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ON THE REFINEMENT OF QUANTUM HERMITE–HADAMARD INEQUALITIES FOR CONTINUOUS CONVEX FUNCTIONS

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Abstract. The purpose of this paper is to establish some new refinement of quantum Hermite-Hadamard inequalities for continuous convex functions. Several known results are reduced as special cases.

1. Introduction

Quantum calculus (or q-calculus) is centered on the idea of deriving q-analogous results to the usual calculus, without the use of limits. Many of the formulas of q-calculus become the classical mathematical formulas when q tends to 1. The first mathematician who introduced q-calculus was Euler, starting in the eighteenth century. Jackson [15] started the study of q-calculus in a systematic way and presented q-definite integrals. The subject of q-calculus has large applications in various areas of pure and applied sciences. In recent years, many researchers have been increasingly interested in the topic of q-calculus. For recent developments on q-calculus, we refer to [4, 10, 16, 17, 11, 22, 21, 23, 25] and the references therein.

Convex functions are important in many areas of mathematics. There are many results in theory of inequalities established for convex functions and one of the most important and fundamental inequality in analysis is the following Hermite-Hadamard inequality:

THEOREM 1.1. *Let $\phi : [u,v] \to \mathbb{R}$ be a convex function with $u < v$. Then*

$$\phi\left(\frac{u+v}{2}\right) \leqslant \frac{1}{v-u}\int_u^v \phi(z)dz \leqslant \frac{\phi(u)+\phi(v)}{2}. \tag{1}$$

Hermite-Hadamard inequality was first discovered by Hermite [13] in 1883 and rediscovered ten years later by Hadamard [12]. Over the years many researchers have investigated several inequalities related to Hermite-Hadamard's inequality and a variety of refinement of Hermite-Hadamard's inequality see [1, 2, 5, 6, 14, 18, 19, 20] and references therein for more information.

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In 2014, Tariboon and Ntouyas [27] introduced the concept of q-calculus on finite interval of $[u,v]$. The q-derivative and q-integral are defined and some basic properties are given. Also, q-Hermite-Hadamard inequality, as well as various other important integral inequalities are obtained. Alp *et al.* in [3] improved the q-Hermite-Hadamard inequality.

In 2019, Prabseang *et al.* [24] established the q-Hermite-Hadamard inequality for double integral and obtained refinement of Hermite-Hadamard inequality for q-differentiable convex functions.

The main purpose of this paper is to establish refinements of quantum Hermite-Hadamard inequalities for continuous convex functions. The obtained results, in special cases when q tends to 1, led to known results.

Our main theorem are presented in Section 3, while some needed concepts, definitions, and results from q-calculus are recalled in Section 2.

2. Preliminaries

In this section, we recall some known concepts and basic results from q-calculus. Throughout this section, we let $J=[u,v]\subset\mathbb{R}$ be an interval and q be a constant with $0<q<1$.

DEFINITION 2.1. Let $\phi: J\to\mathbb{R}$ be a continuous function and $z\in J$. Then the q-derivative of ϕ on J at z is defined by

$$ {}_uD_q\phi(z)=\frac{\phi(z)-\phi(qz+(1-q)u)}{(1-q)(z-u)},\ \text{for } z\neq u. \tag{1}$$

For $z=u$, we define ${}_uD_q\phi(u)=\lim_{z\to u}{}_uD_q\phi(z)$.

A function ϕ is q-differentiable on J if ${}_uD_q\phi(z)$ exists for all $z\in J$. Moreover, if $u=0$ in (2.1), then ${}_0D_q\phi=D_q\phi$, where D_q is the well-known q-derivative of the function $\phi(z)$, which is defined by

$$D_q\phi(z)=\frac{\phi(z)-\phi(qz)}{(1-q)z}, \tag{2}$$

see [17], for more details.

In addition, we shall define higher-order q-derivatives of functions on J.

