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TAUBERIAN THEOREMS UNDER STATISTICALLY NÖRLUND–CESÁRO SUMMABILITY METHOD

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Abstract. Let (p_n) and (q_n) be any two non-negative real sequences with

$$R_n := \sum_{k=0}^{n} p_k q_{n-k} \neq 0 \ (n \in \mathbb{N}).$$

And C_n^1- Cesáro summability method. Let (x_n) be a sequence of real or complex numbers and set

$$N_{p,q}^n C_n^1 := \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v$$

for $n \in \mathbb{N}$. In this paper, we present necessary and sufficient conditions under which the existence of the limit $st - \lim_{n\to\infty} x_n = L$ follows from that of $st - \lim_{n\to\infty} N_{p,q}^n C_n^1 = L$. These conditions are one-sided or two-sided if (x_n) is a sequence of real or complex numbers, respectively.

1. Introduction

Let (p_n) and (q_n) be any two non-negative real sequences with

$$R_n := \sum_{k=0}^{n} p_k q_{n-k} \neq 0 \ (n \in \mathbb{N}).$$

And $(C,1)-$ Cesáro summability method. Let (x_n) be a sequence of real or complex numbers and set

$$N_{p,q}^n C_n^1 := \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v$$

for $n \in \mathbb{N}$. In this paper, we present necessary and sufficient conditions under which the existence of the limit $\lim_{n\to\infty} x_n = L$ follows from that of $\lim_{n\to\infty} N_{p,q}^n C_n^1 = L$. These conditions are one-sided or two-sided if (x_n) is a sequence of real or complex numbers, respectively.

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In what follows we give the concept of the summability method known as the generalized Nörlund summability method (N,p,q) (see [1, 4]). Given two non-negative sequences (p_n) and (q_n), the convolution $(p\star q)$ is defined by

$$R_n := (p\star q)_n = \sum_{k=0}^{n} p_k q_{n-k} = \sum_{k=0}^{n} p_{n-k} q_k.$$

With $(C,1)-$ we will denote the Cesáro summability method. Let (x_n) be a sequence. When $(p\star q)_n \neq 0$ for all $n \in \mathbb{N}$, the generalized Nörlund-Cesáro transform of the sequence (x_n) is the sequence $N^n_{p,q}C^1_n$ obtained by putting

$$N^n_{p,q}C^1_n = \frac{1}{(p\star q)_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v. \tag{1.1}$$

We say that the sequence (x_n) is generalized Nörlund-Cesáro summable to L determined by the sequences (p_n) and (q_n) or briefly summable $N^n_{p,q}C^1_n$ to L if

$$\lim_{n\to\infty} N^n_{p,q}C^1_n = L. \tag{1.2}$$

Suppose throughout the paper we assume that the sequences (q_n) and (p_n) are satisfying the following conditions:

$$q_n \geqslant 1, \sum_{k=0}^{n} p_k \sim n,\ n\in\mathbb{N}, \tag{1.3}$$

$$q_{\lambda_n - k} \leqslant 2q_{n-k}, k = 0,1,2,3,\cdots,n; \lambda > 1, \tag{1.4}$$

$$q_{n-k} \leqslant 2q_{\lambda_n - k}, k = 0,1,2,3,\cdots,\lambda_n; 0<\lambda<1, \tag{1.5}$$

where $\lambda_n = [\lambda\cdot n]$, $a_n \sim b_n$, means that there are constants C, C_1 such that $a_n \leqslant Cb_n \leqslant C_1 a_n$.

If

$$\lim_{n\to\infty} x_n = L \tag{1.6}$$

implies (1.2), then the method $N^n_{p,q}C^1_n$ is called to be regular. However, the converse is not always true. We can show by the following example

EXAMPLE 1.1. Let us consider that $p_n = q_n = 1$ for all $n\in\mathbb{N}$. Also we define the following sequence $x = (x_k) = (-1)^k$, then we have

$$\frac{1}{n+1}\left|\sum_{k=0}^{n}\frac{1}{k+1}\sum_{v=0}^{k}(-1)^v\right| \leqslant \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{k+1}\sum_{v=0}^{k} 1 \to 1 \quad \text{as} \quad n\to\infty.$$

And as we know $x = (x_k)$, is not convergent.

