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NEW INEQUALITIES FOR ROTOR FRAMES IN HILBERT SPACE

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Abstract. In this paper, a new identity of the Weyl-Heisenberg frame using a rotation operator has been investigated in Hilbert space. The characterization and significance of Rotor frame inequalities have been discussed by using rotation and translation operators. Also discussed the application of Rotor frames using a rotation operator. Finally, the reconstruction theorem has been investigated for recovers original data. In this work, we would like to highlight that the reconstruction theorem used to obtain the energy of the signal and reconstruct the original signal with eradicated the garbage vector using frame operator in Hilbert space. Today, this technique is very useful in communication systems.

1. Introduction

Frames are a generalization of orthonormal basis and redundant systems of vectors in Hilbert space. Linear independence property allows for a basis that every vector in the space to be written as linear combinations. It has some difficulties to process for practical problems. However, a frame allows each element in the space to be written as a linear combination of the vector in the frames. Here, linear independence between the frame elements is not required. This fact becomes important in communication systems, signal processing, image processing, coding theory, sampling theory, and cryptography. Today, frames play an important role not only in theory but also in many applications in Engineering and Technology.

Frames for Hilbert spaces were formally defined by R. J. Duffin and A. C. Schaffer [1] in 1952 to deal with no harmonic Fourier series. Daubechies, Grossmann and Meyer discussed the context of Painless non-orthogonal expansions [3] in 1986. Peter G. Casazza and Ole Christensen provided some results and properties for operators and frames have been discussed in [5]. The five name team Radu Balan, Peter Casazza and Dan Edidin, Laura Walters, and Eric Weber have discussed about signal reconstruction and they provided some results in frame theory without phase as the paper [6]. The authors K. Raju Pillai and S. Palaniammal have discussed linearity, stability, and properties of frames in Hilbert space with application in communication systems in [4, 7, 11]. The matrix absolute value and some properties in the matrix theory has been discussed by Rajendra Bhatia in the paper [8]. A theorem to recover the signal

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for frames has been discussed in [9, 12, 15]. Parseval frames for ICC groups and approximating the inverse frame multiplier or operator from localized frames have been investigated in [2, 10, 13]. Harp frame and Harp frame operator has been discussed in finite-dimensional Hilbert space and provided some result and properties of Harp frame operator on Hilbert space in [14]. Some characterization Weyl-Heisenberg frame has been discussed by C. Easwaran Nambudiri and K. Parthasarathy in [16, 17].

Zhong-Qi Xiang has established several new inequalities for g-frames in Hilbert C^*-modules which are different in structure and also present some equalities and inequalities for g-frames in Hilbert C^*-modules in [18, 19].

In this work, we have investigated a new identity of Weyl-Heisenberg frame in Hilbert space. The characterization and significance of Rotor frames have been provided in Hilbert space. Finally, Signal Reconstruction Technique has been discussed in the Hilbert space.

2. Preliminaries and notations

Let H be finite dimensional Hilbert space, $L(H)$ be a set of all bounded linear operator on H and J is finite or countable index set and it has been used thought paper. We can define the operators $T : l^2 \to H$ as follows,

$$T : l^2 \to H, \quad Ta = \sum_{n \in J} a_n f_n, \text{ for all } a = \{a_n\} \in l^2 \tag{2.1}$$

is called synthesis operator or pre frame operator and the ad joint operator is given by

$$T^* : H \to l^2, \quad T^* f = \{\langle f, f_n \rangle\}_{n \in J} \tag{2.2}$$

which is the analysis operator.

The composition operator T with its adjoint T^* and it denoted by $S = TT^*$ such that

$$S : H \to H \quad Sf = \sum_{n \in J} \langle f, f_n \rangle f_n, \text{ for all } f \in H \tag{2.3}$$

is called the frame operator. The frame operator S satisfies linearity, positive and invertible condition in Hilbert space.

The distance between an element $f \in H$ and a subspace $H_1 \subseteq H$ such that $d(f, H_1) = \inf_{g \in H_1} \|f - g\|$. We can find a condition implying that $\{f_n\}_{n \in J}$ is a conditional Riesz frame.

