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A PRIORI AND POSTERIORI ERROR ESTIMATES OF LEGENDRE GALERKIN SPECTRAL METHODS FOR GENERAL ELLIPTIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, the Legendre Galerkin spectral method is applied to solve the constrained optimal control problems governed by general elliptic equations. Under some reasonable assumptions, by using the orthogonal projection operator, we derive a priori error estimates for the spectral approximation of optimal control problems. Then, we obtain a posteriori error estimates for both the state and the control approximation, where we use the L^2-norm for estimating the control approximation error, and the H^1-norm or L^2-norm for the state and co-state approximation error. Finally, some numerical experiments are presented to test our theoretical results.

1. Introduction

During the past decades, optimal control problems have wide applications in the operation of physical, social, economic processes, and other fields. Therefore, the study of optimal control problems have received considerable attention. Finite element methods seem to be the most popular used numerical method in solving optimal control problems (see, e.g., [17, 21, 20]). Spectral method as a kind of numerical method also can be used to approximate the solutions of optimal control problems (see, e.g., [22, 13, 8, 4]). Meanwhile, other numerical methods, such as finite volume method, mixed finite element method etc., have also been applied to approximate the solutions of optimal control problems (see, e.g., [3, 5, 6, 9, 2, 11, 7, 18]).

Spectral method can provide very accurate approximations with a relatively small number of unknowns when the solutions are smooth by employing global polynomials functions. Recently, due to this advantage, spectral method has attracted thousands of scholars and researchers. There are some literatures to study the optimal control problems governed by partial differential equations. In [12], the spectral method has

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been used to approximate unconstrained optimal control problems. From [8, 4], we knew that the authors derived a priori and posterior error estimates for the spectral approximation of constrained optimal control problems. The authors investigated the Legendre Galerkin spectral approximation of elliptic optimal control problems with integral state and control constraints in [13]. The goals of this paper is to establish a priori error estimate and a posteriori error estimates for the Legendre Galerkin spectral methods for general elliptic optimal control problems.

In this paper, we study the integral constraint optimal control problems, where the objective function is general functional. Under some reasonable assumptions, we obtain an optimal a priori error estimate and a posteriori error estimates for general optimal control problems governed by elliptic equation. As far as I known, the authors have obtained a priori and a posteriori error estimates for the Legendre-Galerkin spectral methods for quadratic elliptic optimal control problems in [8], where the objective functional is quadratic functional $\min\limits_{u\in K}\left\{\frac{1}{2}\int_\Omega(y-y_0)^2+\frac{1}{2}\int_\Omega u^2\right\}$. Moreover, the elliptic equations in this paper are much more general than the equations in [8].

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|v\|_{W^{m,p}(\Omega)}$ given by $\|v\|_{W^{m,p}(\Omega)}=\sum\limits_{|\alpha|\leqslant m}\|D^\alpha v\|^p_{L^p(\Omega)}$, and the semi-norm $|\, v\,|_{W^{m,p}(\Omega)}=\sum\limits_{|\alpha|=m}\|D^\alpha v\|^p_{L^p(\Omega)}$. We set $W_0^{m,p}(\Omega)=\{v\in W^{m,p}(\Omega): v\,|_{\partial\Omega}=0\}$. For $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $H_0^m(\Omega)=W_0^{m,2}(\Omega)$. As usual, we use $(\cdot,\cdot)$ to denote the $L^2(\Omega)$-inner product.

Now, we will discuss the following general elliptic optimal control problems:

$$\min_{u\in K}\{g(y)+j(u)\}, \tag{1.1}$$

$$-\mathrm{div}(A\nabla y)=f+Bu,\ \text{ in }\Omega,\quad y|_{\partial\Omega}=0, \tag{1.2}$$

where $\Omega\in\mathbb{R}^2$ be a convex open domain with a Lipschitz boundary $\partial\Omega$, B is a bounded continuous linear operator, g and j are convex functionals. Furthermore, we assume that the coefficient matrix $A\in W^{2,\infty}(\Omega)$ is a symmetric positive definite matrix and there is a constant $c>0$ satisfying for any vector $\mathbf{X}\in\mathbb{R}^2$, $\mathbf{X}^tA\mathbf{X}\geqslant c\|\mathbf{X}\|^2_{\mathbb{R}^2}$. K is a set defined by

$$K=\left\{v\in L^2(\Omega):\int_\Omega v dx\geqslant 0\right\}.$$

This paper embraces seven parts as follows. In Section 2, we present some notations and the Legendre Galerkin spectral approximation for the optimal control problems. In Section 3, we derive a priori error estimate between the exact solution and the spectral approximation. In Section 4, we show the L^2-H^1 posteriori error estimates for both the state and the control approximation. And in Section 5, we obtain the L^2-L^2 posteriori error estimates. Next, two numerical experiments are presented to test our theoretical results in the Section 6. Finally, we give a conclusion and some possible future work in the Section 7.

2. Legendre Gelerkin spectral approximation

In this section, we study Legendre Gelerkin spectral approximation of the general elliptic optimal control problems (1.1)–(1.2). We first introduce a weak formulation of the problem. Let $V = H_0^1(\Omega)$ and $U = L^2(\Omega)$. Let

$$a(y,w) = \int_\Omega (A\nabla y)\cdot(\nabla w)dx, \quad \forall y,w \in V,$$
$$(u,v)_U = \int_\Omega uvdx, \quad \forall (u,v) \in U \times U.$$

Then the standard weak formulation for the state equation reads: find $y(u) \in V$ such that

$$a(y(u),w) = (f + Bu, w), \quad \forall w \in V.$$

Therefore, the optimal control problems (1.1)–(1.2) can be restated as: find (y,u) such that

$$\min_{u\in K} \{g(y) + j(u)\}, \tag{2.1}$$
$$a(y(u),w) = (f + Bu, w), \quad \forall w \in V. \tag{2.2}$$

From [16], we can know that the optimal control problems (2.1)–(2.2) has a unique solution (y^*,u^*) and (y^*,u^*) is the solution if and only if there is a co-state $p^* \in V$ such that the triplet (y^*,p^*,u^*) satisfies the following optimality conditions:

$$a(y^*,w) = (f + Bu^*, w), \quad \forall w \in V, \tag{2.3}$$
$$a(q,p^*) = (g'(y^*),q), \quad \forall q \in V, \tag{2.4}$$
$$(j'(u^*) + B^*p^*, v - u^*)_U \geqslant 0, \quad \forall v \in K \subset U. \tag{2.5}$$

