac{\partial\rho(z)}{\partial z}\frac{D^3 f(0)(z^3)\rho(z)}{3!}-\mu\Big(2\frac{\partial\rho(z)}{\partial z}\frac{D^2 f(0)(z^2)}{2!}\Big)^2\Big| \\
\leqslant&\frac{1}{2}\Big(2(1-\alpha)\rho^4(z)+\Big(2\mu-\frac{4\alpha-3}{2(\alpha-1)}-\frac{1}{2(1-\alpha)}\Big)\Big|2\frac{\partial\rho(z)}{\partial z}\frac{D^2 g(0)(z^2)}{2!}\Big|^2\Big) \\
\leqslant&\frac{1}{2}\Big(2(1-\alpha)\rho^4(z)+4(1-\alpha)^2\Big(2\mu-\frac{4\alpha-3}{2(\alpha-1)}-\frac{1}{2(1-\alpha)}\Big)\rho^4(z)\Big) \\
=&(1-\alpha)\rho^4(z)[4\alpha+4(1-\alpha)\mu-3].
\end{aligned} \tag{23}$$

Finally, in order to see that the estimations of Theorem 1 are sharp, we can consider the following mappings:

If $|\frac{4\alpha-3}{2(\alpha-1)}-2\mu|\geqslant\frac{1}{2(1-\alpha)}$, $0\leqslant\alpha<1$, then

$$f(z)=z\exp\int_0^{\frac{z_1}{r}}\left[\frac{1+t}{1+(2\alpha-1)t}-1\right]\frac{1}{t}dt=\begin{cases}z\exp\int_0^{\frac{z_1}{r}}\frac{2-2\alpha}{1+(2\alpha-1)t}dt, & \alpha\neq\frac{1}{2},\\ z\exp\int_0^{\frac{z_1}{r}}dt, & \alpha=\frac{1}{2}\end{cases}$$
$$=\begin{cases}\left(z_1[1+(2\alpha-1)\frac{z_1}{r}]^{\frac{2(1-\alpha)}{2\alpha-1}},\dots,z_n[1+(2\alpha-1)\frac{z_1}{r}]^{\frac{2(1-\alpha)}{2\alpha-1}}\right), & \alpha\neq\frac{1}{2},\\ \left(z_1e^{\frac{z_1}{r}},\dots,z_ne^{\frac{z_1}{r}}\right), & \alpha=\frac{1}{2}.\end{cases} \tag{24}$$

If $|\frac{4\alpha-3}{2(\alpha-1)}-2\mu|\leqslant\frac{1}{2(1-\alpha)}$, $0\leqslant\alpha<1$, then

$$f(z)=z\exp\int_0^{\frac{z_1}{r}}\left[\frac{1+t^2}{1+(2\alpha-1)t^2}-1\right]\frac{1}{t}dt=\begin{cases}z\exp\int_0^{\frac{z_1}{r}}\frac{2(1-\alpha)t}{1+(2\alpha-1)t^2}, & \alpha\neq\frac{1}{2},\\ z\exp\int_0^{\frac{z_1}{r}}t\,dt, & \alpha=\frac{1}{2}\end{cases}$$
$$=\begin{cases}\left(z_1[1+(2\alpha-1)(\frac{z_1}{r})^2]^{\frac{1-\alpha}{2\alpha-1}},\dots,z_n[1+(2\alpha-1)(\frac{z_1}{r})^2]^{\frac{1-\alpha}{2\alpha-1}}\right), & \alpha\neq\frac{1}{2},\\ \left(z_1e^{\frac{1}{2}(\frac{z_1}{r})^2},\dots,z_ne^{\frac{1}{2}(\frac{z_1}{r})^2}\right), & \alpha=\frac{1}{2}.\end{cases} \tag{25}$$

where $r=\sup\{|z_1|:z=(z_1,0,\dots,0)'\in\Omega\}$, $z=(z_1,z_2,...,z_n)\in\Omega$.

