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# BOUNDS FOR THE PERIMETER OF AN ELLIPSE IN TERMS OF POWER MEANS

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*Abstract.* In the article, we provide several precise bounds for the perimeter of an ellipse in terms of the power means, and present new bounds for the complete elliptic integral of the second kind. The given results are the improvements of some earlier results.

## 1. Introduction

Let $a,b,c\in\mathbb{R}$ with $c\neq 0,-1,-2,\cdots$ and $x\in(-1,1)$. Then the Gaussian hypergeometric function $F(a,b;c;x)$ [1-9] is defined by

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}, \tag{1.1}$$

where $(a,0)=1$ for $a\neq 0$, $(a,n)=a(a+1)(a+2)\cdots(a+n-1)=\Gamma(a+n)/\Gamma(a)$ is the shifted factorial function and $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$ $(x>0)$ is the gamma function [10-17].

It is well known that the Gaussian hypergeometric function $F(a,b;c;x)$ has wide applications in mathematics, physics and many other natural sciences. In particular, many special and elementary functions are the particular or limiting cases of Gaussian hypergeometric function, for example, the complete elliptic integrals $\mathscr{K}(r)$ [18-21] and $\mathscr{E}(r)$ $(0<r<1)$ [22-25] of the first and second kinds

$$\mathscr{K}(r)=\int_0^{\pi/2}\frac{dt}{\sqrt{1-r^2\sin^2(t)}},\quad \mathscr{K}(0^+)=\frac{\pi}{2},\quad \mathscr{K}(1^-)=\infty$$

and

$$\mathscr{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2(t)}dt,\quad \mathscr{E}(0^+)=\frac{\pi}{2},\quad \mathscr{E}(1^-)=1$$

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can be expressed by the Gaussian hypergeometric function $F(a,b;c;x)$. Indeed,

$$\mathscr{K}(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n\right)^2}{(n!)^2}r^{2n} \tag{1.2}$$

and

$$\mathscr{E}(r)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2},n\right)\left(\frac{1}{2}\right)_n}{(n!)^2}r^{2n}. \tag{1.3}$$

The perimeter $L(x,y)$ of the ellipse with semi-axes $x,y$ and eccentricity $e=\sqrt{1-(y/x)^2}$ give the formula

$$\begin{aligned} L(x,y) &= 4\int_0^{\pi/2}\left(x^2\cos^2(t)+y^2\sin^2(t)\right)^{1/2}dt \qquad (1.4)\\ &= 4x\int_0^{\pi/2}\sqrt{1-e^2\sin^2(t)}dt=4x\mathscr{E}(e)=2\pi xF\left(-\frac{1}{2},\frac{1}{2};1;e^2\right). \end{aligned}$$

In the past century, the bounds, formulae and methods for estimating and computing the perimeter of an ellipse have attracted the attention of many researchers. Vuorinen [26] conjectured that

$$L(x,y)>2\pi M_{3/2}(x,y) \tag{1.5}$$

for all $x,y>0$ with $x\neq y$, where

$$M_\lambda(x,y)=\left(\frac{x^\lambda+y^\lambda}{2}\right)^{1/\lambda}\quad(\lambda\neq 0),\quad M_0(x,y)=\sqrt{xy} \tag{1.6}$$

is the $\lambda$ th power mean [27-30] of $x$ and $y$. Inequality (1.5) was proved by Barnard, Pearce and Richards in [31].

In [32], Alzer and Qiu proved that the inequality

$$L(x,y)<2\pi M_\lambda(x,y)$$

holds for all $x,y>0$ with $x\neq y$ if and only if $\lambda\leqslant\log 2/(\log\pi-\log 2)=1.5349\cdots$.

Recently, the authors in [33, Theorem 2.3] and [34, Theorem 3.1] proved that the double inequality

$$\frac{\pi\left[23M_1(x,y)-5M_{-1}(x,y)-2M_2(x,y)\right]}{8}<L(x,y)<\frac{\pi\left[21M_1(x,y)-2M_0(x,y)-3M_{-1}(x,y)\right]}{8} \tag{1.7}$$

takes place for all $x,y>0$ with $x\neq y$.

It is the aim of the article to improve the the upper and lower bounds for the perimeter $L(x,y)$ of an ellipse given in (1.7).

## 2. Lemmas

In order to prove our main results, we need several lemmas which we present in this section. In what follows, we denote $\sqrt{1-r^2}$ by $r'$.