3. Definition and classifications of the frames

DEFINITION 3.1. Let H be Hilbert space and a sequence $\{f_n\}_{n \in J} \subset H$ is called an ordinary frames, if there exists constants $A, B > 0$, such that

$$A \|f\|^2 \leqslant \sum_{n \in J} |\langle f, f_n \rangle|^2 \leqslant B \|f\|^2, \text{ for all } f \in H \tag{3.1}$$

where A and B are lower and upper frame bounds for the frames $\{f_n\}_{n \in J}$.

Next, we discussed classifications of the frames as follows:

1. The maximum number $A > 0$ and minimum number $B > 0$ satisfying the frame inequalities for all $f \in H$ are called the optimal bounds.

2. A frame is a tight frame if $A = B$.

3. A frame is normalized tight frame if $A = B = 1$.

4. A frame is exact frames if it ceases to be a frame when any one of its elements is removed.

5. A frame is exact if and only if it is a Riesz basis.

6. In the sense that at least one vector can be abandoned from the frame and the remaining set of vectors or signal will still form a frame for H and that is a non-exact frame is called over complete.

7. A frame $(g_n) \in H$ is called an alternate dual frame (or a pseudo-dual) for $(f_n) \in H$ if $f = \sum_{n \in J} \langle f, f_n \rangle g_n$, for all $f \in H$.

8. $S^{-1} f_n$ is the canonical dual of (f_n) and if (f_n) is a normalized tight frames, then $S = I$ and so the frame equals its canonical dual.

9. If K is any invertible operator in Hilbert space, $K : H \to H$ and it is defined for lower frame bounds, it follows that

$$A \|K^* f\|^2 \leqslant \sum_{n \in J} |\langle f, f_n \rangle|^2 \leqslant B \|f\|^2, \quad \text{for all } f \in H \tag{3.2}$$

where A and B are lower and upper frame bounds for the K-frames $\{f_n\}_{n \in J}$.

The next definition gives Weyl-Heisenberg frame which is used modulation and translation operator as follows.

4. Weyl-Heisenberg frame in Hilbert space

DEFINITION 4.1. The translation and modulation operator can be written as follows. For all $a, b \in R$, $(T_a f)(x) = f(x - a)$ and $(E_b f)(x) = e^{2\pi i b x} f(x)$ acting on $L^2(R)$. The Weyl-Heisenberg frame operator defined as follows

$$S : H \longrightarrow H \text{ and } Sf = \sum_{m,n \in J} \langle f, E_{mb} T_{na} g \rangle E_{mb} T_{na} g, \quad \text{for all } g \in L^2(R) \tag{4.1}$$

and $\{E_{mb} T_{na} g\}$, for $n, m \in J$ generated by a single function through translation and modulation is called Weyl-Heisenberg frame. This is a special class of frame in $L^2(R)$, where $T_{na} g(x) = g(x - na)$ and $E_{mb} g(x) = e^{2\pi i m b x} g(x)$ translation and modulation operator respectively.

Necessary condition for Weyl-Heisenberg frames is given as $g \in L^2(R)$ and $a, b > 0$ in R, we say that (g, a, b) generates a Weyl-Heisenberg frames, if $\{E_{mb} T_{na} g\}_{n,m \in J}$ is a frame in $L^2(R)$.

DEFINITION 4.2. Let H be a Hilbert space and for $g \in L^2(R)$ and there exists constants $A, B > 0$, so that $\{E_{mb}T_{na}g\}$ is called Weyl-Heisenberg frames, if satisfies the inequalities, for all $n, m \in J$ such that

$$A\|f\|^2 \leqslant \sum_{m,n \in J} |\langle f, E_{mb}T_{na}g\rangle|^2 \leqslant B\|f\|^2, \text{ for all } f \in H \tag{4.2}$$

where a and b are frame parameters.

Now, we have provided a special operator to define the Rotor frame in Hilbert space in the next section.