We use Legendre Galerkin spectral method to investigate the spectral approximation of the optimal control problems (2.1)–(2.2), and assume that $\Omega = (-1,1)^2$. Firstly, let us introduce some basic notations which will be used in the sequel. For x_i, $i = 1,2$, we denote by $L_r(x_i)$ the rth degree Legendre polynomial in the variable x_i, and we set

$$X_N^i = \text{span}\{L_0(x_i),\ L_1(x_i),\ \cdots,\ L_N(x_i)\},$$

where $N \geqslant 0$ is an integer. We define a product space such as

$$X_N = \prod_{i=1}^{2} X_N^i.$$

We introduce the finite dimensional spaces $V_N = X_N \bigcap V$, $U_N = X_N \bigcap U$, and $K_N = X_N \bigcap K$. C and c denotes a general positive constant independet of N.

Then the Legendre Galerkin approximation for optimal control problems are:

$$\min_{u_N \in K_N \subset U_N} \{g(y_N) + j(u_N)\}, \tag{2.6}$$

$$a(y_N, w_N) = (f + Bu_N, w_N), \quad \forall w_N \in V_N \subset H_0^1(\Omega). \tag{2.7}$$

This is a finite dimensional optimization problem and may be solved by existing mathematical programming methods such as the steepest decent method, the conjugate gradient method, the trust domain method, and the sequential quadratic programming method.

It follows that the optimal control problems have one solution (y_N^*, u_N^*), and (y_N^*, u_N^*) is the solution if and only if there is a co-state $p_N^* \in V_N$ such that the triplet (y_N^*, p_N^*, u_N^*) satisfies the following optimality conditions:

$$a(y_N^*, w_N) = (f + Bu_N^*, w_N), \quad \forall w_N \in V_N, \tag{2.8}$$
$$a(q_N, p_N^*) = (g'(y_N^*), q_N), \quad \forall q_N \in V_N, \tag{2.9}$$
$$(j'(u_N^*) + B^* p_N^*, v_N - u_N^*)_U \geqslant 0, \quad \forall v_N \in K_N. \tag{2.10}$$

Let $y(u)$ and $y_N(u_N)$ be the solutions of (2.2) and (2.7) respectively. Set

$$J(u) = \{g(y) + j(u)\},$$

$$J_N(u_N) = \{g(y_N) + j(u_N)\}.$$

We show that u_N^* converges to u^* in spectral accuracy provided the data are sufficient smooth. Define $J(\cdot)$ and $J_N(\cdot)$ as before, it can be shown that (see [16]):

$$(J'(u^*), v)_U = (j'(u^*) + B^* p^*, v)_U, \tag{2.11}$$
$$(J_N'(u_N^*), v)_U = (j'(u_N^*) + B^* p_N^*, v)_U, \tag{2.12}$$
$$(J'(u_N^*), v)_U = (j'(u_N^*) + B^* p(u_N^*), v)_U, \tag{2.13}$$

where B^* is the adjoint operator of B and $(y(u_N^*), p(u_N^*))$ is the solution of the auxiliary equation.

$$a(y(u_N^*), w)_U = (f + Bu_N^*, w), \quad \forall w \in V = H_0^1(\Omega), \tag{2.14}$$
$$a(q, p(u_N^*))_U = (g'(y(u_N^*)), q), \quad \forall q \in V = H_0^1(\Omega). \tag{2.15}$$

It is clearly that $J(\cdot)$ is uniformly convex. Then there is a $c > 0$ independent of N, such that

$$(J'(u^*) - J'(u_N^*), u^* - u_N^*)_U \geqslant c\|u^* - u_N^*\|_{L^2(\Omega)}^2, \tag{2.16}$$

where u^* and u_N^* are the solutions of (2.1)–(2.2) and (2.6)–(2.7), respectively.

For any $u \in L^2(\Omega)$, we define the orthogonal projection operator $P_N : L^2(\Omega) \to U_N$ which satisfies

$$(P_N u - u, w_N) = 0, \quad \forall w_N \in U_N.$$

From [10], we can know that

$$P_N u = \sum_{\max\limits_{1 \leqslant i \leqslant 2}\{k_i\} \leqslant N} \hat{u}_k \phi_k, \quad \hat{u}_k = \prod_{i=1}^{2} \left(k_i + \frac{1}{2}\right) \int_\Omega u(x) \phi_k(x) dx,$$

where $k=(k_1,k_2)$ and $\phi_k(x)=L_{k_1}(x_1)L_{k_2}(x_2)$.

For any $u\in H_0^1(\Omega)$, $\quad P_{1,N}^0:\ H_0^1(\Omega)\to V_N$ is defined by

$$\int_\Omega \nabla(u-P_{1,N}^0u)\cdot\nabla w dx=0,\quad \forall w\in V_N.$$

The following Lemma is important for deriving a priori error estimate and a posteriori error estimates of residual type. It can be found in [10].

LEMMA 2.1. *For all $u\in H^m(\Omega)$ $(m\geqslant 0)$, we have*

$$\|u-P_Nu\|_{H^l(\Omega)}\leqslant CN^{\sigma(l)-m}\|u\|_{H^m(\Omega)},\quad 0\leqslant l\leqslant m,$$

where $\sigma(l)=0$ if $l=0$, and $\sigma(l)=2l-\frac{1}{2}$ for $l>0$. If $u\in H_0^1(\Omega)\cap H^m(\Omega)$, $m\geqslant 1$, then we have

$$\|u-P_{1,N}^0u\|_{H^\mu(\Omega)}\leqslant CN^{\mu-m}\|u\|_{H^m(\Omega)},\quad 0\leqslant\mu\leqslant 1.$$

3. An optimal a priori error estimates

In this section, we will discuss a priori error estimate for the spectral approximation of optimal control problems. By using Lemma 2.1, we can derive the following result.