Notice that (1.6) may imply (1.2) under a certain condition, which is called a Tauberian condition. Any theorem which states that convergence of a sequence follows from its $N^n_{p,q}C^1_n$ summability and some Tauberian condition is said to be a Tauberian theorem for the $N^n_{p,q}C^1_n$ summability method. The inclusion and Tauberian type theorems are proved in the papers [4, 5, 2, 3], and some theorems of inclusion, Tauberian and convexity type for certain families of generalized Nörlund methods are obtained in [6].

In this section our aim is to find conditions (so-called Tauberian) under which the converse implication holds, for defined convergence. Exactly, we will prove under which conditions statistical convergence of sequences (x_n), follows from statistically Nörlund-Cesáro summability method.

DEFINITION 1.2. A sequence (x_n) is weighted $N^n_{p,q}C^1_n-$statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{(p\star q)_n}\left|\left\{k \leqslant (p\star q)_n : \left|\frac{1}{(p\star q)_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k} x_v - L\right| \geqslant \varepsilon\right\}\right| = 0.$$

And we say that the sequence (x_n) is statistically summable to L by the weighted summability method $N^n_{p,q}C^1_n$, if $st-\lim_n N^n_{p,q}C^1_n = L$. We denote by $N^n_{p,q}C^1_n(st)$ the set of all sequences which are statistically summable, by the weighted summability method $N^n_{p,q}C^1_n$.

THEOREM 1.3. *If sequence $x=(x_n)$ is $N^n_{p,q}C^1_n$ summable to L, then sequence $x=(x_n)$ is $N^n_{p,q}C^1_n-$ statistically convergent to L. But not conversely.*

Proof. The first part of the proof is obvious. To prove the second part we will show this example:

EXAMPLE 1.4. We will define

$$x_k = \begin{cases} \sqrt{k}\,, & \text{for} \quad k=n^2 \\ 0 \quad , & \text{otherwise} \end{cases}$$

and $p_n = 1 = q_n$. Under this conditions we get:

$$\frac{1}{n+1}\left|\left\{k \leqslant n+1 : \left|\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{P_k}\sum_{v=0}^{k} p_v x_v - 0\right| \geqslant \varepsilon\right\}\right| \leqslant \frac{\sqrt{n+1}}{n+1} \to 0.$$

On the other hand, for $k=n^2$, we have

$$\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{k+1}\sum_{v=0}^{k} x_v \to \infty, \quad \text{as} \quad n\to\infty.$$

From last relation follows that $x=(x_n)$ is not $N^n_{p,q}C^1_n$ summable to 0. □

THEOREM 1.5. *Let us suppose that sequence (x_n)-statistically convergent to L, and $|x_n - L| \leqslant M$ for every $n \in \mathbb{N}$. Then it converges $N^n_{p,q}C^1_n$-statistically to L. Converse is not true.*

Proof. From fact that (x_n) converges statistically to L, we get

$$\lim_{n\to\infty} \frac{|\{k \leqslant n : |x_k - L| \geqslant \varepsilon\}|}{n} = 0.$$

Let us denote by $B_\varepsilon = \{k \leqslant n : |x_k - L| \geqslant \varepsilon\}$ and $\overline{B_\varepsilon} = \{k \leqslant n : |x_k - L| \leqslant \varepsilon\}$. Then

$$\begin{aligned}
&\left|\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k} x_v - L\right| = \left|\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k} (x_v - L)\right| \\
\leqslant &\frac{1}{R_n}\sum_{\substack{k=0\\ k\in B_\varepsilon}}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k} |x_v - L| + \frac{1}{R_n}\sum_{\substack{k=0\\ k\in \overline{B}_\varepsilon}}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k} |x_v - L| \\
\leqslant &M\frac{1}{R_n}\sum_{\substack{k=0\\ k\in B_\varepsilon}}^{n} 1 + \varepsilon \leqslant M\frac{C_2}{n}\sum_{\substack{k=0\\ k\in B_\varepsilon}}^{n} 1 + \varepsilon \to 0 + \varepsilon, \quad \text{as} \quad n \to \infty,
\end{aligned}$$

for some constant C_2. To show that converse is not true we will use into consideration this