5. Rotor frames in Hilbert space

In this section, we have investigated Rotor frames which is a special class of frames in finite dimensional Hilbert space $L^2(R) \in H$ of the form $\left\{R_{n\phi}T_{m\tau}g : n, m \in J\right\}$ which is a generator by a single function through $R_{n\phi}$ and $T_{m\tau}$ are rotation and translation operator respectively.

The frame operator of the Rotor frame commutes with involved $R_{n\phi}$ and $T_{m\tau}$ and satisfies the condition of positive, invertible and bounded linear operator. These two operators are given by $R_{n\phi}g(n) = g(n)e^{i\phi_n}$ and $T_{m\tau}g(m) = g(m-\tau)$, where

$$\begin{aligned}\phi_n &= [H_1, H_2]\\ &= C^{-1}\{\sup|x,y| : x \in H_1\Theta H_2, y \in H_2\Theta H_1\} \text{ and } \|x\| = \|y\| = 1.\end{aligned}$$

and H_1 and H_2 are subspace of Hilbert space. The cosine of the angle of two subspaces is denoted by $C[H_1, H_2]$. If two subspaces are orthogonal, the cosine of the angle is zero. Here, we find the angle between a frame in subspace H_1 and reference frame in subspace H_2 in H. If two subspaces are closed, then its cosine of the angle is less than 1 and its converse is also true.

DEFINITION 5.1. [Rotor frames] Let H be finite dimensional Hilbert space and $\left\{R_{n\phi}T_{m\tau}g : n, m \in J\right\}$ in H is called Rotor frame if there exist constant $A_i > 0$ and $B_i > 0$ and for all $i \in J' \subseteq J$ such that

$$A_i\|f\|^2 \leqslant \sum_{n,m \in J} \left|f, R_{n\phi}T_{m\tau}g\right|^2 \leqslant B_i\|f\|^2, \text{ for all } f \in H \tag{5.1}$$

where $R_{n\phi}$ and $T_{m\tau}$ are rotation and translation operator respectively.

EXAMPLE 5.2. Let H_1 and H_2 be two subspaces of Hilbert space H and here $\phi_n = [H_1, H_2] = C^{-1}(\sup\{|\langle f, g\rangle| : f \in H_1\Theta H_2, g \in H_2\Theta H_1 \text{ and } \|f\| = \|g\| = 1\})$, the cosine of the angle of two subspaces is denoted by $C[H_1, H_2]$.

Let $f_1 = (1,0,0)^T$ and $g_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T$ be any two elements in H, its inner product is $\langle f_1, g_1\rangle = \left\langle (1,0,0)^T, \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T \right\rangle = \frac{1}{\sqrt{2}}$. Therefore, $|\langle f_1, g_1\rangle| = \frac{1}{\sqrt{2}}$.

$$\begin{aligned}\phi_1 = [H_1, H_2] &= C^{-1}\left(\sup\{|\langle f_1, g_1\rangle| : f_1 \in H_1 \Theta H_2, g_1 \in H_2 \Theta H_1\}\right)\\ &= C^{-1}\left(\sup\left\{1/\sqrt{2} : f_1 \in H_1 \Theta H_2, g_1 \in H_2 \Theta H_1\right\}\right)\\ &= (\cos)^{-1}\left(1/\sqrt{2}\right).\end{aligned}$$

Therefore, we get $\phi_1 = \frac{\pi}{4}$.

Suppose $f_2 = (1,0,0)^T$, $g_2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)^T$ in H and its inner product $\langle f_2, g_2\rangle = \left\langle (1,0,0)^T, \left(\frac{1}{2}, \frac{1}{2}, 0\right)^T \right\rangle = \frac{1}{2}$.

Therefore $|\langle f_2, g_2\rangle| = \frac{1}{2}$.

$$\begin{aligned}\phi_2 = [H_1, H_2] &= C^{-1}\left(\sup\{|\langle f_2, g_2\rangle| : f_2 \in H_1 \Theta H_2, g_2 \in H_2 \Theta H_1\}\right)\\ &= C^{-1}\left(\sup\left\{\frac{1}{2} : f_2 \in H_1 \Theta H_2, g_2 \in H_2 \Theta H_1\right\}\right)\\ &= (\cos)^{-1}\left(1/\sqrt{2}\right).\end{aligned}$$

Therefore, we have $\phi_1 = \frac{\pi}{3}$ is angle of rotation.