THEOREM 3.1. *Let (y^*,p^*,u^*) and (y_N^*,p_N^*,u_N^*) be the solutions of (2.3)–(2.5) and (2.8)–(2.10), respectively. Assume that g' and j' are Lipschitz continuous in a neighborhood of y^* and the solution (y^*,p^*,u^*) is sufficient regular. Then for any integer $k>0$ there exists $C>0$, independent of N, such that*

$$\|u^*-u_N^*\|_{L^2(\Omega)}+\|y^*-y_N^*\|_{H^1(\Omega)}+\|p^*-p_N^*\|_{H^1(\Omega)}\leqslant CN^{-k},\quad \forall k\in\mathbb{Z}^+. \tag{3.1}$$

Proof. For any $v_N\in K_N$, from (2.5), (2.10), (2.11), and (2.13), and the Schwartz inequality, we obtain

$$\begin{aligned}
c\|u^*-u_N^*\|^2_{L^2(\Omega)}\leqslant&(J'(u^*),u^*-u_N^*)_U-(J'(u_N^*),u^*-u_N^*)_U\\
=&(j'(u^*)+B^*p^*,u^*-u_N^*)_U+(j'(u_N^*)+B^*p(u_N^*),u_N^*-u^*)_U\\
=&(j'(u^*)+B^*p^*,u^*-v_N)_U+(j'(u^*)+B^*p^*,v_N-u_N^*)_U\\
&+(j'(u_N^*)+B^*p_N^*,u_N^*-v_N)_U+(j'(u_N^*)+B^*p_N^*,v_N-u^*)_U\\
&+(B^*(p_N^*-p(u_N^*)),u^*-u_N^*)_U\\
\leqslant&(j'(u^*)+B^*p^*,v_N-u^*)_U+(j'(u_N^*)-j'(u^*),v_N-u^*)_U\\
&+(j'(u^*)+B^*p^*,v_N-u^*)_U+(B^*(p_N^*-p(u_N^*)),v_N-u^*)_U\\
&+(B^*(p(u_N^*)-p^*),v_N-u^*)_U+(B^*(p_N^*-p(u_N^*)),u^*-u_N^*)_U
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2(j'(u^*)+B^*p^*,v_N-u^*)_U \\
&\quad +\|j'(u_N^*)-j'(u^*)\|_{L^2(\Omega)}\|v_N-u^*\|_{L^2(\Omega)} \\
&\quad +\|p_N^*-p(u_N^*)\|_{L^2(\Omega)}\|v_N-u^*\|_{L^2(\Omega)} \\
&\quad +\|p(u_N^*)-p^*\|_{L^2(\Omega)}\|v_N-u^*\|_{L^2(\Omega)} \\
&\quad +\|p_N^*-p(u_N^*)\|_{L^2(\Omega)}\|u^*-u_N^*\|_{L^2(\Omega)} \\
&\leqslant 2(j'(u^*)+B^*p^*,v_N-u^*)_U+\delta\|j'(u_N^*)-j'(u^*)\|_{L^2(\Omega)}^2 \\
&\quad +C(\delta)\|v_N-u^*\|_{L^2(\Omega)}^2+C(\delta)\|p_N^*-p(u_N^*)\|_{L^2(\Omega)}^2 \\
&\quad +\delta\|p(u_N^*)-p^*\|_{L^2(\Omega)}^2+\delta\|u^*-u_N^*\|_{L^2(\Omega)}^2, \qquad (3.2)
\end{aligned}
$$

where $\delta>0$ is a sufficient small constant. Using the triangle inequality, we have

$$\|y_N^*-y^*\|_{H^1(\Omega)}\leqslant\|y_N^*-y(u_N^*)\|_{H^1(\Omega)}+\|y(u_N^*)-y^*\|_{H^1(\Omega)}, \qquad (3.3)$$
$$\|p_N^*-p^*\|_{H^1(\Omega)}\leqslant\|p_N^*-p(u_N^*)\|_{H^1(\Omega)}+\|p(u_N^*)-p^*\|_{H^1(\Omega)}. \qquad (3.4)$$

We can obtain the following error equation from (2.3) and (2.14):

$$a(y(u_N^*)-y^*,w)=(B(u_N^*-u^*),w), \quad \forall w\in V.$$

Let $w=y(u_N^*)-y^*$ in above equation, we show

$$\|y(u_N^*)-y^*\|_{H^1(\Omega)}\leqslant c\|u_N^*-u^*\|_{L^2(\Omega)}, \qquad (3.5)$$

where $c=c(\Omega)>0$ is the coefficient of the following Poincare inequality

$$\|w\|_{H_0^1(\Omega)}\leqslant C(\Omega)\|\nabla w\|_{L^2(\Omega)},$$

From (2.8) and (2.13), we can prove

$$a(y_N^*-y(u_N^*),w_N)=0, \quad \forall w_N\in V_N. \qquad (3.6)$$

Therefore, $\|y_N^*-y(u_N^*)\|_{H^1(\Omega)}$ can be computed as follows:

$$
\begin{aligned}
\|y_N^*-y(u_N^*)\|_{H^1(\Omega)}^2 &\leqslant Ca(y_N^*-y(u_N^*),y_N^*-y(u_N^*)) \\
&= Ca(y_N^*-y(u_N^*),y_N^*-y(u_N^*)+w_N) \\
&\leqslant C\|y_N^*-y(u_N^*)\|_{H^1(\Omega)}\cdot\inf_{w_N\in V_N}\|y(u_N^*)-w_N\|_{H^1(\Omega)}.
\end{aligned}
$$

By using Lemma 2.1, we obtain

$$
\begin{aligned}
\|y_N^*-y(u_N^*)\|_{H^1(\Omega)} &\leqslant C\inf_{w_N\in V_N}\|y(u_N^*)-w_N\|_{H^1(\Omega)} \\
&\leqslant C\|y(u_N^*)-P_{1,N}^0y(u_N^*)\|_{H^1(\Omega)}\leqslant CN^{-k}, \quad \forall k\in\mathbb{Z}^+,
\end{aligned} \qquad (3.7)
$$

where $P_{1,N}^0$ is the orthogonal projection operator defined in section 2.