We can check that the mappings in (24) and (25) belong to the class $\mathscr{A}\mathscr{S}^*_\alpha(\Omega)$. Indeed, if $\psi(t)=\frac{1+t}{1+(2\alpha-1)t}$ and $|\frac{4\alpha-3}{2(\alpha-1)}-2\mu|\geqslant\frac{1}{2(1-\alpha)}$ in (24), then

$$f(z)=z\exp\int_0^{\frac{z_1}{r}}(\psi(t)-1)\frac{1}{t}dt=z\frac{r\frac{z_1}{r}\exp\int_0^{\frac{z_1}{r}}(\psi(t)-1)\frac{1}{t}dt}{z_1}=z\frac{rF(\frac{z_1}{r})}{z_1},$$

where $F(\frac{z_1}{r})=\frac{z_1}{r}\exp\int_0^{\frac{z_1}{r}}(\psi(t)-1)\frac{1}{t}dt$. By straightforward calculation, we find that $\frac{\hat{z}F'(\hat{z})}{F(\hat{z})}=\psi(\hat{z})\in\psi(\mathbb{U}),\hat{z}\in\mathbb{U}$. Thus, we obtain $f\in\mathscr{A}\mathscr{S}^*_\alpha(\Omega)$ by Theorem 4 in Xu-Liu [24]. Similarly, if $\psi(t)=\frac{1+t^2}{1+(2\alpha-1)t^2}$ and $|\frac{4\alpha-3}{2(\alpha-1)}-2\mu|\leqslant\frac{1}{2(1-\alpha)}$, we also can give $f\in\mathscr{A}\mathscr{S}^*_\alpha(\Omega)$.

It is not difficult to verify that the $f(z)$ defined in (24) and (25) satisfy the condition of Theorem 1. Taking $z=Ru(0<R<1)$, where $u=(u_1,u_2,\dots,u_n)'\in\partial\Omega,u_1=r$, we deduce that the equalities in (20) and (23) hold true when f is defined in (24). Also, the equalities in (21) and (22) hold true when f is defined in (25). This completes the proof of Theorem 1.

The following natural question arises:

OPEN QUESTION 3.1. Whether or not obtain the corresponding result in Theorem 1 without using the condition (6)?

By checking the process of proof in Theorem 1, we find that it is difficult to solve the Question 3.1. However, an interesting thing can be done along this line. Dropping off the condtion (6), we can give an answer to the Question 3.1 for a subclass of almost starlike mapping of order α.

THEOREM 2. *Suppose that* $f:\Omega\to\mathbb{C}, F(z)=zf(z)\in\mathscr{A}\mathscr{S}^*_\alpha(\Omega)$. *Then for* $z\in\Omega\setminus\{0\}$, *we have*

$$\left|A_3-\mu A_2^2\right|\leqslant\begin{cases}(1-\alpha)[3-4\alpha-4(1-\alpha)\mu], & if \quad \mu\leqslant\frac{4\alpha-2}{4(\alpha-1)},\\ 1-\alpha, & if \quad \frac{4\alpha-2}{4(\alpha-1)}\leqslant\mu\leqslant 1,\\ (1-\alpha)[4\alpha+4(1-\alpha)\mu-3], & if \quad \mu\geqslant 1,\end{cases}$$

where

$$A_3=2\frac{\partial\rho(z)}{\partial z}\frac{D^3F(0)(z^3)}{3!\rho^3(z)},\quad A_2=2\frac{\partial\rho(z)}{\partial z}\frac{D^2F(0)(z^2)}{2!\rho^2(z)}.$$

The above estimates are sharp for each real μ.