The following theorem shown that the relation between the ordinary frames and Rotor frames.

THEOREM 5.3. *Let $\{f_n\}_{n=1}^m$ be a frame in H_m, for all $m \in J$ with A and B are optimal frame bounds. If $R_{x_n\theta_n}$ is rotation operator in H_m, then $\{R_{x_n\theta_n}(f_n)\}_{n=1}^m$ is a frame and $R_{x_n\theta_n} S R_{x_n\theta_n}^*$ is frame operator.*

Proof. Since norm of rotation operator is unity, then $R_{x_n\theta_n}$ is non singular and also invertible operator.

$$\begin{aligned}\left\langle R_{x_n\theta_n} S R_{x_n\theta_n}^* f, f\right\rangle &= \left\langle S R_{x_n\theta_n}^* f, R_{x_n\theta_n}^* f\right\rangle\\ &\leqslant \left\langle S R_{x_n\theta_n}^* f, R_{x_n\theta_n}^* f\right\rangle\\ &\leqslant \left\langle R_{x_n\theta_n}^* S f, R_{x_n\theta_n}^* f\right\rangle\\ &\leqslant \left\langle S f, R_{x_n\theta_n} R_{x_n\theta_n}^* f\right\rangle\\ &\leqslant \langle S f, f\rangle.\end{aligned}$$

Hence, as follows

$$\left\langle R_{x_n\theta_n} S R_{x_n\theta_n}^* f, f\right\rangle \leqslant \sum_{n=1}^{m} |\langle f, f_n\rangle|^2. \tag{5.2}$$

Similarly, as follows

$$\left\langle R_{x_n\theta_n} S R_{x_n\theta_n}^* f, f\right\rangle \geqslant \sum_{n=1}^{m} |\langle f, f_n\rangle|^2. \tag{5.3}$$

From (5.2) and (5.3), as follows

$$\left\langle R_{x_n\theta_n} S R^*_{x_n\theta_n} f, f\right\rangle = \sum_{n=1}^{m} |\langle f, f_n\rangle|^2 .$$

Therefore, it satisfies frames inequality condition and $\{R_{x_n\theta_n}(f_n)\}_{n=1}^{m}$ is a frame and $R_{x_n\theta_n} S R^*_{x_n\theta_n}$ is frame operator.

Next, we need to show that rotation operator commutes with frame operator:

$$\begin{aligned} S\left(R_{x_n\theta_n} f\right) &= \sum_{n=1}^{m} \langle R_{x_n\theta_n} f, f_n\rangle f_n \\ &\leqslant \sum_{n=1}^{m} \left\langle f, R^*_{x_n\theta_n} f_n\right\rangle f_n \\ &\leqslant \sum_{n=1}^{m} \langle f, f_n\rangle R^*_{x_n\theta_n} f_n \\ &\leqslant R^*_{x_n\theta_n} \sum_{n=1}^{m} \langle f, f_n\rangle f_n . \end{aligned}$$

Therefore we have,

$$S R_{x_n\theta_n} \leqslant R_{x_n\theta_n} S. \tag{5.4}$$

Similarly we have,

$$S R_{x_n\theta_n} \geqslant R_{x_n\theta_n} S \tag{5.5}$$

from above two inequalities (5.4) and (5.5) and we get $S R_{x_n\theta_n} = R_{x_n\theta_n} S$.

Hence, the proof is completed. □

The following corollary discussed that sum of the spectral values and angle of rotation of the signal in H.

COROLLARY 5.4. *If $R_{x_n\theta_n}$ is a rotation operator in H_m and S is frame operator of the frames $\{f_n\}_{n=1}^{m}$ in H_m which is real or complex finite Hilbert space, then $\left\|R^*_{x_n\theta_n} S R_{x_n\theta_n} f_n\right\|^2 = TraceS$ and $\theta = cos^{-1}\left(\frac{\|R^*_{x_n\theta_n} S R_{x_n\theta_n} f_n\|^2 - 1}{2}\right)$, where θ is rotation angle in R.*

Proof. Let $f = e_i$, product of Eigen values of rotation operator is unity and it has been used Rotation matrix theorem. Hence the proof is completed. □

THEOREM 5.5. *Let $R_{x_n\theta_n}$ be a rotation operator in H_m and H_m be real or image finite Hilbert space, then*

(i) $R_{x_n\theta_n}$ is orthogonal.