For the co-state variable, we can obtain the error equation from 2.4 and 2.15

$$a(q, p(u_N^*) - p^*) = (g'(y(u_N^*) - g'(y^*), q). \tag{3.8}$$

Similarly, assuming that g' is Lipschitz continuous in a neighborhood of y^*, we have

$$\|p(u_N^*) - p^*\|_{H^1(\Omega)} \leqslant c\|y(u_N^*) - y^*\|_{L^2(\Omega)} \leqslant c\|u_N^* - u^*\|_{L^2(\Omega)}. \tag{3.9}$$

Due to (2.9) and (2.15), we have

$$\begin{aligned}\|p(u_N^*) - p_N^*\|_{H^1(\Omega)}^2 \leqslant& C(\|p(u_N^*) - p_N^* - P_N(p(u_N^*) - p_N^*)\|_{H^1(\Omega)}\|p(u_N^*) - p_N^*\|_{H^1(\Omega)}\\ &+\|g'(y(u_N^*)) - g'(y_N^*)\|_{L^2(\Omega)}\|p(u_N^*) - p_N^* - P_N(p(u_N^*) - p_N^*)\|_{L^2(\Omega)}\\ &+\|g'(y(u_N^*)) - g'(y_N^*)\|_{L^2(\Omega)}\|p(u_N^*) - p_N^*\|_{L^2(\Omega)})\\ \leqslant& C(\|p(u_N^*) - p_N^* - P_N(p(u_N^*) - p_N^*)\|_{H^1(\Omega)}\|p(u_N^*) - p_N^*\|_{H^1(\Omega)}\\ &+\|y(u_N^*) - y_N^*\|_{L^2(\Omega)}\|p(u_N^*) - p_N^* - P_N(p(u_N^*) - p_N^*)\|_{L^2(\Omega)}\\ &+\|y(u_N^*) - y_N^*\|_{L^2(\Omega)}\|p(u_N^*) - p_N^*\|_{L^2(\Omega)})\\ \leqslant& CN^{-2k}, \quad \forall k \in \mathbb{Z}^+.\end{aligned} \tag{3.10}$$

Then, applying (3.9)–(3.10) in (3.2) to obtain

$$\begin{aligned}c\|u^* - u_N^*\|_{L^2(\Omega)}^2 \leqslant& 2(j'(u^*) + B^*p^*, v_N - u^*)_U + C(\delta)\|v_N - u^*\|_{L^2(\Omega)}^2\\ &+C(\delta)N^{-2k} + (1+2c)\delta\|u^* - u_N^*\|_{L^2(\Omega)}^2,\end{aligned} \tag{3.11}$$

where $\delta > 0$ is a sufficient small constant. So, we choose $\delta = c/2(1+2c)$ to derive that

$$\|u^* - u_N^*\|_{L^2(\Omega)}^2 \leqslant 2(j'(u^*) + B^*p^*, v_N - u^*)_U + C(\delta)\|v_N - u^*\|_{L^2(\Omega)}^2 + C(\delta)N^{-2k}. \tag{3.12}$$

Now, setting $v_N = P_N u^* \in U_N$ in (3.12), where P_N is the L^2 orthogonal projection operator defined in section 2. We have

$$\begin{aligned}(j'(u^*) + B^*p^*, v_N - u^*)_U =& (j'(u^*) + B^*p^* - P_N(j'(u^*) + B^*p^*), P_N u^* - u^*)_U\\ \leqslant& \|j'(u^*) + B^*p^* - P_N(j'(u^*) + B^*p^*)\|_{L^2(\Omega)}\|P_N u^* - u^*\|_{L^2(\Omega)}\\ \leqslant& CN^{-2k},\end{aligned} \tag{3.13}$$

where $P_N(j'(u^*) + B^*p^*) \in U_N$. And further, we also have

$$(u^* - P_N u^*, v) = 0, \quad \forall v \in P_N,$$

especially letting $v = 1 \in P_N$ we have

$$(u^* - P_N u^*, v) = \int_\Omega (u^* - P_N u^*) dx = 0,$$

thus

$$\int_\Omega P_N u^* dx = \int_\Omega u^* dx \geqslant 0.$$

Then $v_N \in K_N \subset U_N$. It follows from Lemma 2.1 that

$$\|u^* - v_N\|_{L^2(\Omega)} \leqslant CN^{-k}. \tag{3.14}$$

Applying (3.13) and (3.14) in (3.12), we can obtain

$$\|u^* - u_N^*\|_{L^2(\Omega)} \leqslant CN^{-k}. \tag{3.15}$$

Finally, we deduce the error estimates (3.1) from (3.5)–(3.10), (3.13) and (3.15). □

4. $L^2 - H^1$ posteriori error estimates

In this section, we derive a posteriori error estimates for the spectral approximation of optimal control problems. We use the L^2-norm for estimating the control approximation error, and the H^1-norm for the state and co-state approximation error.

The following Theorem is important to obtain the error estimates of the intermediate variable.

THEOREM 4.1. *Let $(y(u_N^*),\ p(u_N^*))$ and $(\ y_N^*,\ p_N^*)$ be the solutions of (2.14)–(2.15) and (2.8)–(2.9), respectively. Assume that g' is Lipschitz continuous in a neighborhood of y^*. Then*

$$\|y(u_N^*) - y_N^*\|_{H^1(\Omega)} + \|p(u_N^*) - p_N^*\|_{H^1(\Omega)} \leqslant C\eta_1, \tag{4.1}$$

where the estimator η_1 is defined as

$$\eta_1 = N^{-1}\|g'(y_N^*) + \Delta p_N^*\|_{L^2(\Omega)} + N^{-1}\|f + Bu_N^* + \Delta y_N^*\|_{L^2(\Omega)}.$$

Proof. Setting $e_y = y(u_N^*) - y_N^*$, and letting $e_y^N = P_{1,N}^0 e_y \in V_N$ be the orthogonal projection of e_y, where $P_{1,N}^0$ is the orthogonal projection operator defined in section 2. Then, using (3.6), (2.8), and (2.14), we have

$$\begin{aligned} c^{-1}\|y(u_N^*) - y_N^*\|_{H^1(\Omega)}^2 \leqslant &(\nabla e_y, \nabla e_y) = (\nabla(e_y - e_y^N), \nabla e_y) \\ = &(f + Bu_N^* + \Delta y_N^*, e_y - e_y^N) \\ \leqslant &CN^{-2}\|f + Bu_N^* + \Delta y_N^*\|_{L^2(\Omega)}^2 + \delta\|y(u_N^*) - y_N^*\|_{H^1(\Omega)}^2. \end{aligned} \tag{4.2}$$