Proof. Taking similar arguments as those in Theorem 7.1.14 by Graham-Kohr [6], we have

$$(DF(z))^{-1}=\frac{1}{f(z)}\left(I-\frac{\frac{zDf(z)}{f(z)}}{1+\frac{Df(z)z}{f(z)}}\right),\ z\in\Omega\setminus\{0\}. \tag{26}$$

Hence, after some computations with (26), we have

$$\frac{\rho(z)}{2\frac{\partial\rho(z)}{\partial z}(DF(z))^{-1}F(z)}=1+\frac{Df(z)z}{f(z)},\ z\in\Omega\setminus\{0\}. \tag{27}$$

Using (8) and (27), we have

$$\mathscr{P}(\zeta)=\frac{\rho(\zeta z_0)}{2\frac{\partial\rho(\zeta z_0)}{\partial z}(Df(\zeta z_0))^{-1}f(\zeta z_0)}=1+\frac{Df(\zeta z_0)\zeta z_0}{f(\zeta z_0)},\ \zeta\in\mathbb{U}. \tag{28}$$

From (28), it yields

$$\mathscr{P}(\zeta)f(\zeta z_0)=f(\zeta z_0)+Df(\zeta z_0)\zeta z_0. \tag{29}$$

Considering the Taylor series expansions with ξ in (29), we have

$$\begin{aligned}&\left(1+(1-\alpha)r_1\zeta+\left((1-\alpha)\left(r_2-\frac{r_1^2}{2}\right)+\frac{1}{2}(\alpha-1)(2\alpha-1)r_1^2\right)\zeta^2+\dots\right)\cdot\mathfrak{P}\\ =&\mathfrak{P}+\left(Df(0)(z_0)\zeta+D^2f(0)(z_0^2)\zeta^2+\dots\right),\end{aligned} \tag{30}$$

where

$$\mathfrak{P}=1+Df(0)(z_0)\zeta+\frac{D^2f(0)(z_0^2)}{2}\zeta^2+\dots.$$

Comparing the homogeneous expansions of two sides in (30), we have

$$(1-\alpha)r_1=Df(0)(z_0) \tag{31}$$

and

$$(1-\alpha)\left(r_2-\frac{r_1^2}{2}\right)+\frac{1}{2}(\alpha-1)(2\alpha-1)r_1^2=D^2f(0)(z_0^2)-(Df(0)(z_0))^2. \tag{32}$$

Using the fact that $z_0=\frac{z}{\rho(z)}$, (31) and (32) imply that

$$Df(0)(z)=(1-\alpha)r_1\rho(z) \tag{33}$$

and

$$D^2f(0)(z^2)=\left[(1-\alpha)\big(r_2-\frac{r_1^2}{2}\big)+\frac{1}{2}(\alpha-1)(2\alpha-1)r_1^2\right]\rho^2(z)+(1-\alpha)^2r_1^2\rho^2(z). \tag{34}$$

Also, since $F(z)=zf(z)$, it gives that

$$\frac{D^3F(0)(z^3)}{3!}=\frac{D^2f(0)(z^2)}{2!}z,\ \frac{D^2F(0)(z^2)}{2!}=Df(0)(z)z. \tag{35}$$

From Lemma 1 and (35), we can know that

$$2\frac{\partial\rho(z)}{\partial z}\frac{D^3F(0)(z^3)}{3!}=\frac{D^2f(0)(z^2)\rho(z)}{2!} \tag{36}$$

and

$$2\frac{\partial\rho(z)}{\partial z}\frac{D^2F(0)(z^2)}{2!}=Df(0)(z)\rho(z) \tag{37}$$