(ii) If $\{f_n\}_{n=1}^{m}$ is orthogonal vector in M_m and for all $R_{x_n\theta_n}$ in $R^{3\times 3} \in H_m$, then $\sum_{n=1}^{m} \|R_{x_n\theta_n}\|^2 \leqslant 3\|f_n\|^2$ it satisfies the condition of Bessel's sequence with upper bound 3.

THEOREM 5.6. *Let* $\{g_n\}_{n=1}^{m}$ *be a frames in* H_m *with lower and upper frame bounds are* A *and* B *respectively. If* $R_{x_n\theta_n}$ *and* $T_{m\tau}$ *are rotation and translation operator in* H_m*, then* $R_{t_m}(g_n,x_n,\theta_n) = \left\{\left(R_{x_n\theta_n/T_{m\tau}}\right)g_n\right\}_{n\in J}$ *is Rotor frame in* H_m.

Proof. Let $R_{t_m}(g_n,x_n,\theta_n) = \{f_n\}_{n\in J}$ and $\{g_n\}_{n=1}^{m}$ is a frames in H_m, then,

$$Sf = \sum_{n=1}^{m} \langle f, g_n\rangle g_n, \ \text{ for all } \ f \in H_m$$

and

$$\left|\left\langle f, \left(R_{x_n\theta_n/T_{m\tau}}\right)g_n\right\rangle\right| \leqslant \|f\| \left\|\left(R_{x_n\theta_n/T_{m\tau}}\right)g_n\right\|$$

and if $B_n = \left\|\left(R_{x_n\theta_n/T_{m\tau}}\right)g_n\right\|^2$, then,

$$\sum_{n\in J} |\langle f, f_n\rangle|^2 \leqslant B_n\|f\|^2, \ \text{ for all } \ f \in H_m. \tag{5.6}$$

For all $f \in H_m$,

$$\begin{aligned}
A\|f\|^2 &= \langle sf, f\rangle \\
&= \sum_{n=1}^{m} |\langle f, g_n\rangle|^2 \\
&\leqslant k\sum_{n=1}^{m} \left|\left\langle f, \left(R_{x_n\theta_n/T_{m\tau}}\right)g_n\right\rangle\right|^2 \\
A\|f\|^2 &\leqslant k\sum_{n=1}^{m} \left|\left\langle f, \left(R_{x_n\theta_n/T_{m\tau}}\right)g_n\right\rangle\right|^2 \\
\frac{A}{k}\|f\|^2 &\leqslant k\sum_{n=1}^{m} \left|\left\langle f, \left(R_{x_n\theta_n/T_{m\tau}}\right)g_n\right\rangle\right|^2.
\end{aligned}$$

Therefore,

$$A_n\|f\|^2 \leqslant k\sum_{n=1}^{m} \left|\left\langle f, \left(R_{x_n\theta_n/T_{m\tau}}\right)g_n\right\rangle\right|^2 \tag{5.7}$$

from above two inequalities (5.6) and (5.7), such that

$$A_n\|f\|^2 \leqslant k\sum_{n=1}^{m} \left|\left\langle f, \left(R_{x_n\theta_n/T_{m\tau}}\right)g_n\right\rangle\right|^2 \leqslant B_n\|f\|^2.$$

Therefore $\left\{\left(R_{x_n\theta_n/T_{m\tau}}\right)g_n\right\}_{n\in J}$ is a Rotor frames in H_m, where A_n and B_n are lower and upper Rotor frame bounds. □

5.1. What is the necessary and sufficient condition for Rotor frames in H?

REMARK. Converse of above theorem is true when $\left(R_{x_n\theta_n/T_{m\tau}}\right)$ is orthogonal and determinant of rotation operator is 1 and we have $R_{x_n\theta_n/T_{m\tau}}$ is orthogonal if and only if $R^{-1}_{x_n\theta_n/T_{m\tau}} = R^{T}_{x_n\theta_n/T_{m\tau}}$.