LEMMA 2.1. *(See [35, Theorem 1.25]) Let* $\sigma,\tau\in\mathbb{R}$ *with* $\sigma<\tau$, $\Phi,\Psi:[\sigma,\tau]\to\mathbb{R}$ *be two continuous functions on the interval* $[\sigma,\tau]$ *and differentiable on* $(\sigma,\tau)$ *such that* $\Psi'(u)\neq 0$ *for all* $u\in(\sigma,\tau)$. *Then both the functions* $[\Phi(u)-\Phi(\sigma)]/[\Psi(u)-\Psi(\sigma)]$ *and* $[\Phi(u)-\Phi(\tau)]/[\Psi(u)-\Psi(\tau)]$ *are (strictly) increasing (decreasing) on* $(\sigma,\tau)$ *if the function* $\Phi'(u)/\Psi'(u)$ *is (strictly) increasing (decreasing) on* $(\sigma,\tau)$.

LEMMA 2.2. *The following statements are true:*

*(1) The function* $r\to r'^{\mu}\mathscr{K}(r)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,\pi/2)$ *if* $\mu\geqslant 1/2$.

*(2) The function* $r\to\left[\mathscr{E}(r)-r'^2\mathscr{K}(r)\right]/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4,1)$.

*(3) The function* $r\to\left[\left(1+r'^2\right)\mathscr{K}(r)-2\mathscr{E}(r)\right]/r^4$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/16,\infty)$.

*(4) The function*

$$\varphi(r)=\frac{4\left[(\mathscr{K}(r)-\mathscr{E}(r))-\left(\mathscr{E}(r)-r'^2\mathscr{K}(r)\right)\right]-r^2[\mathscr{K}(r)-\mathscr{E}(r)]}{r^6} \tag{2.1}$$

*is strictly increasing from* $(0,1)$ *onto* $(3\pi/32,\infty)$.

*Proof.* Parts (1)-(3) can be found in the literature [35, Theorem 3.21(7) and Exercises 3.43(26) and (29)].

Next, we prove part (4). It follows from (1.1)-(1.3) and (2.1) that

$$\begin{aligned}
\frac{2}{\pi}r^6\varphi(r)=&8\sum_{n=0}^{\infty}\frac{\left(\frac12,n\right)^2}{(n!)^2}r^{2n}-8\sum_{n=0}^{\infty}\frac{\left(-\frac12,n\right)\left(\frac12,n\right)}{(n!)^2}r^{2n}\\
&-5\sum_{n=0}^{\infty}\frac{\left(\frac12,n\right)^2}{(n!)^2}r^{2(n+1)}+\sum_{n=0}^{\infty}\frac{\left(-\frac12,n\right)\left(\frac12,n\right)}{(n!)^2}r^{2(n+1)}\\
=&8\sum_{n=0}^{\infty}\frac{\left(\frac12,n+1\right)\left(\frac12,n\right)}{(n+1)!n!}r^{2(n+1)}-5\sum_{n=0}^{\infty}\frac{\left(\frac12,n\right)^2}{(n!)^2}r^{2(n+1)}+\sum_{n=0}^{\infty}\frac{\left(-\frac12,n\right)\left(\frac12,n\right)}{(n!)^2}r^{2(n+1)}\\
=&8\sum_{n=0}^{\infty}\frac{\left(\frac12,n+2\right)\left(\frac12,n+1\right)}{(n+2)!(n+1)!}r^{2(n+2)}-5\sum_{n=0}^{\infty}\frac{\left(\frac12,n+1\right)^2}{[(n+1)!]^2}r^{2(n+2)}\\
&+\sum_{n=0}^{\infty}\frac{\left(-\frac12,n+1\right)\left(\frac12,n+1\right)}{[(n+1)!]^2}r^{2(n+2)}
\end{aligned}$$

$$
\begin{aligned}
=&8\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n+2\right)\left(\frac{1}{2},n+1\right)}{(n+2)!(n+1)!}r^{2(n+2)}-\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n+1\right)\left(\frac{1}{2},n\right)(5n+3)}{[(n+1)!]^2}r^{2(n+2)}\\
=&8\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n+3\right)\left(\frac{1}{2},n+2\right)}{(n+3)!(n+2)!}r^{2(n+3)}-\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n+2\right)\left(\frac{1}{2},n+1\right)(5n+8)}{[(n+2)!]^2}r^{2(n+3)}\\
=&3\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n+1\right)\left(\frac{1}{2},n+2\right)}{(n+3)!(n)!}r^{2(n+3)},
\end{aligned}
$$

which leads to

$$
\varphi(r)=\frac{3\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n+1\right)\left(\frac{1}{2},n+2\right)}{(n+3)!(n)!}r^{2n}. \tag{2.2}
$$