Following Lemma 2 in Xiong-Feng-Zhang [23], (31),(32), (36) and (37) give us

$$\begin{aligned}
&\Big|2\frac{\partial\rho(z)}{\partial z}\frac{D^3F(0)(z^3)\rho(z)}{3!}-\mu\Big(2\frac{\partial\rho(z)}{\partial z}\frac{D^2F(0)(z^2)}{2!}\Big)^2\Big|\\
=&\Big|\frac{D^2f(0)(z^2)\rho^2(z)}{2!}-\mu\rho^2(z)(Df(0)(z))^2\Big|\\
=&\frac{1}{2}\rho^2(z)\Big|D^2f(0)(z^2)-2\mu(Df(0)(z))^2\Big|\\
=&\frac{1}{2}\rho^4(z)\Big|(1-\alpha)\big(r_2-\frac{r_1^2}{2}\big)+\frac{1}{2}(\alpha-1)(2\alpha-1)r_1^2+(1-\alpha)^2r_1^2-2\mu(1-\alpha)^2r_1^2\Big|\\
=&\frac{1-\alpha}{2}\rho^4(z)\Big|r_2-\frac{r_1^2}{2}-\frac{1}{2}(2\alpha-1)r_1^2+(1-\alpha)r_1^2-2\mu(1-\alpha)r_1^2\Big|\\
\leqslant&\frac{1-\alpha}{2}\rho^4(z)\Big(2-\frac{1}{2}|r_1|^2+\frac{1}{2}|r_1|^2|3-4\alpha-4\mu(1-\alpha)|\Big).
\end{aligned} \tag{38}$$

The rest of the proof is similar to the case in Theorem 1 (see, (19)), and thus, we omit the details. The proof is completed.

If we take $\Omega=\mathbb{B}^n$, then $\rho(z)=\|z\|$. Following Lemma 1, it is easy to get the corollary below by Theorems 1 and 2.

COROLLARY 1. **(i)** *Suppose that* $f \in \mathscr{A}\mathscr{S}_{\alpha}^{*}(\mathbb{B}^n)$ *and*

$$\frac{1}{2}D^2 f(0)\Big(z, \frac{D^2 f(0)(z^2)}{2!}\Big)\overline{z} = \Big(\frac{1}{2\|z\|}D^2 f(0)(z^2)\overline{z}\Big)^2, \ z \in \mathbb{B}^n. \tag{39}$$

Then for $z \in \mathbb{B}^n \backslash \{0\}$ *and* $\mu \in \mathbb{R}$, *we have*

$$\left|A_3 - \mu A_2^2\right| \leqslant \begin{cases} (1-\alpha)[3-4\alpha-4(1-\alpha)\mu], & if \quad \mu \leqslant \frac{4\alpha-2}{4(\alpha-1)}, \\ 1-\alpha, & if \quad \frac{4\alpha-2}{4(\alpha-1)} \leqslant \mu \leqslant 1, \\ (1-\alpha)[4\alpha+4(1-\alpha)\mu-3], & if \quad \mu \geqslant 1, \end{cases}$$

where

$$A_3 = 2\frac{1}{\|z\|^4}\frac{D^3 f(0)(z^3)}{3!}\overline{z}, \quad A_2 = \frac{1}{\|z\|^3}\frac{D^2 f(0)(z^2)}{2!}\overline{z}.$$

The above estimates are sharp for each real μ.

(ii) *Suppose that* $f : \mathbb{B}^n \to \mathbb{C}, F(z) = zf(z) \in \mathscr{A}\mathscr{S}_{\alpha}^{*}(\mathbb{B}^n)$. *Then for* $z \in \mathbb{B}^n \backslash \{0\}$ *and* $\mu \in \mathbb{R}$, *we have*

$$\left|A_3 - \mu A_2^2\right| \leqslant \begin{cases} (1-\alpha)[3-4\alpha-4(1-\alpha)\mu], & if \quad \mu \leqslant \frac{4\alpha-2}{4(\alpha-1)}, \\ 1-\alpha, & if \quad \frac{4\alpha-2}{4(\alpha-1)} \leqslant \mu \leqslant 1, \\ (1-\alpha)[4\alpha+4(1-\alpha)\mu-3], & if \quad \mu \geqslant 1. \end{cases}$$

where

$$A_3 = 2\frac{1}{\|z\|^4}\frac{D^3 F(0)(z^3)}{3!}\overline{z}, \quad A_2 = \frac{1}{\|z\|^3}\frac{D^2 F(0)(z^2)}{2!}\overline{z}.$$

The above estimates are sharp for each real μ.

REMARK 1. If we take $\Omega = \mathbb{U}$ in Theorems 1 and 2, then the results coincide with Theorem B in one dimension.

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