In finite dimensional space, any rotation can be expressed as combination of finite rotation about their axis. The rows (or columns) of rotation are orthogonal vectors and rotation preserves the length of the vector and angle between two vectors.

The Rotor frames concept is effectively used in robot's spatial localization. In order to locate a robot in space, it is needed to have such mathematical tool that allows the space localization of its elements or sequences. In two dimension and three dimension, the positioning have two and three degree of freedom respectively. Therefore elements or sequences position is defined by two linear independent components in two dimension space. In three dimension space, we need to use three linear independent components. The more Instinctive form used to specifying the position of elements is the Cartesian coordinate systems.

An element completed is defined in the space through its position. In the case of a robot, its end link position and indicate their orientation which is defined by three linear independent components. Here, rotation operator has been used to transform coordinates of one system to other system. The Euclidean transformation is defined by $f_n = Rg_n + T$ and it becomes the following form

$$f_n = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} g_n$$

The rotation and translation operator have been used in Robot spatial localization for its movement in various angles. Rotor frames and these two operator has been used for accurate activities of Robot. For locate a robot in space it is required to have a mathematical concept that allows the space localization of its sequence.

The positioning have two degrees of freedom in two dimensional space and therefore a point's position will be defined by two linear independent vector. In the case of a three-dimensional space it will be required to use three linear independent vector and so on. In this section, the rotation matrix and the homogeneous transformation matrix concept has been used to find robot's localization.

NUMERICAL EXAMPLE 5.7. A system $OUVW$ has been rotated 90° around the OX axis and then translated by vector $(4,-2,6)$ which regards to system $OXYZ$. Let $g_n = (-1,2,-5)^T$ be coordinates (*vector*) and the equation has been used to obtain rotation and translated coordinates as follows:

$$f_n = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 8 \\ 1 \end{bmatrix}.$$

In this way, it has been observed that rotation is followed by translation and vice versa which then calculates the coordinate vectors using the concept of translation followed by rotation, but, both are not equal in finite dimensional space. To obtain reconstruction the original coordinates or position which has used rotation operator R to reach the coordinate from another coordinate in the finite dimensional space in H_m, It has been used following way, for all $n \in N$ and R is rotation operator in H_m such that $R^n_{\theta_n} f_m = f_m$, for all $\theta_n = \frac{2\pi}{n}$. Here, n depends upon the angle which number of rotations in finite dimensional space.

6. Signal reconstruction theorem in Hilbert space

Next, we have discussed that how do obtain the energy of signal and also provided the reconstruction of filtered signal. The two orthogonal frames $F = (f_n) \in H_1$ and $G = (g_n) \in H_2$,for all $n \in J$ with analysis and synthesis operator has been vividly discussed.

The application of frames inequalities in communication systems have been provided in this section. Suppose, we denoted the original signal $\tau \in H_1$ and garbage vector as $\sigma \in H_2$. Here analysis operator T^* is ad joint operator of T which generated the keys in this system. Here, the analysis operator is assumed as the tensor product of finite number of matrices. The analysis operator can be formed from just first row of entries in each Hadamard array which is used to minimize the size of the key. However, to recover the original signal, it is essential that the recipient of the signal with noise to know the analysis operator T_{f_n} and synthesis operator ${T^*}_{f_n}$. Composition of analysis and synthesis operator is denoted by $S = {T^*}_{f_n} T_{f_n}$. For encoding the original signal, the sender multiplies the original signal τ by T_{f_n} and add the garbage vector g to generated the noisy signal $h_{n,m} = T_{f_n}\tau + T_{g_n}\sigma \in H$ and the reconstruction technique which is vulnerable to a known original signal and it has been transformed or encoded the information using analysis operator. We want to reconstruct the original information from the noisy signal which used synthesis operator.

The energy and reconstructed the original information using the frame operator have been discussed in the following theorem.