DEFINITION 2.2. Let $\phi: J\to\mathbb{R}$ be a continuous function. The second-order q-derivative of ϕ on J, denoted by ${}_uD_q^2\phi$ (provided that ${}_uD_q\phi$ is q-differentiable on J), is defined by

$$ {}_uD_q^2\phi={}_uD_q({}_uD_q\phi). \tag{3}$$

Similarly, provided that ${}_uD_q^{n-1}\phi$ is q-differentiable on J for some integer $n>2$, the n^{th}-order q-derivative of ϕ on J is defined by

$$ {}_uD_q^n\phi={}_uD_q({}_uD_q^{n-1}\phi). \tag{4}$$

EXAMPLE 2.1. Let $\phi : J \to \mathbb{R}$ with $\phi(z) = z^2 + c$, where $c \in \mathbb{R}$ and $0 < q < 1$. Then for $z \neq u$, we have

$$\begin{aligned} {}_uD_q(z^2+c) &= \frac{(z^2+c) - \left[(qz+(1-q)u)^2+c\right]}{(1-q)(z-u)} \\ &= \frac{(1+q)z^2 - 2quz - (1-q)u^2}{(z-u)} \\ &= (1+q)z + (1-q)u. \end{aligned} \tag{5}$$

For $z = u$, we have ${}_uD_q\phi(u) = \lim_{z\to u} {}_uD_q\phi(z) = 2u$.

DEFINITION 2.3. Let $\phi : J \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the q-integral on J is defined by

$$\int_u^z \phi(t)\ {}_ud_qt = (1-q)(z-u)\sum_{n=0}^{\infty} q^n \phi(q^n z + (1-q^n)u) \tag{6}$$

for $x \in J$.

If $u = 0$ in (6), then we have the classical q-integral of the function $\phi(z)$, which is defined by

$$\int_0^z \phi(t)\ {}_0d_qt = (1-q)z\sum_{n=0}^{\infty} q^n \phi(q^n z) \tag{7}$$

for $z \in [0,\infty)$; see [17] for more details.

EXAMPLE 2.2. Let $\phi : J \to \mathbb{R}$ with $\phi(z) = cz$, where $c \in \mathbb{R}$ and $0 < q < 1$. Then we have

$$\begin{aligned} \int_u^v \phi(z)\ {}_ud_qz &= \int_u^v cz\ {}_ud_qz \\ &= c(1-q)(v-u)\sum_{n=0}^{\infty} q^n(q^n v + (1-q^n)u) \\ &= \frac{c(v-u)(v+qu)}{1+q}. \end{aligned} \tag{8}$$

Note that if $q \to 1$, then we have the classical integration

$$\int_u^v \phi(z)dz = \int_u^v czdz = \frac{c(v^2-u^2)}{2}. \tag{9}$$

THEOREM 2.1. Let $\phi : J \to \mathbb{R}$ be a continuous function. Then we have the following:

(i) ${}_uD_q \int_u^z \phi(t)\ {}_ud_qt = \phi(z)$;

(ii) $\int_c^z {}_uD_q\phi(t)\ {}_ud_qt = \phi(z) - \phi(c)$ for $c \in (u,z)$.

THEOREM 2.2. Let $\phi, g : J \to \mathbb{R}$ be continuous functions and $\alpha \in \mathbb{R}$. Then we have the following:

(i) $\int_u^z [\phi(t)+g(t)]\ {}_ud_qt = \int_u^z \phi(t)\,{}_ud_qt + \int_u^z g(t)\,{}_ud_qt$;

(ii) $\int_u^z (\alpha\phi)(t)\ {}_ud_qt = \alpha \int_u^z \phi(t)\ {}_ud_qt$;

(iii) $\int_c^z \phi(t)\,{}_uD_qg(t)\ {}_ud_qt = (\phi g)|_c^z - \int_c^z g(qt+(1-q)u)\,{}_uD_q\phi(t)\ {}_ud_qt$ for $c \in (u,z)$.

For the proofs of Theorem 2.1 and Theorem 2.2, see [26].