Therefore,

$$
\begin{aligned}
\varphi\left(0^{+}\right)&=\frac{3\pi}{2}\frac{\left(\frac{1}{2},1\right)\left(\frac{1}{2},2\right)}{3!}=\frac{3\pi}{32},\\
\varphi(1^{-})&=\lim_{r\to 1^{-}}\frac{\left(8-5r^2\right)\mathscr{K}(r)-\left(8-r^2\right)\mathscr{E}(r)}{r^6}=\infty
\end{aligned}
$$

and $\varphi(r)$ is strictly increasing on $(0,1)$ follows from (2.1) and (2.2). □

LEMMA 2.3. *Let* $k\in\mathbb{N}$. *Then*

$$
\frac{5\sqrt{\pi}(k^2+19k+20)}{32(2k+1)(k+3)}-\frac{\sqrt{\pi}(k+1)(k+2)}{24(2k+1)2^{k+1}}>\frac{1}{\sqrt{k+1/4}}
$$

*for* $k\geqslant 8$.

*Proof.* Let

$$
\begin{aligned}
\psi(k)=&\big[900\pi k^5+(34425\pi-147456)k^4+(369450\pi-1032192)k^3\\
&+(774225\pi-2248704)k^2+(531000\pi-1548288)k+90000\pi-331776\big]2^{2(k+1)}\\
&+64\pi(k+1)(k+2)(k+3)\left(k+\frac{1}{4}\right)\left[k^3+6k^2+11k+6-15(k^2+19k+20)2^k\right].
\end{aligned} \tag{2.3}
$$

Then elaborated computations lead to

$$
\begin{aligned}
&\left[\frac{5\sqrt{\pi}(k^2+19k+20)}{32(2k+1)(k+3)}-\frac{\sqrt{\pi}(k+1)(k+2)}{24(2k+1)2^{k+1}}\right]^2-\left(\frac{1}{\sqrt{k+1/4}}\right)^2\\
=&\frac{\psi(k)}{9216(k+3)^2(2k+1)^2(4k+1)2^{2(k+1)}}.
\end{aligned} \tag{2.4}
$$

Note that

$$
9\pi=28.2743\cdots>28,\qquad \frac{34425\pi-147456}{100}=-393.0667\cdots>-395, \tag{2.5}
$$

$$\frac{369450\pi-1032192}{100}=1284.6940>1280, \tag{2.6}$$

$$\frac{774225\pi-2248704}{100}=1835.9557>1835, \tag{2.7}$$

$$\frac{531000\pi-1548288}{100}=1198.9769\cdots>1195, \tag{2.8}$$

$$\frac{90000\pi-331776}{100}=-490.3266\cdots>-495, \quad \frac{12\pi}{5}=7.5398\cdots<8, \tag{2.9}$$

$$\begin{aligned}2^x &= 256e^{(x-8)\log 2} \\ &\geqslant 256\left[1+\frac{\log 2}{1!}(x-8)+\frac{\log^2 2}{2!}(x-8)^2\right] \\ &\geqslant 256+177(x-8)+60(x-8)^2\end{aligned} \tag{2.10}$$

for $x \geqslant 8$.

We clearly see that

$$28k^5-395k^4+1280k^3+1835k^2+1195k-495>0 \tag{2.11}$$

for $k \geqslant 8$. Indeed, it is not difficult to verify that (2.11) is valid for $k=8$ and $k=9$, and

$$28k^5-395k^4+1280k^3>0$$

for $k \geqslant 10$.