THEOREM 6.1. [Reconstruction theorem] *If S is invertible and self adjoint operator in Hilbert space H and $\{f_n\}_{n\in J}$ be Parseval frame in H, the following inequalities are satisfies*

1. *If $h_{n,m} = SR^T_{x_n\theta_n}$ and g are frames and garbage vector in H, then*

$$\|p\| \leqslant \sqrt{\|h_{n,m}\|^2 - \|g\|^2}. \tag{6.1}$$

2. *If ${T^*}_{f_n}$ is rotation operator and Parseval frames f_n in H, then*

$$ {T^*}_{f_n}(h_{n,m}) \leqslant \tau, \tag{6.2}$$

where $T_{f_n} = R^T_{x_n\theta_n}$, τ is the original signal and ${T^*}_{f_n}(h_{n,m})$ is filtered signal in H.

Proof. Since H_1 and H_2 are closed subspaces of Hilbert space H which is orthonormal and theorem 5.3. Suppose for all $\tau \in H_1$ and (garbage vector) $\sigma \in H_2$ and by using synthesis operator and adding some garbage vector with the original information we have noisy signal that is $h_{n,m} = T_{f_n}\tau + T_{g_n}\sigma \in H$. We proved that $h_{n,m}$ is also Parseval frames in H as follows, As we know S is frame operator in H and it becomes $\langle Sf, f\rangle \leqslant \sum_{m,n\in N} |\langle f, h_{n,m}\rangle|^2$ and ${T^*}_{f_n}T_{g_n} = \begin{cases} I & \text{if } f_n = g_n, \\ 0 & \text{if } f_n \neq g_n. \end{cases}$ A vector is removed from Parseval frames $h_{n,m}$ (also frame) that is exact frame and we have $E_{n,m} = h_{n,m} - T_{g_n}\sigma$ is also frames in H.

Table 1: *Wavelength (in meter) for Sampling Frequency to speech signal*

S.NO	$F_c Hz$	$F_s Hz$	Wavelength (in meter)
1	625	1308	2.29×10^5
2	625	1418	2.12×10^5
3	645	1428	2.10×10^5
4	885	1900	1.58×10^5
5	965	2000	1.50×10^5
6	1165	3000	1.00×10^5

Table 2: *Lower and upper sideband frequencies for speech signal*

S.NO	$F_c Hz$	$F_s Hz$	$\lvert F_c - kF_s \rvert$	$\lvert F_c + kF_s \rvert$
1	625	1308	683	1933
2	625	1418	128	2708
3	645	1428	567	3423
4	885	1900	1640	5440
5	965	2000	2825	6825
6	1165	3000	3990	9990

Since $\{f_n\}_{n\in J}$ is Parseval frames, $S = T^*{}_{f_n} T_{f_n} = I$ and we have, the filtered information is reconstructed using the operator which less than or equal to the original information τ.

Hence, the proof of the above Theorem 6.1 is completed. □

Suppose, we would like to transmit the signal $f(t) = t^3 sinc(t-2)$, for all $t \in [-4,4]$ and we have good representation of signal. It has been used by the rotation operator to modulate the original signal and reconstruct the original signal using the operator as shown in following figures. The information signal is shown in the top of all figures and encoded signal is displayed in the middle of every figure, then the reconstruct signal or filter signal wave form is displayed in the bottom of all figures with different frequencies and it has been reduced the reconstruction error by using the operator.

Modulation is enormously required in communication systems due to the following motivation that is practical antenna length, operating range and wireless communication. The energy of signal depends upon its frequency and the greater frequency of the signal and greater energy possessed by it. If the speech signal frequency is small, then these cannot be transmitted over large distance of radiated directly into space. The technique of changing frequency of a carrier wave in accordance with the intensity of the signal. Amplitude, frequency and phase modulations are three type of the modulation. We have used frequency modulation to speech signal with power spectral density in various frequencies.

We conclude that the Rotor frames, rotation and translation operators have been effectively used in communication system in finite dimensional Hilbert space and also used in signal reconstruction and spectrum analysis in communication systems by using rotation and translation operators in Hilbert space.

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