3. Main results

In this section, we present refinement of q-Hermite-Hadamard inequalities on the interval $J = [u,v]$.

THEOREM 3.1. *Let $\phi : J \to \mathbb{R}$ be a continuous convex function on J and $0 < q < 1$. Then the following inequalities:*

$$\begin{aligned}\phi\left(\frac{qu+v}{1+q}\right) &\leqslant \frac{1}{(v-u)^2}\int_u^v\int_u^v \phi\left(\frac{z+y}{2}\right)\ {}_ud_qz\,{}_ud_qy \\ &\leqslant \frac{1}{(v-u)^2}\int_u^v\int_u^v \frac{1}{2}\left[\phi\left(\frac{\alpha z+\beta y}{\alpha+\beta}\right)+\phi\left(\frac{\beta z+\alpha y}{\alpha+\beta}\right)\right]\ {}_ud_qz\,{}_ud_qy \\ &\leqslant \frac{1}{v-u}\int_u^v \phi(z)\ {}_ud_qz \end{aligned} \tag{1}$$

are valid for all $\alpha, \beta \geqslant 0$ with $\alpha+\beta > 0$.

Proof. Since f is convex on J, it follows that

$$\phi\left(\frac{z+y}{2}\right) \leqslant \frac{1}{2}\left[\phi\left(\frac{\alpha z+\beta y}{\alpha+\beta}\right)+\phi\left(\frac{\beta z+\alpha y}{\alpha+\beta}\right)\right] \leqslant \frac{\phi(z)+\phi(y)}{2} \tag{2}$$

for all $z, y \in J$ and $\alpha, \beta \geqslant 0$ with $\alpha+\beta > 0$. Taking double q-integration on both sides of (2) on $J \times J$, we obtain the second part of (1).

On the other hand, by using the Jensen's inequality, we have

$$\phi\left(\frac{1}{(v-u)^2}\int_u^v\int_u^v \left(\frac{z+y}{2}\right)\ {}_ud_qz\,{}_ud_qy\right) \leqslant \frac{1}{(v-u)^2}\int_u^v\int_u^v \phi\left(\frac{z+y}{2}\right)\ {}_ud_qz\,{}_ud_qy.$$

Since

$$\frac{1}{(v-u)^2}\int_u^v\int_u^v \left(\frac{z+y}{2}\right)\ {}_ud_qz\,{}_ud_qy = \frac{qu+v}{1+q},$$

this yields the first part of (1).

REMARK 3.1. If $q \to 1$, then (1) reduces to

$$\begin{aligned}\phi\left(\frac{u+v}{2}\right) &\leqslant \frac{1}{(v-u)^2}\int_u^v\int_u^v \phi\left(\frac{z+y}{2}\right)dzdy\\ &\leqslant \frac{1}{(v-u)^2}\int_u^v\int_u^v \frac{1}{2}\left[\phi\left(\frac{\alpha z+\beta y}{\alpha+\beta}\right)+\phi\left(\frac{\beta z+\alpha y}{\alpha+\beta}\right)\right]dzdy\\ &\leqslant \frac{1}{v-u}\int_u^v \phi(z)dz,\end{aligned}$$

which readily appeared in [8].

THEOREM 3.2. *Let $\phi : J \to \mathbb{R}$ be a continuous convex function on J and $0 < q < 1$. Then the following inequalities*

$$\begin{aligned}(i)\quad &\frac{1}{(v-u)^n}\int_u^v\cdots\int_u^v \phi\left(\frac{z_1+z_2+\cdots+z_n}{n}\right)\, {}_ud_qz_1\, {}_ud_qz_2\cdots\, {}_ud_qz_n\\ &\leqslant \frac{1}{(v-u)^{n-1}}\int_u^v\cdots\int_u^v \phi\left(\frac{z_1+z_2+\cdots+z_{n-1}}{n-1}\right)\, {}_ud_qz_1\, {}_ud_qz_2\cdots\, {}_ud_qz_{n-1},\end{aligned} \tag{3}$$

and

$$\begin{aligned}(ii)\quad \phi\left(\frac{qu+v}{1+q}\right) &\leqslant \frac{1}{(v-u)^n}\int_u^v\cdots\int_u^v \phi\left(\frac{z_1+z_2+\cdots+z_n}{n}\right)\, {}_ud_qz_1\, {}_ud_qz_2\cdots\, {}_ud_qz_n\\ &\leqslant \frac{1}{v-u}\int_u^v \phi(z)\, {}_ud_qz,\end{aligned} \tag{4}$$

are valid for all $n \in \mathbb{N}$ with $n \geqslant 3$.