Choosing that $\delta = \frac{1}{2c}$, we have

$$\|y(u_N^*) - y_N^*\|_{H^1(\Omega)}^2 \leqslant CN^{-2}\|f + Bu_N^* + \Delta y_N^*\|_{L^2(\Omega)}^2. \tag{4.3}$$

Similarly, setting $e_p = p(u_N^*) - p_N^*$, and letting $e_p^N = P_{1,N}^0 e_p \in V_N$ be the orthogonal projection of e_p. Then it follows from (2.9) and (2.15) that

$$\begin{aligned}
c^{-1}\|p(u_N^*) - p_N^*\|_{H^1(\Omega)}^2 \leqslant &(\nabla e_p, \nabla e_p) = (\nabla(e_p - e_p^N), \nabla e_p) + (\nabla e_p^N, \nabla e_p)\\
=&(g'(y(u_N^*)) + \Delta p_N^*, e_p - e_p^N) + (g'(y(u_N^*)) - g'(y_N^*), e_p^N)\\
=&(g'(y_N^*) + \Delta p_N^*, e_p - e_p^N) + (g'(y(u_N^*)) - g'(y_N^*), e_p)\\
\leqslant&\|g'(y_N^*) + \Delta p_N^*\|_{L^2(\Omega)} \cdot \|e_p - e_p^N\|_{L^2(\Omega)}\\
&+ (g'(y(u_N^*)) - g'(y_N^*), e_p)\\
\leqslant&CN^{-1}\|g'(y_N^*) + \Delta p_N^*\|_{L^2(\Omega)}\|e_p\|_{H^1(\Omega)}\\
&+ \|g'(y(u_N^*)) - g'(y_N^*)\|_{L^2(\Omega)}\|e_p\|_{L^2(\Omega)}\\
\leqslant&CN^{-2}\|g'(y_N^*) + \Delta p_N^*\|_{L^2(\Omega)}^2 + C\|y(u_N^*) - y_N^*\|_{L^2(\Omega)}^2\\
&+ 2\delta\|e_p\|_{H^1(\Omega)}^2.
\end{aligned} \tag{4.4}$$

Choosing that $\delta = \frac{1}{4c}$, we have

$$\|p(u_N^*) - p_N^*\|_{H^1(\Omega)}^2 \leqslant CN^{-2}\|g'(y_N^*) + \Delta p_N^*\|_{L^2(\Omega)}^2 + C\|y(u_N^*) - y_N^*\|_{L^2(\Omega)}^2. \tag{4.5}$$

Then, we combine (4.3) and (4.5) to derive our estimate (4.1). □

We now prove the following theorem.

THEOREM 4.2. *Let (y^*, p^*, u^*) and (y_N^*, p_N^*, u_N^*) be the solutions of (2.3)–(2.5) and (2.8)–(2.10), respectively. Assume that g' is Lipschitz continuous in a neighborhood of y^*. Then we have that*

$$\|u^* - u_N^*\|_{L^2(\Omega)} + \|y^* - y_N^*\|_{H^1(\Omega)} + \|p^* - p_N^*\|_{H^1(\Omega)} \leqslant C\eta_1, \tag{4.6}$$

where η_1 is defined in Theorem 4.1.

Proof. Let $y(u_N^*)$ and $p(u_N^*)$ be the intermediate variables defined in (2.14) and (2.15). For $v_N = P_N u^* \in X_N$, it follows from (2.11)–(2.13), and (2.16) that

$$\begin{aligned}
c\|u^* - u_N^*\|_{L^2(\Omega)}^2 \leqslant&(J'(u^*), u^* - u_N^*)_U - (J'(u_N^*), u^* - u_N^*)_U\\
\leqslant&-(J'(u_N^*), u^* - u_N^*)_U\\
=&(J_N'(u_N^*), u_N^* - u^*)_U + (J_N'(u_N^*) - J'(u_N^*), u^* - u_N^*)_U\\
\leqslant&(J_N'(u_N^*), v_N - u^*)_U + (J_N'(u_N^*) - J'(u_N^*), u^* - u_N^*)_U\\
=&(J_N'(u_N^*) - J'(u_N^*), u^* - u_N^*)_U\\
=&(B^*(p_N^* - p(u_N^*)), u^* - u_N^*)\\
\leqslant&C\|p(u_N^*) - p_N^*\|_{L^2(\Omega)}\|u^* - u_N^*\|_{L^2(\Omega)}.
\end{aligned} \tag{4.7}$$

Therefore, from Theorem 4.1, we have

$$\|u^* - u_N^*\|_{L^2(\Omega)} \leqslant C\|p_N^* - p(u_N^*)\|_{L^2(\Omega)}. \tag{4.8}$$

Using the triangle inequality, we have

$$\begin{aligned}\|y^* - y_N^*\|_{H^1(\Omega)} &\leqslant \|y^* - y(u_N^*)\|_{H^1(\Omega)} + \|y(u_N^*) - y_N^*\|_{H^1(\Omega)},\\ \|p^* - p_N^*\|_{H^1(\Omega)} &\leqslant \|p^* - p(u_N^*)\|_{H^1(\Omega)} + \|p(u_N^*) - p_N^*\|_{H^1(\Omega)}.\end{aligned} \tag{4.9}$$

By using (3.5), (3.9), (4.8)–(4.9), and Theorem 4.1, we can obtain the result (4.6). □

5. $L^2 - L^2$ posteriori error estimates

In this section, we use the L^2-norm for estimating the control, the state and co-state approximation error. In order to obtain $L^2 - L^2$ posteriori error estimates, we first assume an auxiliary problem, then we use the orthogonal projection operator $P_{1,N}^0$ to get this estimates.