It follows (2.3) and (2.5)-(2.11) that

$$\begin{aligned}\psi(k) >& \big[900\pi k^5+(34425\pi-147456)k^4+(369450\pi-1032192)k^3 \\ &+(774225\pi-2248704)k^2+(531000\pi-1548288)k+90000\pi-331776\big]2^{2(k+1)} \\ &-960\pi(k+1)(k+2)(k+3)\left(k+\frac{1}{4}\right)(k^2+19k+20)2^k \\ >& \big[900\pi k^5+(34425\pi-147456)k^4+(369450\pi-1032192)k^3 \\ &+(774225\pi-2248704)k^2+(531000\pi-1548288)k+90000\pi-331776\big]2^{2(k+1)} \\ &-960\pi(k+1)^2(k+2)(k+3)(k^2+19k+20)2^k\end{aligned}$$

and

$$\begin{aligned}&\frac{\psi(k)}{100\times 2^{k+2}} \\ >& \left[9\pi k^5+\frac{34425\pi-147456}{100}k^4+\frac{369450\pi-1032192}{100}k^3\right. \\ &\left.+\frac{774225\pi-2248704}{100}k^2+\frac{531000\pi-1548288}{100}k+\frac{90000\pi-331776}{100}\right]2^k \\ &-\frac{12\pi}{5}(k+1)^2(k+2)(k+3)(k^2+19k+20)\end{aligned} \tag{2.12}$$

$$
\begin{aligned}
&> \left[28k^5-395k^4+1280k^3+1835k^2+1195k-495\right]\left[256+177(k-8)+60(k-8)^2\right]\\
&\quad -8(k+1)^2(k+2)(k+3)(k^2+19k+20)\\
&=1680(k-8)^7+48448(k-8)^6+528501(k-8)^5+2783460(k-8)^4\\
&\quad +8154255(k-8)^3+16135583(k-8)^2+15415465(k-8)+4028864>0
\end{aligned}
$$

for $k \geqslant 8$.

Therefore, Lemma 2.3 follows from (2.4) and (2.12) together with the fact that

$$\frac{5\sqrt{\pi}(k^2+19k+20)}{32(2k+1)(k+3)} > \frac{\sqrt{\pi}(k+1)(k+2)}{24(2k+1)2^{k+1}}$$

for $k \geqslant 8$. □

LEMMA 2.4. *Let $n \in \mathbb{N}$ and $c_n$ be defined by*

$$
\begin{aligned}
c_n=&\left(\frac{1}{2},n\right)\left\{(n+2)!+\left[32\left(\frac{3}{2},n\right)(n+2)-3(n+1)!(5n+24)\right]2^n\right\} \qquad (2.13)\\
&+3\left(\frac{3}{4},n\right)(n+2)!2^n.
\end{aligned}
$$

*Then $c_0=c_1=c_2=c_3=c_4=c_5=c_6=0$ and $c_n>0$ for $n \geqslant 7$.*

*Proof.* It follows from (2.13) that

$$c_0=c_1=c_2=c_3=c_4=c_5=c_6=0,\ c_7=\frac{49116375}{4}>0,\ c_8=\frac{33153553125}{4}>0. \qquad (2.14)$$

Next, we use mathematical induction to prove the remain results. Let $n \geqslant 8$ and assume that $c_k>0$ for $k=8,9,\cdots,n$. Then it follows from (2.13) and the induction hypothesis that

$$3\left(\frac{3}{4},n\right)(n+2)!2^n > \left(\frac{1}{2},n\right)\left[3(5n+24)(n+1)!2^n-32(n+2)\left(\frac{3}{2},n\right)2^n-(n+2)!\right]$$

and

$$
\begin{aligned}
&c_{n+1} \qquad (2.15)\\
&=\left(\frac{1}{2},n+1\right)\left\{(n+3)!+\left[32\left(\frac{3}{2},n+1\right)(n+3)-3(n+2)!(5n+29)\right]2^{n+1}\right\}\\
&\quad +3\left(\frac{3}{4}n+1\right)(n+3)!2^{n+1}\\
&=2\left(n+\frac{3}{4}\right)(n+3)\left[3\left(\frac{3}{4},n\right)(n+2)!2^n\right]+\left(\frac{1}{2},n+1\right)(n+3)!\\
&\quad +32\left(\frac{1}{2},n+1\right)\left(\frac{3}{2},n+1\right)(n+3)2^{n+1}-3\left(\frac{1}{2},n+1\right)(n+2)!(5n+29)2^{n+1}
\end{aligned}
$$