EXAMPLE 1.6. Let us consider that $(p_n) = n+1$, $(q_n) = 1$ for $n \in \mathbb{N}\cup\{0\}$, and we define the sequence $x = (x_n)$, as follows:

$$x_k = \begin{cases} 1 & , \quad \text{for} \quad k = m^2 - m, \cdots, m^2 - 1 \\ -\frac{1}{m} & , \quad \text{for} \quad k = m^2, m = 2, \cdots \\ 0 & , \quad \text{otherwise} \end{cases}$$

Under this conditions, after some calculations we get:

$$\left|\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n} 1\cdot\sum_{v=0}^{k} x_v - 1\right| \leqslant \left|\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n} 1\cdot\sum_{v=0}^{k} - 1\right| \to 0, \quad \text{as} \quad n \to \infty.$$

From last relation follows that $x = (x_n)$ is $N^n_{p,q}C^1_n-$ summable to 1. Hence from Theorem 1.5, (x_n) is $N^n_{p,q}C^1_n-$statistically convergent. On the other hand, the sequence $(m^2; m = 2, 3\cdots,)$ has natural density zero and it is clear that $st - \liminf_n x_n = 0$ and $st - \limsup_n x_n = 1$. Thus, (x_k) is not statistically convergent. $\square$

2. Tauberian theorems under statistical Nörlund-Cesáro summability method

In the following theorem we characterize the converse implication when the statistically convergence follows from its $N^n_{p,q}C^1_n-$ statistically convergence.

THEOREM 2.1. *Let* (p_n) *and* (q_n) *be two non-negative real sequences, defined as above, and*

$$st-\liminf_{n\to\infty}\frac{R_{\lambda_n}}{R_n}>1, \quad \text{for every} \quad \lambda>1, \tag{2.1}$$

where $\lambda_n:=[\lambda n]$ *denotes the integral part of* λn *for every* $n\in\mathbb{N}$, *and let* (x_n) *be a sequence of real numbers which is* $N^n_{p,q}C^1_n-$ *statistically convergent to a finite number* L. *Then* (x_n) *is* st*-convergent to the same number* L *if and only if the following two conditions hold:*

$$\inf_{\lambda>1}\limsup_n\frac{1}{R_n}\left|\left\{k\leqslant R_n:\frac{1}{R_{\lambda_k}-R_k}\sum_{j=k+1}^{\lambda_k}p_jq_{\lambda_k-j}\frac{1}{j+1}\sum_{v=0}^{k}(x_j-x_k)\leqslant-\varepsilon\right\}\right|=0, \tag{2.2}$$

and

$$\inf_{0<\lambda<1}\limsup_n\frac{1}{R_n}\left|\left\{k\leqslant R_n:\frac{1}{R_k-R_{\lambda_k}}\sum_{j=\lambda_k+1}^{k}p_jq_{k-j}\frac{1}{j+1}\sum_{v=0}^{k}(x_k-x_j)\leqslant-\varepsilon\right\}\right|=0. \tag{2.3}$$

REMARK 2.2. Let us suppose that $st-\lim_k x_k=L$; (x_n) is $N^n_{p,q}C^1_n-$ statistically convergent and relation (2.1) satisfies, then for every $t>1$, is valid the following relation:

$$st-\lim_k\frac{1}{R_{\lambda_k}-R_k}\sum_{j=k+1}^{\lambda_k}p_jq_{\lambda_k-j}\frac{1}{j+1}\sum_{v=0}^{k}(x_j-x_k)=0 \tag{2.4}$$

and in case where $0<t<1$,

$$st-\lim_k\frac{1}{R_k-R_{\lambda_k}}\sum_{j=\lambda_k+1}^{k}p_jq_{k-j}\frac{1}{j+1}\sum_{v=0}^{k}(x_k-x_j)=0. \tag{2.5}$$

In the next result we will consider the case where $x=(x_n)$ is a sequence of complex numbers.