We assume that the auxiliary problem:

$$-\Delta\varphi = f, \quad x \in \Omega, \quad \psi|_{\partial\Omega} = 0, \tag{5.1}$$

possesses the following regularity estimates (see [14], for example):

$$\|\varphi\|_{H^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}. \tag{5.2}$$

Then, we can show that:

THEOREM 5.1. *Let $(y(u_N^*), p(u_N^*))$ and (y_N^*, p_N^*) be the solutions of (2.14)–(2.15) and (2.8)–(2.9), respectively. Assume that g' is Lipschitz continuous in a neighborhood of y^*. Then*

$$\|p(u_N^*) - p_N^*\|_{L^2(\Omega)} + \|y(u_N^*) - y_N^*\|_{L^2(\Omega)} \leqslant C\eta_2, \tag{5.3}$$

where

$$\eta_2 = N^{-2}\|g'(y_N^*) + \Delta p_N^*\|_{L^2(\Omega)} + N^{-2}\|f + Bu_N^* + \Delta y_N^*\|_{L^2(\Omega)}.$$

Proof. Let φ be the solution of (5.1) with $f(x) = y(u_N^*) - y_N^*$, letting $\varphi^N = P_{1,N}^0\varphi \in V_N$ be the orthogonal projection of φ, where $P_{1,N}^0$ is the orthogonal projection operator. Then it follows from (2.8), (2.14), and Lemma 2.1 that

$$\begin{aligned}\|y(u_N^*) - y_N^*\|_{L^2(\Omega)}^2 =& (\nabla\varphi, \nabla(y(u_N^*) - y_N^*))\\ =& (\nabla(\varphi - \varphi^N), \nabla(y(u_N^*) - y_N^*)) + (\nabla\varphi^N, \nabla(y(u_N^*) - y_N^*))\\ =& (f + Bu_N^* + \Delta y_N^*, \varphi - \varphi^N)\\ \leqslant& CN^{-2}\|f + Bu_N^* + \Delta y_N^*\|_{L^2(\Omega)} \cdot \|\psi\|_{H^2(\Omega)}\\ \leqslant& CN^{-2}\|f + Bu_N^* + \Delta y_N^*\|_{L^2(\Omega)} \cdot \|f\|_{L^2(\Omega)}.\end{aligned} \tag{5.4}$$

Then

$$\|y(u_N^*) - y_N^*\|_{L^2(\Omega)} \leqslant CN^{-2}\|f + Bu_N^* + \Delta y_N^*\|_{L^2(\Omega)}. \tag{5.5}$$

Similarly, let φ be the solution of (5.1) with $f(x) = p(u_N^*) - p_N^*$, let $\varphi^N = P_{1,N}^0 \varphi \in V_N$ be the orthogonal projection of φ. Then it follows from (2.9), (2.14), and Lemma 2.1 that

$$\begin{aligned}
\|p(u_N^*) - p_N^*\|_{L^2(\Omega)}^2 =& (\nabla\varphi, \nabla(p(u_N^*) - p_N^*)) \\
=& (\nabla(\varphi - \varphi^N), \nabla(p(u_N^*) - p_N^*)) + (\nabla\varphi^N, \nabla(p(u_N^*) - p_N^*)) \\
=& (g'(y(u_N^*)) + \Delta p_N^*, \varphi - \varphi^N) + (g'(y(u_N^*)) - g'(y_N^*), \varphi^N) \\
=& (g'(y_N^*) + \Delta p_N^*, \varphi - \varphi^N) + (g'(y(u_N^*)) - g'(y_N^*), \varphi) \qquad (5.6) \\
\leqslant& C\Big(N^{-2}\|g'(y_N^*) + \Delta p_N^*\|_{L^2(\Omega)} + \|y(u_N^*) - y_N^*\|_{L^2(\Omega)}\Big) \cdot \|\varphi\|_{H^2(\Omega)} \\
\leqslant& C\Big(N^{-2}\|g'(y_N^*) + \Delta p_N^*\|_{L^2(\Omega)} + \|y(u_N^*) - y_N^*\|_{L^2(\Omega)}\Big) \cdot \|f\|_{L^2(\Omega)}.
\end{aligned}$$

So that

$$\|p(u_N^*) - p_N^*\|_{L^2(\Omega)} \leqslant CN^{-2}\|g'(y_N^*) + \Delta p_N^*\|_{L^2(\Omega)} + C\|y(u_N^*) - y_N^*\|_{L^2(\Omega)}. \qquad (5.7)$$

Thus (5.5)–(5.6) implies (5.3). □

According to above Theorem, we can derive the following Theorem.

THEOREM 5.2. *Let (y^*, p^*, u^*) and (y_N^*, p_N^*, u_N^*) be the solutions of (2.3)–(2.5) and (2.8)–(2.10) respectively. Assume that g' is Lipschitz continuous in a neighborhood of y^*. Then, we have*

$$\|u^* - u_N^*\|_{L^2(\Omega)} + \|y^* - y_N^*\|_{L^2(\Omega)} + \|p^* - p_N^*\|_{L^2(\Omega)} \leqslant C\eta_2, \qquad (5.8)$$

where η_2 is defined in Theorem 5.1.

Proof. Using the triangle inequality, we have

$$\begin{aligned}
\|y^* - y_N^*\|_{L^2(\Omega)} &\leqslant \|y^* - y(u_N^*)\|_{L^2(\Omega)} + \|y(u_N^*) - y_N^*\|_{L^2(\Omega)}, \\
\|p^* - p_N^*\|_{L^2(\Omega)} &\leqslant \|p^* - p(u_N^*)\|_{L^2(\Omega)} + \|p(u_N^*) - p_N^*\|_{L^2(\Omega)}.
\end{aligned} \qquad (5.9)$$

From (3.5), (5.9), (4.8), and Theorem 5.1, we have

$$\|y^* - y_N^*\|_{L^2(\Omega)} \leqslant C\eta_2, \qquad (5.10)$$

By using (3.9), (5.9), (4.8), and Theorem 5.1, we get

$$\|p^* - p_N^*\|_{L^2(\Omega)} \leqslant C\eta_2, \qquad (5.11)$$

Finally, we can obtain the result (5.8) from (5.10)–(5.11). □

6. Numerical examples

In this section, we will carry out two numerical examples for the one-demensional and two-dimensional cases to demonstrate our theoretical estimates. We consider the following elliptic optimal control problem

$$\min_{u\in K} J(u) = \left\{\tfrac{1}{2}\textstyle\int_\Omega (y-y_0)^2 + \tfrac{1}{2}\int_\Omega u^2\right\}, \tag{6.1}$$

$$-\Delta y = f+u, \text{ in } \Omega, \quad y|_{\partial\Omega} = 0, \tag{6.2}$$

where K is a set defined by

$$K = \left\{ v \in L^2(\Omega) : \int_\Omega v \geqslant 0 \right\}.$$

By using the gradient projection algorithm, we can solve the control problem (6.1)–(6.2), and give the interative scheme as follows:

$$\begin{cases} (u_{n+\frac{1}{2}}, v) = (u_n, v) - \rho_n (J'(u_n), v), & \forall\, v \in K, \\ u_{n+1} = P_k(u_{n+\frac{1}{2}}), \end{cases} \tag{6.3}$$

where ρ_n will be specified later, and the projection operator $P_K : U \to K$ satisfies for any $w \in U$ that $(P_K w - w, P_K w - w) = \min_{u\in K}(u-w, -w)$, which is equivalent to $(P_K w - w, P_K w - w) \geqslant 0$.