$$
\begin{aligned}
>&2\left(\frac{1}{2},n\right)\left(n+\frac{3}{4}\right)(n+3)\left[3(5n+24)(n+1)!2^{n}-32(n+2)\left(\frac{3}{2},n\right)2^{n}-(n+2)!\right]\\
&+\left(\frac{1}{2},n\right)\Bigg[\left(n+\frac{1}{2}\right)(n+3)!+32\left(n+\frac{1}{2}\right)\left(\frac{3}{2},n+1\right)(n+3)2^{n+1}\\
&\qquad-3\left(n+\frac{1}{2}\right)(5n+29)(n+2)!2^{n+1}\Bigg]\\
=&\left(\frac{1}{2},n\right)2^{n+1}\Bigg[32\left(n+\frac{1}{2}\right)(n+3)\left(\frac{3}{2},n+1\right)-3\left(n+\frac{1}{2}\right)(5n+29)(n+2)!\\
&\qquad+3\left(n+\frac{3}{4}\right)(n+3)(5n+24)(n+1)!\\
&\qquad-32\left(n+\frac{3}{4}\right)(n+2)(n+3)\left(\frac{3}{2},n\right)-\frac{(n+3)!(n+1)}{2^{n+1}}\Bigg]\\
=&\left(\frac{1}{2},n\right)2^{n+1}\Bigg[(n+1)!\left(\frac{15}{4}(n^2+19n+20)-\frac{(n+1)(n+2)(n+3)}{2^{n+1}}\right)\\
&\qquad-24(n+1)(n+3)\left(\frac{3}{2},n\right)\Bigg]\\
=&\frac{24(2n+1)(n+3)\left(\frac{1}{2},n\right)(n+1)!2^{n+1}}{\sqrt{\pi}}\\
&\times\left[\frac{5\sqrt{\pi}(n^2+19n+20)}{32(2n+1)(n+3)}-\frac{\sqrt{\pi}(n+1)(n+2)}{24(2n+1)2^{n+1}}-\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right].
\end{aligned}
$$

Therefore, $c_n > 0$ for $n \geqslant 9$ follows from Lemma 2.3, (2.14), (2.15) and the Wallis inequality [36]

$$
\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} < \frac{1}{\sqrt{n+\frac{1}{4}}}
$$

for $n \geqslant 1$ together with the mathematical induction. □

LEMMA 2.5. *Let $r \in (0,1)$ and $f(r)$ be defined by*

$$
f(r) = \frac{\frac{32}{\pi}\left[2\mathscr{E}(r)-r'^2\mathscr{K}(r)\right]+5r'^2+2\sqrt{1+r^2}-23}{2\left(\sqrt{1+r^2}+r'^2-r'-1\right)}. \tag{2.16}
$$

*Then $f(r)$ is strictly decreasing from $(0,1)$ onto* $\left(\left[64-\left(23-2\sqrt{2}\right)\pi\right]/\left[2\left(\sqrt{2}-1\right)\pi\right],3/4\right)$.

*Proof.* Let

$$
f_1(r) = \frac{32}{\pi}\left[2\mathscr{E}(r)-r'^2\mathscr{K}(r)\right]+5r'^2+2\sqrt{1+r^2}-23,
$$

$$g_1(r) = 2\left(\sqrt{1+r^2} + r'^2 - r' - 1\right),$$
$$f_2(r) = \frac{16\left[\mathscr{E}(r) - r'^2\mathscr{K}(r)\right]}{\pi r^2} + \frac{1}{\sqrt{1+r^2}} - 5,$$
$$g_2(r) = \frac{1}{\sqrt{1+r^2}} + \frac{1}{r'} - 2,$$
$$f_3(r) = \frac{16r'^3\left[\left(1+r'^2\right)\mathscr{K}(r) - 2\mathscr{E}(r)\right]}{\pi r^4} - \frac{r'^3}{(1+r^2)^{3/2}},$$
$$g_3(r) = 1 - \frac{r'^3}{(1+r^2)^{3/2}}.$$

Then from Lemma 2.2(1), (2), (3), (2.16) and the derivatives formulas

$$\frac{d\mathscr{K}(r)}{dr} = \frac{\mathscr{E}(r) - r'^2\mathscr{K}(r)}{rr'^2}, \quad \frac{d\mathscr{E}(r)}{dr} = \frac{\mathscr{E}(r) - \mathscr{K}(r)}{r}$$