THEOREM 2.3. *Let condition* (2.1) *be satisfied and let* (x_n) *be a sequence of complex numbers which is* $N^n_{p,q}C^1_n$ *summable to a finite number* L. *Then* (x_n) *is convergent to the same number* L *if and only if one of the following two conditions holds:*

$$\inf_{\lambda>1}\limsup_n\frac{1}{R_n}\left|\left\{k\leqslant R_n:\left|\frac{1}{R_{\lambda_k}-R_k}\sum_{j=k+1}^{\lambda_k}p_jq_{\lambda_k-j}\frac{1}{j+1}\sum_{v=0}^{k}(x_j-x_k)\right|\geqslant\varepsilon\right\}\right|=0, \tag{2.6}$$

and

$$\inf_{0<\lambda<1}\limsup_n\frac{1}{R_n}\left|\left\{k\leqslant R_n:\left|\frac{1}{R_k-R_{\lambda_k}}\sum_{j=\lambda_k+1}^{k}p_jq_{k-j}\frac{1}{j+1}\sum_{v=0}^{k}(x_k-x_j)\right|\geqslant\varepsilon\right\}\right|=0. \tag{2.7}$$

In what follows we list some auxiliary lemmas which are needful in the sequel.

LEMMA 2.4. *The condition given by relation (2.1) is equivalent to the condition*

$$st-\liminf_{n\to\infty}\frac{R_n}{R_{\lambda_n}}>1,\quad 0<\lambda<1. \tag{2.8}$$

Proof. Suppose that relation (2.1) is valid, $0<\lambda<1$ and $m=\lambda_n=[\lambda n],\ n\in\mathbb{N}$. Then it follows that

$$\frac{1}{\lambda}>1\Rightarrow\frac{m}{\lambda}=\frac{[\lambda n]}{t}\leqslant n.$$

From above relation and definition of the sequences (p_n) and (q_n), we obtain:

$$\frac{R_n}{R_{\lambda_n}}\geqslant\frac{R_{\left[\frac{m}{\lambda}\right]}}{R_{\lambda_n}}\Rightarrow st-\liminf_{n\to\infty}\frac{R_n}{R_{\lambda_n}}\geqslant st-\liminf_{n\to\infty}\frac{R_{\left[\frac{m}{\lambda}\right]}}{R_{\lambda_n}}>1.$$

Conversely, suppose that relation (2.8) is valid. Let $\lambda>1$ be given number and let λ_1 be chosen such that $1<\lambda_1<\lambda$. Set $m=\lambda_n=[\lambda n]$. From $0<\frac{1}{\lambda}<\frac{1}{\lambda_1}<1$, it follows that:

$$n\leqslant\frac{\lambda n-1}{\lambda_1}<\frac{[\lambda n]}{\lambda_1}=\frac{m}{\lambda_1},$$

provided $\lambda_1\leqslant\lambda-\frac{1}{n}$, which is a case where if n is large enough. Under this conditions we have:

$$\frac{R_{\lambda_n}}{R_n}\geqslant\frac{R_{\lambda_n}}{R_{\left[\frac{m}{\lambda_1}\right]}}\Rightarrow st-\liminf_{n\to\infty}\frac{R_{\lambda_n}}{R_n}\geqslant st-\liminf_{n\to\infty}\frac{R_{\lambda_n}}{R_{\left[\frac{m}{\lambda_1}\right]}}>1.\quad\square$$

PROPOSITION 2.5. *Let us suppose that relation (2.1) is satisfied and let $x=(x_k)$ be a sequence of complex numbers which is $N^n_{p,q}C^1_n-$statistically convergent to L. Then*

$$st-\lim_n\frac{1}{R_{\lambda_n}-R_n}\sum_{j=n+1}^{\lambda_n}p_jq_{t_n-j}\frac{1}{j+1}\sum_{v=0}^{j}x_v=L,\quad \textit{for}\quad \lambda>1 \tag{2.9}$$

and

$$st-\lim_n\frac{1}{R_n-R_{\lambda_n}}\sum_{j=\lambda_n+1}^{n}p_jq_{n-j}\frac{1}{j+1}\sum_{v=0}^{j}x_v=L,\quad \textit{for}\quad 0<\lambda<1. \tag{2.10}$$