Proof. Define

$$y_1 = \frac{z_1+\cdots+z_{n-1}}{n-1},\ y_2 = \frac{z_2+\cdots+z_n}{n-1},\ \ldots,\ y_n = \frac{z_n+\cdots+z_{n-2}}{n-1}.$$

Note that

$$\frac{y_1+y_2+\cdots+y_n}{n} = \frac{z_1+z_2+\cdots+z_n}{n},$$

and thus, by Jensen's inequality, we have

$$\phi\left(\frac{y_1+y_2+\cdots+y_n}{n}\right) \leqslant \frac{1}{n}\Big[\phi(y_1)+\cdots+\phi(y_n)\Big],$$

that is

$$\begin{aligned}\phi\left(\frac{z_1+z_2+\cdots+z_n}{n}\right) \leqslant \frac{1}{n}\Bigg[&\phi\left(\frac{z_1+\cdots+z_{n-1}}{n-1}\right)+\phi\left(\frac{z_2+\cdots+z_n}{n-1}\right)+\cdots\\ &+\phi\left(\frac{z_n+\cdots+z_{n-2}}{n-1}\right)\Bigg].\end{aligned}$$

Taking q-integration on both sides of the above inequality on J^n, we obtain

$$\begin{aligned}&\int_u^v \cdots \int_u^v \phi\left(\frac{z_1+z_2+\cdots+z_n}{n}\right) {}_ud_qz_1\, {}_ud_qz_2 \cdots {}_ud_qz_n\\ \leqslant&\frac{1}{n}\left[\int_u^v \cdots \int_u^v \phi\left(\frac{z_1+\cdots+z_{n-1}}{n-1}\right) {}_ud_qz_1 \cdots {}_ud_qz_n + \cdots\right.\\ &\left.+\int_u^v \cdots \int_u^v \phi\left(\frac{z_n+\cdots+z_{n-2}}{n-1}\right) {}_ud_qz_1 \cdots {}_ud_qz_n\right].\end{aligned}$$

Since

$$\begin{aligned}&\int_u^v \cdots \int_u^v \phi\left(\frac{z_1+\cdots+z_{n-1}}{n-1}\right) {}_ud_qz_1 \cdots {}_ud_qz_n\\ =&\int_u^v \cdots \int_u^v \phi\left(\frac{z_2+\cdots+z_n}{n-1}\right) {}_ud_qz_1 \cdots {}_ud_qz_n\\ &\vdots\\ =&\int_u^v \cdots \int_u^v \phi\left(\frac{z_n+\cdots+z_{n-2}}{n-1}\right) {}_ud_qz_1 \cdots {}_ud_qz_n\\ =&(v-u)\int_u^v \cdots \int_u^v \phi\left(\frac{z_1+\cdots+z_{n-1}}{n-1}\right) {}_ud_qz_1 \cdots {}_ud_qz_{n-1},\end{aligned}$$

we get

$$\begin{aligned}&\frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \phi\left(\frac{z_1+z_2+\cdots+z_n}{n}\right) {}_ud_qz_1\, {}_ud_qz_2 \cdots {}_ud_qz_n\\ \leqslant&\frac{1}{(v-u)^{n-1}}\int_u^v \cdots \int_u^v \phi\left(\frac{z_1+z_2+\cdots+z_{n-1}}{n-1}\right) {}_ud_qz_1\, {}_ud_qz_2 \cdots {}_ud_qz_{n-1}\end{aligned}$$

which proves (3).