we get

$$f_1(0^+) = \frac{32}{\pi}\left(\pi - \frac{\pi}{2}\right) + 5 + 2 - 23 = 0, \quad g_1(0^+) = 0,$$
$$f_2(0^+) = \frac{16}{\pi} \times \frac{\pi}{4} + 1 - 5 = 0, \quad g_2(0^+) = 0,$$
$$f_3(0^+) = \frac{16}{\pi} \times \frac{\pi}{16} - 1 = 0, \quad g_3(0^+) = 0,$$
$$f(r) = \frac{f_1(r)}{g_1(r)} = \frac{f_1(r) - f_1(0^+)}{g_1(r) - g_1(0^+)}, \quad \frac{f_1'(r)}{g_1'(r)} = \frac{f_2(r)}{g_2(r)} = \frac{f_2(r) - f_2(0^+)}{g_2(r) - g_2(0^+)}, \tag{2.17}$$
$$\frac{f_2'(r)}{g_2'(r)} = \frac{f_3(r)}{g_3(r)} = \frac{f_3(r) - f_3(0^+)}{g_3(r) - g_3(0^+)}, \quad \frac{f_3'(r)}{g_3'(r)} = 1 - \frac{8}{3\pi}(1+r^2)^{5/2}\varphi(r), \tag{2.18}$$

where $\varphi(r)$ is defined by (2.1).

Therefore,

$$f(0^+) = 1 - \frac{8}{3\pi} \times \frac{3\pi}{32} = \frac{3}{4}, \quad f(1^-) = \frac{64 - (23 - 2\sqrt{2})\pi}{2(\sqrt{2}-1)\pi} = 0.2417\cdots$$

and $f(r)$ is strictly decreasing on $(0,1)$ follow from Lemma 2.1, Lemma 2.2(1) and (4), (2.16)-(2.18) and the L'Hôpital's rule. □

## 3. Main results

THEOREM 3.1. *The double inequality*

$$\frac{21\sqrt{2}\pi - 64}{8(\sqrt{2}-1)}M_1(x,y) - \frac{64 - (23 - 2\sqrt{2})\pi}{8(\sqrt{2}-1)}M_0(x,y) - \frac{3(6+\sqrt{2})\pi - 64}{8(\sqrt{2}-1)}M_{-1}(x,y) \tag{3.1}$$

$$-\frac{21\pi-64}{8(\sqrt{2}-1)}M_2(x,y)<L(x,y)<\frac{43\pi}{16}M_1(x,y)-\frac{3\pi}{16}M_0(x,y)-\frac{7\pi}{16}M_{-1}(x,y)-\frac{\pi}{16}M_2(x,y)$$

*holds for all* $x,y>0$ *with* $x\neq y$.

*Proof.* It is not difficult to very that inequality (3.1) is equivalent to

$$\frac{64-(23-2\sqrt{2})\pi}{2(\sqrt{2}-1)\pi}<\frac{\frac{8}{\pi}L(x,y)+5M_{-1}(x,y)+2M_2(x,y)-23M_1(x,y)}{2\left[M_2(x,y)+M_{-1}(x,y)-M_0(x,y)-M_1(x,y)\right]}<\frac{3}{4}. \tag{3.2}$$

Without loss of generality, we assume that $x>y>0$. Let $r=(x-y)/(x+y)\in(0,1)$. Then from (1.4), (1.6) and the Landen identity [35, Appendix E, Formulary (16)]

$$\mathscr{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathscr{E}(r)-r'^2\mathscr{K}(r)}{1+r},$$

we have

$$\frac{8}{\pi}L(x,y)=\frac{2x\left[2\mathscr{E}(r)-r'^2\mathscr{K}(r)\right]}{\pi(1+r)},\quad M_{-1}(x,y)=x(1-r),$$
$$M_2(x,y)=\frac{x\sqrt{1+r^2}}{1+r},\quad M_1(x,y)=\frac{x}{1+r},\quad M_0(x,y)=\frac{xr'}{1+r}$$

and

$$\frac{\frac{8}{\pi}L(x,y)+5M_{-1}(x,y)+2M_2(x,y)-23M_1(x,y)}{2\left[M_2(x,y)+M_{-1}(x,y)-M_0(x,y)-M_1(x,y)\right]}=f(r), \tag{3.3}$$

where $f(r)$ is defined by (2.16).

Therefore, inequality (3.2) follows from Lemma 2.5 and (3.3). □

REMARK 3.2. Let $\alpha=\left[64-\left(23-2\sqrt{2}\right)\pi\right]/\left[2\left(\sqrt{2}-1\right)\pi\right]=0.2417\cdots$ and $\beta=3/4$. Then inequality (3.1) can be rewritten as

$$\begin{aligned}&\alpha\frac{\pi\left[21M_1(x,y)-2M_0(x,y)-3M_{-1}(x,y)\right]}{8}\\&\quad+(1-\alpha)\frac{\pi\left[23M_1(x,y)-5M_{-1}(x,y)-2M_2(x,y)\right]}{8}<L(x,y)\\&<\beta\frac{\pi\left[21M_1(x,y)-2M_0(x,y)-3M_{-1}(x,y)\right]}{8}\\&\quad+(1-\beta)\frac{\pi\left[23M_1(x,y)-5M_{-1}(x,y)-2M_2(x,y)\right]}{8}.\end{aligned} \tag{3.4}$$

From (3.4) we clearly see that the upper and lower bounds for the perimeter of an ellipse given in Theorem 3.1 are better than that given (1.7).