Proof. (I) Let us consider the case where $\lambda>1$. Then we obtain

$$\frac{1}{R_{\lambda_n}-R_n}\sum_{k=n+1}^{\lambda_n}p_kq_{\lambda_n-k}\frac{1}{k+1}\sum_{v=0}^{k}(x_v-L)$$
$$=\frac{R_{\lambda_n}}{R_{\lambda_n}-R_n}\frac{1}{R_{\lambda_n}}\sum_{k=0}^{\lambda_n}p_kq_{\lambda_n-k}\frac{1}{k+1}\sum_{v=0}^{k}(x_v-L)-\frac{R_n}{R_{\lambda_n}-R_n}\frac{1}{R_n}\sum_{k=0}^{n}p_kq_{\lambda_n-k}\frac{1}{k+1}\sum_{v=0}^{k}(x_v-L)$$

$$
\begin{aligned}
&=\frac{R_{\lambda_n}}{R_{\lambda_n}-R_n}\frac{1}{R_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k q_{\lambda_n-k}\frac{1}{k+1}\sum_{v=0}^{k}(x_v-L)\\
&\quad-\frac{R_n}{R_{\lambda_n}-R_n}\frac{1}{R_n}\sum_{k=0}^{n} p_k(q_{\lambda_n-k}+q_{n-k}-q_{n-k})\frac{1}{k+1}\sum_{v=0}^{k}(x_v-L)\\
&=\frac{R_{\lambda_n}}{R_{\lambda_n}-R_n}\frac{1}{R_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k q_{\lambda_n-k}\frac{1}{k+1}\sum_{v=0}^{k}(x_v-L)-\frac{R_n}{R_{\lambda_n}-R_n}\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k}(x_v-L)\\
&\quad-\frac{R_n}{R_{\lambda_n}-R_n}\frac{1}{R_n}\sum_{k=0}^{n} p_k(q_{\lambda_n-k}-q_{n-k})\frac{1}{k+1}\sum_{v=0}^{k}(x_v-L).
\end{aligned}
\tag{2.11}
$$

From relation (2.11), definition of the sequence (q_n), and relation

$$
\limsup_n \frac{R_{\lambda_n}}{R_{\lambda_n}-R_n}<\infty,
$$

we get relation (2.9).

(II) In this case we have that $0<\lambda<1$. Then

$$
\begin{aligned}
&\frac{1}{R_n-R_{\lambda_n}}\sum_{k=\lambda_n+1}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k} x_v\\
&=\frac{R_n}{R_n-R_{\lambda_n}}\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k} x_v-\frac{R_{\lambda_n}}{R_n-R_{\lambda_n}}\frac{1}{R_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k} x_v\\
&=\frac{R_n}{R_n-R_{\lambda_n}}\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k} x_v-\frac{R_{\lambda_n}}{R_n-R_{\lambda_n}}\frac{1}{R_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k} x_v\\
&\quad-\frac{R_{\lambda_n}}{R_n-R_{\lambda_n}}\frac{1}{R_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k(q_{n-k}-q_{\lambda_n-k})\frac{1}{k+1}\sum_{v=0}^{k} x_v.
\end{aligned}
$$

Now proof of the proposition is similar to the first part. □

Proof of Theorem 2.1.

Necessity. Suppose that $\lim_{n\to\infty} x_n = L$, and (2.1) holds. Following Proposition 2.5, we have

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{R_{\lambda_n}-R_n}\sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n-k}\frac{1}{k+1}\sum_{v=0}^{k}(x_v-x_n)\\
&=\lim_{n\to\infty}\left\{\left(\frac{1}{R_{\lambda_n}-R_n}\sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n-k}\frac{1}{k+1}\sum_{v=0}^{k} x_v\right)-x_n\right\}=0,
\end{aligned}
$$

for every $\lambda>1$. In case where $0<\lambda<1$, we find that

$$
\lim_{n\to\infty}\frac{1}{R_n-R_{\lambda_n}}\sum_{k=\lambda_n+1}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k}(x_n-x_v)
$$

$$= \lim_{n\to\infty} \left\{ x_n - \left(\frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v \right) \right\} = 0.$$