On the other hand, by the Jensen's inequality, we have

$$\begin{aligned}&\phi\left(\frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \left(\frac{z_1+z_2+\cdots+z_n}{n}\right) {}_ud_qz_1\, {}_ud_qz_2 \cdots {}_ud_qz_n\right)\\ \leqslant&\frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \phi\left(\frac{z_1+z_2+\cdots+z_n}{n}\right) {}_ud_qz_1\, {}_ud_qz_2 \cdots {}_ud_qz_n.\end{aligned}$$

Since

$$\frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \left(\frac{z_1+z_2+\cdots+z_n}{n}\right) {}_ud_qz_1\, {}_ud_qz_2 \cdots {}_ud_qz_n = \frac{qu+v}{1+q},$$

the first inequality in (4) is proved. The second inequality follows from (i). The proof is completed.

REMARK 3.2. If $q \to 1$, then (4) reduces to

$$\phi\left(\frac{u+v}{2}\right) \leqslant \frac{1}{(v-u)^n} \int_u^v \cdots \int_u^v \phi\left(\frac{z_1+z_2+\cdots+z_n}{n}\right) dz_1 dz_2 \cdots dz_n \leqslant \frac{1}{v-u} \int_u^v \phi(z) dz,$$

which readily appeared in [8].

COROLLARY 3.1. *Let $\phi : J \to \mathbb{R}$ be a continuous convex function on J and $0 < q < 1$. Then we have*

$$\phi\left(\frac{qu+v}{1+q}\right) \leqslant \frac{1}{(v-u)^2} \int_u^v \int_u^v \phi\left(\frac{z_1+z_2}{2}\right) {}_u d_q z_1 \, {}_u d_q z_2 \leqslant \frac{1}{v-u} \int_u^v \phi(z) \, {}_u d_q z. \quad (5)$$

REMARK 3.3. If $q \to 1$, then (5) reduces to

$$\phi\left(\frac{u+v}{2}\right) \leqslant \frac{1}{(v-u)^2} \int_u^v \int_u^v \phi\left(\frac{z_1+z_2}{2}\right) dz_1 dz_2 \leqslant \frac{1}{v-u} \int_u^v \phi(z) dz,$$

which readily appeared in [7].

THEOREM 3.3. *Let $\phi : J \to \mathbb{R}$ be a continuous convex function and $0 < q < 1$. Then the following inequalities*

$$\begin{aligned} \phi\left(\frac{qu+v}{1+q}\right) &\leqslant \frac{1}{(v-u)^n} \int_u^v \cdots \int_u^v \phi\left(\frac{t_1 z_1 + t_2 z_2 + \cdots + t_n z_n}{T_n}\right) {}_u d_q z_1 \, {}_u d_q z_2 \cdots {}_u d_q z_n \\ &\leqslant \frac{1}{v-u} \int_u^v \phi(z) {}_u d_q z \end{aligned} \quad (6)$$

are valid for all $t_i \geqslant 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} t_i = T_n > 0$ and $n \in \mathbb{N}$.

Proof. By the Jensen's inequality, we have

$$\phi\left(\frac{t_1 z_1 + t_2 z_2 + \cdots + t_n z_n}{T_n}\right) \leqslant \frac{1}{T_n} [t_1 \phi(z_1) + t_2 \phi(z_2) + \cdots + t_n \phi(z_n)]$$

for all $z_i \in J$ and $t_i \geqslant 0$ where $i = 1, 2, \ldots, n$. Taking q-integration on both sides of the above inequality on J^n, we obtain

$$\begin{aligned} &\int_u^v \cdots \int_u^v \phi\left(\frac{t_1 z_1 + t_2 z_2 + \cdots + t_n z_n}{T_n}\right) {}_u d_q z_1 \, {}_u d_q z_2 \cdots {}_u d_q z_n \\ \leqslant &\frac{1}{T_n} \int_u^v \cdots \int_u^v [t_1 \phi(z_1) + t_2 \phi(z_2) + \cdots + t_n \phi(z_n)] \, {}_u d_q z_1 \, {}_u d_q z_2 \cdots {}_u d_q z_n \\ = &(v-u)^{n-1} \int_u^v \phi(z) \, {}_u d_q z, \end{aligned}$$

which yields the second part of (6).