THEOREM 3.3. *Let* $\lambda = 55/2^{30} = 5.1222\cdots\times 10^{-8}$ *and* $\mu = (43-\sqrt{2})/64 - 2/\pi = 0.013158\cdots$. *Then the double inequality*

$$\begin{aligned}&\frac{43\pi}{16}M_1(x,y)-\frac{3\pi}{16}M_0(x,y)-\frac{7\pi}{16}M_{-1}(x,y)-\frac{\pi}{16}M_2(x,y)-2\mu\pi x\left(1-\frac{y^2}{x^2}\right)^8 \quad (3.5)\\ &<L(x,y)<\frac{43\pi}{16}M_1(x,y)-\frac{3\pi}{16}M_0(x,y)-\frac{7\pi}{16}M_{-1}(x,y)-\frac{\pi}{16}M_2(x,y)-2\lambda\pi x\left(1-\frac{y^2}{x^2}\right)^8\end{aligned}$$

*holds for all* $x>y>0$.

*Proof.* Let $t\in(0,1)$ and $Q(t)$ be defined by

$$\begin{aligned}Q(t)=&\frac{43}{32}M_1(1,\sqrt{1-t})-\frac{3}{32}M_0(1,\sqrt{1-t})-\frac{7}{32}M_{-1}(1,\sqrt{1-t}) \quad (3.6)\\ &-\frac{1}{32}M_2(1,\sqrt{1-t})-F\left(-\frac{1}{2},\frac{1}{2};1;t\right).\end{aligned}$$

Then from (1.1), (1.6) and the power series formula

$$(1-t)^q=\sum_{n=0}^{\infty}\frac{(-q,n)}{n!}t^n$$

we get

$$F\left(-\frac{1}{2},\frac{1}{2};1;t\right)=\sum_{n=0}^{\infty}\frac{(-\frac{1}{2},n)(\frac{1}{2},n)}{(n!)^2}t^n=1-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2},n)(\frac{1}{2},n+1)}{[(n+1)!]^2}t^{n+1},$$

$$\begin{aligned}&\frac{43}{32}M_1(1,\sqrt{1-t})-\frac{3}{32}M_0(1,\sqrt{1-t})-\frac{7}{32}M_{-1}(1,\sqrt{1-t})-\frac{1}{32}M_2(1,\sqrt{1-t})\\ =&\frac{43}{64}\left(1+\sqrt{1-t}\right)-\frac{3}{32}(1-t)^{1/4}-\frac{7}{16}\left(1-\frac{1-\sqrt{1-t}}{t}\right)-\frac{1}{32}\left(1-\frac{1}{2}t\right)^{1/2}\\ =&\frac{43}{64}\left[2+\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2},n+1\right)}{(n+1)!}t^{n+1}\right]-\frac{3}{32}\left[1+\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{4},n+1\right)}{(n+1)!}t^{n+1}\right]\\ &-\frac{7}{16}\left[\frac{1}{2}+\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2},n+2\right)}{(n+2)!}t^{n+1}\right]-\frac{1}{32}\left[1+\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2},n+1\right)}{2^{n+1}(n+1)!}t^{n+1}\right]\\ =&1+\frac{1}{64}\left[43\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2},n+1\right)}{(n+1)!}t^{n+1}-6\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{4},n+1\right)}{(n+1)!}t^{n+1}\right.\\ &\left.-28\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2},n+2\right)}{(n+2)!}t^{n+1}-2\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2},n+1\right)}{2^{n+1}(n+1)!}t^{n+1}\right]\\ =&1+\frac{1}{64}\sum_{n=0}^{\infty}\left[\frac{43\left(-\frac{1}{2},n+1\right)}{(n+1)!}-\frac{6\left(-\frac{1}{4},n+1\right)}{(n+1)!}-\frac{28\left(-\frac{1}{2},n+2\right)}{(n+2)!}-\frac{2\left(-\frac{1}{2},n+1\right)}{2^{n+1}(n+1)!}\right]t^{n+1}\end{aligned}$$