Sufficiency. Assume that conditions (2.2) and (2.3) are satisfied. In what follows we will prove that $\lim_{n\to\infty} x_n = L$. Given any $\varepsilon > 0$, by relation (2.2) we can choose $\lambda_1 > 0$ such that

$$\liminf_{n\to\infty} \frac{1}{R_{\lambda_{n_1}} - R_n} \sum_{k=n+1}^{\lambda_{n_1}} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - x_n) \geqslant -\varepsilon, \tag{2.12}$$

where $\lambda_{n_1} = [\lambda_1 n]$. By the assumed summability $N^n_{p,q}C^1_n$ of (x_n), Proposition 2.5 and relation (2.12), we have

$$\limsup_{n\to\infty} x_n \leqslant L + \varepsilon, \tag{2.13}$$

for any $\lambda > 1$.

On the other hand, if $0 < \lambda < 1$, for every $\varepsilon > 0$, we can choose $0 < \lambda_2 < 1$ such that

$$\liminf_{n\to\infty} \frac{1}{R_n - R_{\lambda_{n_2}}} \sum_{k=\lambda_{n_2}+1}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} (x_n - x_v) \geqslant -\varepsilon, \tag{2.14}$$

where $\lambda_{n_2} = [\lambda_2 n]$. By the assumed summability $N^n_{p,q}C^1_n$ of (x_n), Proposition 2.5 and (2.14), we have

$$\liminf_{n\to\infty} x_n \geqslant L - \varepsilon, \tag{2.15}$$

for any $0 < \lambda < 1$.

Since $\varepsilon > 0$ is arbitrary, combining relations (2.13) and(2.15) we obtain

$$\lim_{n\to\infty} x_n = L. \qquad \square$$

Proof of Theorem 2.3.

Necessity. If both (1.2) and (1.6) hold, then Proposition 2.5 yields (2.6) for every $\lambda > 1$ and (2.7) for every $0 < \lambda < 1$.

Sufficiency. Suppose that (1.2), (2.1) and one of the conditions (2.6) and (2.7) are satisfied. For any given $\varepsilon > 0$, there exists some $\lambda_1 > 1$ such that

$$\limsup_{n\to\infty} \left| \frac{1}{R_{\lambda_{n_1}} - R_n} \sum_{k=n+1}^{\lambda_{n_1}} p_k q_{\lambda_{n_1} - k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - x_n) \right| \leqslant \varepsilon,$$

where $\lambda_{n_1} = [\lambda_1 n]$. Taking into account fact that (x_n) is $N^n_{p,q}C^1_n$ summable to L and Proposition 2.5, we get the following estimation

$$\limsup_{n\to\infty} |L - x_n| \leqslant \limsup_{n\to\infty} \left| L - \frac{1}{R_{\lambda_{n_1}} - R_n} \sum_{k=n+1}^{\lambda_{n_1}} p_k q_{\lambda_{n_1} - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v \right|$$

$$+\limsup_{n\to\infty}\left|\frac{1}{R_{\lambda_{n_1}}-R_n}\sum_{k=n+1}^{\lambda_{n_1}} p_k q_{\lambda_{n_1}-k}\frac{1}{k+1}\sum_{v=0}^{k}(x_v-x_n)\right|\leqslant \varepsilon.$$

For any given $\varepsilon > 0$, there exists some $0 < \lambda_2 < 1$ such that

$$\limsup_{n\to\infty}\left|\frac{1}{R_n-R_{\lambda_{n_2}}}\sum_{k=\lambda_{n_2}+1}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k}(x_n-x_v)\right|\leqslant \varepsilon,$$

where $\lambda_{n_2} = [\lambda_2 n]$. Taking into account the fact that (x_n) is $N^n_{p,q}C^1_n$ summable to L and Proposition 2.5, we obtain the following

$$\limsup_{n\to\infty}|L-x_n| \leqslant \limsup_{n}\left|L-\frac{1}{R_n-R_{\lambda_{n_2}}}\sum_{k=\lambda_{n_2}+1}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k}x_v\right|$$
$$+\limsup_{n\to\infty}\left|\frac{1}{R_n-R_{\lambda_{n_2}}}\sum_{k=\lambda_{n_2}+1}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k}(x_n-x_v)\right|\leqslant \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, in either case we get $\lim_{n\to\infty} x_n = L$. □

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