On the other hand, by the Jensen's inequality and

$$\frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \left(\frac{t_1z_1+t_2z_2+\cdots+t_nz_n}{T_n}\right) {}_ud_qz_1\ {}_ud_qz_2\cdots\ {}_ud_qz_n = \frac{qu+v}{1+q},$$

we have

$$\begin{aligned}\phi\left(\frac{qu+v}{1+q}\right) &= \phi\left(\frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \left(\frac{t_1z_1+t_2z_2+\cdots+t_nz_n}{T_n}\right) {}_ud_qz_1\ {}_ud_qz_2\cdots\ {}_ud_qz_n\right)\\ &\leqslant \frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \phi\left(\frac{t_1z_1+t_2z_2+\cdots+t_nz_n}{T_n}\right) {}_ud_qz_1\ {}_ud_qz_2\cdots\ {}_ud_qz_n.\end{aligned}$$

This completes the proof.

REMARK 3.4. If $q\to 1$, then (6) reduces to

$$\begin{aligned}\phi\left(\frac{u+v}{2}\right) &\leqslant \frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \phi\left(\frac{t_1z_1+t_2z_2+\cdots+t_nz_n}{T_n}\right) dz_1dz_2\cdots dz_n\\ &\leqslant \frac{1}{v-u}\int_u^v \phi(z)dz,\end{aligned}$$

which readily appeared in [8].

COROLLARY 3.2. *Let $\phi: J\to\mathbb{R}$ be a continuous convex function on J and $0<q<1$. Then we have*

$$\begin{aligned}\phi\left(\frac{qu+v}{1+q}\right) &\leqslant \frac{1}{(v-u)^2}\int_u^v\int_u^v \phi\left(t_1z_1+t_2z_2\right) {}_ud_qz_1\ {}_ud_qz_2\\ &\leqslant \frac{1}{v-u}\int_u^v \phi(z)\ {}_ud_qz\\ &\leqslant \frac{q\phi(u)+\phi(v)}{1+q}.\end{aligned} \tag{7}$$

REMARK 3.5. If $q\to 1$, then (7) reduces to

$$\begin{aligned}\phi\left(\frac{u+v}{2}\right) &\leqslant \frac{1}{(v-u)^2}\int_u^v\int_u^v \phi\left(t_1z_1+t_2z_2\right)dz_1dz_2\\ &\leqslant \frac{1}{v-u}\int_u^v \phi(z)dz\\ &\leqslant \frac{\phi(u)+\phi(v)}{2},\end{aligned}$$

which readily appeared in [7].

THEOREM 3.4. *Let $\phi: J\to\mathbb{R}$ be a continuous convex function and $0<q<1$. Then the following inequalities*

$$\phi\left(\frac{qu+v}{1+q}\right) \leqslant \frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \phi\left(\frac{z_1+z_2+\cdots+z_n}{n}\right) {}_ud_qz_1\ {}_ud_qz_2\cdots\ {}_ud_qz_n$$

$$\leqslant \frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \phi\left(\frac{t_1z_1+t_2z_2+\cdots+t_nz_n}{T_n}\right) {}_ud_qz_1\ {}_ud_qz_2\cdots\ {}_ud_qz_n$$
$$\leqslant \frac{1}{v-u}\int_u^v \phi(z)_ud_qz, \tag{8}$$

are valid for all $t_i \geqslant 0\ \ (i=1,2,\ldots,n)$ *with* $\sum_{i=1}^{n} t_i = T_n > 0$ *and* $n \in \mathbb{N}$.