Let U_n be the coordinates of u_n in $\mathbb{R}^n$ and $J'(U_n)$ be the gradient of $J(U_n)$, then the matrix form of the above algorithm reads as $U_{n+\frac{1}{2}} = U_n - \rho_n M_n^{-1} J'(U_n)$, where M_n is the preconditioner, for above scheme we have the following convergence result [15].

LEMMA 6.1. *Assume that J' is lipschitz and uniformly monotone in the sense that there are positive constants C, c such that*

$$|J'(u) - J'(v)| \leqslant C\|u-v\|_U, \quad \forall\, u, v \in U,$$
$$(J'(u) - J'(v), u-v) \geqslant c\|u-v\|_U^2, \quad \forall\, u, v \in U.$$

Then there are $0<\delta<1$, $\varepsilon>0$ such that

$$\|u-u_n\| \leqslant \delta^n \|u-u_0\|, \quad n = 0, 1, 3, \cdots,$$

provided $\rho_n \leqslant \varepsilon$.

We will select ρ_n satisfying $0 \leqslant 1+\rho_n(C\rho_n - c) \leqslant \delta$ to guarantee the convergence of the algorithm (6.3). It is not difficult to prove that

$$J(u) = \frac{1}{2}\int_\Omega (y(u)-y_0)^2 + \frac{1}{2}\int_\Omega u^2,$$

satisfies all of the conditions in Theorem 3.1. Thus we can apply the above algorithm to solve the discrete elliptic optimal control problem (2.8)–(2.10). Let U_N and K_N be defined in section 2, and $P_{K_N}: U_N \to K_N$ be the discrete projection operator satisfying for any $w \in U_N$ that

$$(P_{K_N} w - w, P_{K_N} w - w) = \min_{u \in K}(u - w, u - w),$$

which is equivalent to

$$(P_{K_N} w - w, v - P_{K_N} w) \geqslant 0, \quad \forall\, v \in K_N.$$

It follows from Theorem 3.2 that for any $u_N \in U_N$

$$P_{K_N} u_N = -\min\{0, \overline{u_N}\} + u_N.$$

Then, from (6.3), we have

$$u_{n+1} = P_{K_N} u_{n+\frac{1}{2}} = -\min\{0, \overline{u_{n+\frac{1}{2}}}\} + u_{n+\frac{1}{2}}.$$

Thus, we can derive the following algorithm for solving the discrete general linear elliptic control problem (2.8)–(2.10).

Algorithm 1: Let $u_N^0 = 0$, and

$$\begin{aligned}
&(\nabla y_N^n, \nabla w_N) = (u_N^n, w_N), \quad \forall\, w_N \in V_N,\\
&(\nabla q_N, \nabla P_N^n) = (y_N - y_0, q_N), \quad \forall\, q_N \in V_N,\\
&(u_N^{n+\frac{1}{2}}, v_N) = (u_N^n, v_N) - \rho(P_N^n + \alpha(u_N^n - u_0), v_N), \quad \forall\, v_N \in U_N,\\
&u_N^{n+1} = P_K(u_N^{n+\frac{1}{2}}) = -\min\left(0, \overline{u_N^{n+\frac{1}{2}}} + u_N^{n+\frac{1}{2}}\right).
\end{aligned}$$

6.1. One-dimensional case

We now consider the problem (6.1)–(6.2) on domain $\Omega = (-1, 1)$, associated with the exact solutions

$$\begin{aligned}
&u = \pi^2 \sin \pi x, \quad p = \sin 2\pi x,\\
&y = \sin \pi x, \quad f = 0,\\
&y_0 = 4\pi^2 \sin 2\pi x + \sin \pi x.
\end{aligned}$$

We denote $L_n(x)$ be the nth-degree Legendre polynomial [19] and

$$\phi_k(x) = \tfrac{1}{\sqrt{4k+6}}(L_k(x) - L_{k+2}(x)), \quad Y_N = \mathrm{span}\{\phi_i(x)\}_{i=0}^{N-2},$$
$$U_N = \mathrm{span}\{L_i(x);\ i = 0, \cdots, N\}.$$

It can be verified that

$$a_{jk} = (\phi_k', \phi_j') = \begin{cases} 1, & k = j,\\ 0, & k \neq j. \end{cases}$$

$$b_{jk} = b_{kj} = (\phi_k, \phi_j) = \begin{cases} c_k c_j \left(\frac{2}{2j+1} + \frac{2}{2j+5} \right), & k = j, \\ -c_k c_j \frac{2}{2k+1}, & k = j+2, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can present y_N, y_N^*, u_N as follows

$$y_N = \sum_{i=0}^{N-2} y_i \phi_i(x), \quad y_N^* = \sum_{i=0}^{N-2} y_i^* \phi_i(x), \quad u_N = \sum_{i=0}^{N} u_i L_i(x).$$

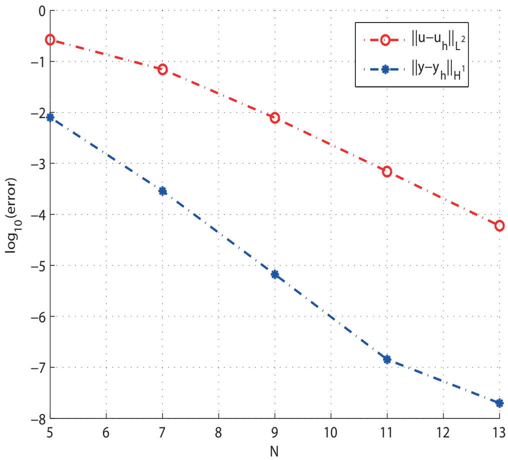

Figure 1: *The exact solutions and its spectral solutions of u in one-dimensional case*