$$=1+\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n\right)\left[(n+2)-3(5n+24)2^n\right]+3\left(\frac{3}{4},n\right)(n+2)2^n}{128(n+2)!2^n}t^{n+1}$$

and

$$Q(t)=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n\right)\left[(n+2)-3(5n+24)2^n\right]+3\left(\frac{3}{4},n\right)(n+2)2^n}{128(n+2)!2^n}t^{n+1} \tag{3.7}$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2},n)(\frac{1}{2},n+1)}{[(n+1)!]^2}t^{n+1}=\sum_{n=0}^{\infty}\frac{c_n}{128(n+1)!(n+2)!2^n}t^{n+1},$$

where $c_n$ is defined by (2.13).

It follows from Lemma 2.4, (2.13), (3.6) and (3.7) that

$$\frac{Q(t)}{t^8}=\frac{c_7}{128(n+1)!(n+2)!2^n}+\sum_{n=8}^{\infty}\frac{c_n}{128(n+1)!(n+2)!2^n}t^{n-7},$$
$$\lim_{t\to 0^+}\frac{Q(t)}{t^8}=\frac{c_7}{128(n+1)!(n+2)!2^n}=\frac{55}{2^{30}}, \tag{3.8}$$
$$\lim_{t\to 1^-}\frac{Q(t)}{t^8}=\lim_{t\to 1^-}Q(1)=\frac{43-\sqrt{2}}{64}-\frac{2}{\pi} \tag{3.9}$$

and $Q(t)/t^8$ is strictly increasing on $(0,1)$.

Let $t=1-y^2/x^2$. Then inequality (3.5) follows from (1.4), (3.6), (3.8), (3.9) and the monotonicity of the function $Q(t)/t^8$ on the interval $(0,1)$. □

Let $r\in(0,1)$, $x=1$ and $y=r'=\sqrt{1-r^2}$. Then from (1.4), (1.6), and Theorems 3.1 and 3.3 we get Corollary 3.4 immediately.

COROLLARY 3.4. *The double inequalities*

$$\frac{21\sqrt{2}\pi-64}{64(\sqrt{2}-1)}(1+r')-\frac{64-(23-2\sqrt{2})\pi}{32(\sqrt{2}-1)}r'^{1/2}-\frac{3(6+\sqrt{2})\pi-64}{16(\sqrt{2}-1)}\frac{r'}{1+r'}$$
$$-\frac{21\pi-64}{32(2-\sqrt{2})}\left(1+r'^2\right)^{1/2}<\mathscr{E}(r)<\frac{43\pi}{128}(1+r')-\frac{3\pi}{64}r'^{1/2}-\frac{7\pi}{32}\frac{r'}{1+r'}-\frac{\sqrt{2}\pi}{128}\left(1+r'^2\right)^{1/2}$$

*and*

$$\frac{43\pi}{128}(1+r')-\frac{3\pi}{64}r'^{1/2}-\frac{7\pi}{32}\frac{r'}{1+r'}-\frac{\sqrt{2}\pi}{128}\left(1+r'^2\right)^{1/2}-2\mu\pi r^{16}<\mathscr{E}(r) \tag{3.10}$$
$$<\frac{43\pi}{128}(1+r')-\frac{3\pi}{64}r'^{1/2}-\frac{7\pi}{32}\frac{r'}{1+r'}-\frac{\sqrt{2}\pi}{128}\left(1+r'^2\right)^{1/2}-2\lambda\pi r^{16}$$

*hold for all* $r\in(0,1)$, *where* $\lambda=55/2^{30}=5.1222\cdots\times 10^{-8}$ *and* $\mu=(43-\sqrt{2})/64-2/\pi=0.013158\cdots$.

REMARK 3.5. From (3.10) we clearly see that the absolute error is less than $1.27\times 10^{-6}$ if $r\in(0,1/2]$ and we use

$$\frac{43\pi}{128}\left(1+r'\right)-\frac{3\pi}{64}r'^{1/2}-\frac{7\pi}{32}\frac{r'}{1+r'}-\frac{\sqrt{2}\pi}{128}\left(1+r'^{2}\right)^{1/2}$$

to represent the approximation of $\mathscr{E}(r)$ due to

$$2\pi(\mu-\lambda)r^{16}\leqslant\frac{\pi}{2^{15}}(\mu-\lambda)=1.2615\cdots\times 10^{-6}.$$

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