Proof. Let us consider the elements

$$y_1 = \frac{t_1z_1+t_2z_2+\cdots+t_nz_n}{T_n},\ y_2 = \frac{t_nz_1+t_1z_2+\cdots+t_{n-1}z_n}{T_n},\cdots,$$
$$y_n = \frac{t_2z_1+t_3z_2+\cdots+t_1z_n}{T_n}.$$

A simple calculation shows that

$$\frac{y_1+y_2+\cdots+y_{n-1}+y_n}{n} = \frac{z_1+z_2+\cdots+z_{n-1}+z_n}{n}.$$

By using the Jensen's inequality, we can write

$$\phi\left(\frac{z_1+z_2+\cdots+z_n}{n}\right) \leqslant \frac{1}{n}\left[\phi\left(\frac{t_1z_1+t_2z_2+\cdots+t_nz_n}{T_n}\right)+\cdots\right.$$
$$\left. +\phi\left(\frac{t_2z_1+t_3z_2+\cdots+t_1z_n}{T_n}\right)\right].$$

Taking q-integration on both sides of the above inequality on J^n, we obtain

$$\int_u^v \cdots \int_u^v \phi\left(\frac{z_1+z_2+\cdots+z_n}{n}\right) {}_ud_qz_1\ {}_ud_qz_2\cdots\ {}_ud_qz_n$$
$$\leqslant \frac{1}{n}\left[\int_u^v \cdots \int_u^v \phi\left(\frac{t_1z_1+\cdots+t_nz_n}{T_n}\right) {}_ud_qz_1\cdots\ {}_ud_qz_n+\cdots\right.$$
$$\left. +\int_u^v \cdots \int_u^v \phi\left(\frac{t_2z_1+t_3z_2+\cdots+t_1z_n}{T_n}\right) {}_ud_qz_1\cdots\ {}_ud_qz_n\right].$$

Since

$$\int_u^v \cdots \int_u^v \phi\left(\frac{t_1z_1+t_2z_2+\cdots+t_nz_n}{T_n}\right) {}_ud_qz_1\cdots\ {}_ud_qz_n$$
$$=\int_u^v \cdots \int_u^v \phi\left(\frac{t_1z_1+t_2z_2+\cdots+t_nz_n}{T_n}\right) {}_ud_qz_1\cdots\ {}_ud_qz_n$$
$$\vdots$$
$$=\int_u^v \cdots \int_u^v \phi\left(\frac{t_2z_1+t_3z_2+\cdots+t_1z_n}{T_n}\right) {}_ud_qz_1\cdots\ {}_ud_qz_n,$$

we get

$$\frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \phi\left(\frac{z_1+z_2+\cdots+z_n}{n}\right) {}_u d_q z_1 \, {}_u d_q z_2 \cdots {}_u d_q z_n$$
$$\leqslant \frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \phi\left(\frac{t_1 z_1+t_2 z_2+\cdots+t_n z_n}{T_n}\right) {}_u d_q z_1 \, {}_u d_q z_2 \cdots {}_u d_q z_n.$$

Using Theorems 3.3 and 3.2, we can get the desired result.

REMARK 3.6. If $q \to 1$, then (8) reduces to

$$\phi\left(\frac{u+v}{2}\right) \leqslant \frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \phi\left(\frac{z_1+z_2+\cdots+z_n}{n}\right) dz_1 dz_2 \cdots dz_n$$
$$\leqslant \frac{1}{(v-u)^n}\int_u^v \cdots \int_u^v \phi\left(\frac{t_1 z_1+t_2 z_2+\cdots+t_n z_n}{T_n}\right) dz_1 dz_2 \cdots dz_n$$
$$\leqslant \frac{1}{v-u}\int_u^v \phi(z) dz,$$

which readily appeared in [9].

4. Conclusion

In this paper, we have obtained some new results on refinements of quantum Hermite–Hadamard inequalities for continuous convex functions. Our results can be reduced to the classical inequality formulas as special cases when $q \to 1$. It is expected that this paper may stimulate further research in this field.

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