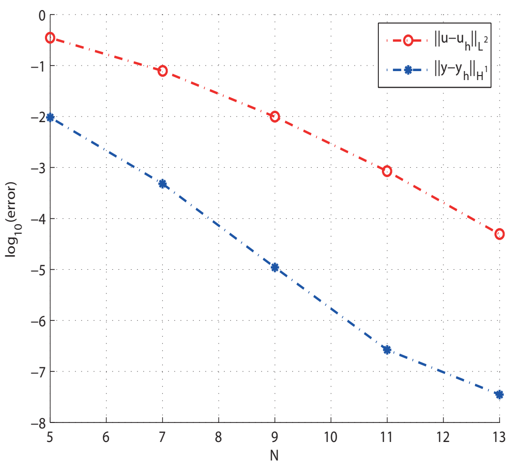

Figure 2: *The exact solutions and its spectral solutions of y in one-dimensional case*

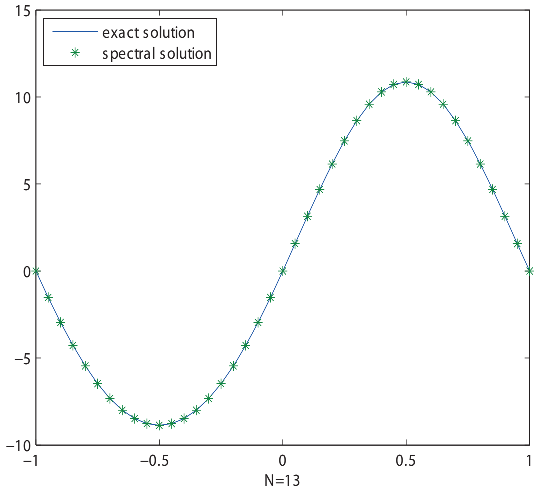

Figure 3: *The errors of different* $N(N = 5,7,9,11,13)$ *base on different norms*

The approximation errors are presented in Table 1 and Figure 3, which show that the errors decrease rapidly. We also plot the exact solutions and the spectral solutions of u and y with $N = 13$ in Figures 1-2. Both Table 1 and Figures 1-2 suggest that the spectral method provides very accurate approximation for the elliptic optimal control problems with a relatively small number of unknowns as the approximated solutions are sufficiently regular.

Table 1: *The values of discretization errors*

N	5	7	9	11	13
$\|u-u_N\|_{L^2}$	2.643E-01	6.982E-02	7.481E-03	6.953E-04	5.970E-05
$\|y-y_N\|_{H^1}$	7.976E-03	2.879E-04	6.658E-06	1.429E-07	1.987E-08

6.2. Tow-dimensional case

By using [8], we can construct the discrete space for state and co-state and the equivalent matrix equations for the algorithm 1. In this example, we consider the optimal control problem (6.1)–(6.2) on domain $\Omega = (-1,1)^2$, associated with the exact solutions

$$
\begin{aligned}
&u = 2\pi^2 \sin \pi x_1 \sin \pi x_2, \quad p = 2\sin \pi x_1 \sin \pi x_2,\\
&y = \sin \pi x_1 \sin \pi x_2, \quad f = 0,\\
&y_0 = 5\sin \pi x_1 \sin \pi x_2.
\end{aligned}
$$

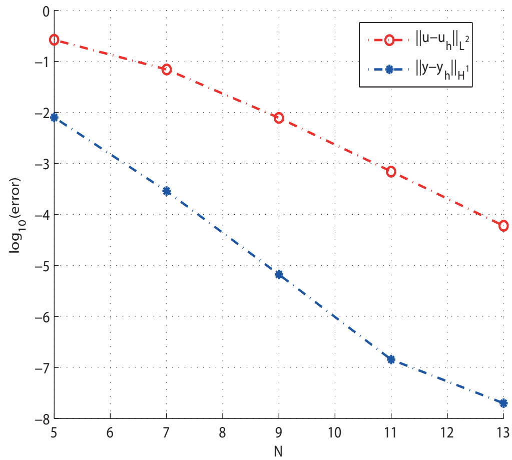

Figure 4: *The errors of different* N $(N=5,7,9,11,13)$ *base on different norms*

Let Y_N and U_N be as follows

$$
\begin{aligned}
Y_N &= \mathrm{span}\{\phi_i(x_1)\phi_j(x_2)\}_{i,j=0}^{N-2},\\
U_N &= \mathrm{span}\{L_i(x_1)L_j(x_2);\ i,j=0,\cdots,N\}.
\end{aligned}
$$

The approximation errors are presented in Table 2 and Figure 4. From the numerical example, we can find that the numerical results demonstrate our theoretical results.

Table 2: *The values of discretization errors in two-dimensional case*

N	5	7	9	11	13
$\|u-u_N\|_{L^2}$	3.489E-01	7.839E-02	9.853E-03	8.467E-04	4.931E-05
$\|y-y_N\|_{H^1}$	9.572E-03	4.795E-04	1.097E-05	2.657E-07	3.515E-08

7. Conclusion and future work

In this paper, we consider a priori error estimate and a posteriori error estimates for the Legendre Galerkin spectral element approximation of optimal control problems governed by general elliptic equations. Under some reasonable assumptions, we obtain a priori error estimate for both the state and the control approximation. Then we also obtain a posteriori error estimates, where use the L^2-norm for estimating the control approximation error, and the H^1-norm for the state and co-state approximation error. Furthermore, we derive the L^2-L^2 error estimate for the control, the state and co-state approximation in L^2-norm. To our best knowledge in the context of optimal control problems, these error estimates for the general elliptic optimal control problems are new.

In future, we shall consider the Legendre Galerkin spectral method for nonlinear parabolic optimal control problems. Furthermore, we shall consider a posteriori error estimates and superconvergence of the Legendre Galerkin spectral element solutions for nonlinear parabolic optimal control problems.

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