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ON UNICITY OF MEROMORPHIC FUNCTIONS CONCERNING THE SHIFTS AND DERIVATIVES

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Abstract. This paper is devoted to studying the sharing value problem for the derivative of a meromorphic function with its shift and q-difference. The results in the paper improve and generalize the recent result due to Qi, Li and Yang [28].

1. Introduction, preliminaries and main results

By a meromorphic function we shall always mean a non-constant meromorphic function in the complex plane. By a constant we shall always mean a complex valued constant. Let k be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. Set $E(a,f) = \{z : f(z) - a = 0\}$, where each a-point of f with multiplicity k is counted k times in the set. If each a-point of f with multiplicity k are only counted once, then we denote the set by $\overline{E}(a,f)$.

Let f and g be two non-constant meromorphic functions. If $E(a,f) = E(a,g)$, then we say that f and g share the value a CM (counting multiplicities); if $\overline{E}(a,f) = \overline{E}(a,g)$, then we say that f and g share the value a IM (ignoring multiplicities). We denote by $E_{k)}(a,f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a,f)$ the set of distinct a-points of f with multiplicities not greater than k. We denote by $N_{k)}(r,1/(f-a))$ the counting function for zeros of $f-a$ with multiplicity less than or equal to k, and by $\overline{N}_{k)}(r,1/(f-a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r,1/(f-a))$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r,1/(f-a))$ the corresponding one for which multiplicity is not counted. We assume that the reader is familiar with the standard definitions and notations used in the Nevanlinna value distribution theory, such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $S(r,f)$ (see [14] page 4, 34 and 42 or [36] page 6).

Around 2001, I. Lahiri introduced the notion of weighted sharing, which measures how close a shared value is to being shared CM or to being shared IM. The definition is as follows.

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DEFINITION 1.1. ([16], page 195) For a complex number $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point with mutiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$ for a complex number $a \in \mathbb{C} \cup \{\infty\}$ we say that f and g share the value a with weight k.

The definition implies that if f and g share a value a with weight k, then z_0 is a zero of $f-a$ with multiplicity $m(\leqslant k)$ if and only if it is a zero of $g-a$ with multiplicity $m(\leqslant k)$ and z_0 is a zero of $f-a$ with multiplicity $m(>k)$ if and only if it is a zero of $g-a$ with multiplicity $n(>k)$, where m is not necessarily equal to n. We write f and g share (a,k) to mean that f and g share the value a with weight k. Clearly if f and g share (a,k) then f and g share (a,p) for all integer p, $0 \leqslant p \leqslant k$. Also we note that f and g share a value a IM or CM if and only if f and g share $(a,0)$ or (a,∞) respectively.

Mermorphic functions sharing values with their derivatives has become a subject of great interest in uniqueness theory. The paper by Rubel and Yang is the starting point of this topic, along with the following.

THEOREM 1.2. ([30], page 101) *Let f be a non-constant entire function. If f and f' share two distinct finite values CM, then $f = f'$.*

Now one may ask the following question: Can we change the number 2 of shared values to 1 in the Theorem 1.2? The function $f = e^{e^z} \int_0^z e^{-e^t}(1-e^t)dt$ from [4] show that the answer is negative. Indeed, clearly f and f' share 1 CM but $f \neq f'$. In a special case, we recall a well-known conjecture by Brück:

CONJECTURE 1.3. ([4], page 22) *Let f be a non-constant entire function such that hyper-order $\rho_2(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$ is not a positive integer or infinity. If f and f' share the finite value a CM, then $\frac{f'-a}{f-a} = c$, where c is nonzero constant.*

The conjecture has been verified in the special cases when $a = 0$ [4], or when f is of finite order [12], or when $\rho_2(f) < \frac{1}{2}$ [7]. Many results have been obtained for this and related topics (See [1], [5], [11], [17], [18], [23]–[27], [31], [32], [34], [35], [37], [39], [41]–[44] and the references therein).

Heittokangas et al. considered analogues of Brück's conjecture for meromorphic functions concerning their shifts, and proved the following theorem.

THEOREM 1.4. ([15], Theorem 1, page 353) *Let f be a meromorphic function of order*

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} < 2$$

and let $c \in C$. If $f(z)$ and $f(z+c)$ share the values $a \in C$ and ∞ CM, then

$$\frac{f(z+c)-a}{f(z)-a} = \tau,$$

for some constant τ.

Since then, many mathematicians considered this topic (See [6], [8], [10], [19]–[22], [29], [38] and the references therein). In 2018, Qi, Li and Yang considered the value sharing problem related to $f'(z)$ and $f(z+c)$, where c is a complex number. They obtained the following result.

THEOREM 1.5. ([28], Theorem 1.5, page 570) *Let f be a non-constant meromorphic function of finite order and $n \geqslant 9$ be an integer. If $[f'(z)]^n$ and $f^n(z+c)$ share $a(\neq 0)$ and ∞ CM, then $f'(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

It is natural to ask whether the f' can be extended to $f^{(k)}$ in Theorem 1.5. Here f^n denotes the nth power of the function f and $f^{(k)}$ stands for the kth derivative of f, where k is a non-negative integer. Considering this question, we prove the following results.

THEOREM 1.6. *Let f be a non-constant meromorphic function of finite order and n be a positive integer. If one of the following conditions is satisfied:*

(I) $[f^{(k)}(z)]^n$ and $f^n(z+c)$ share $(1,2)$, $(\infty,0)$ and $n \geqslant 2k+8$;

(II) $[f^{(k)}(z)]^n$ and $f^n(z+c)$ share $(1,2)$, (∞,∞) and $n \geqslant 2k+7$;

(III) $[f^{(k)}(z)]^n$ and $f^n(z+c)$ share $(1,0)$, $(\infty,0)$ and $n \geqslant 3k+14$;

then $f^{(k)}(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.

If we consider entire function instead of meromorphic function, the counting functions related to the poles of $[f^{(k)}(z)]^n$ and $f^n(z+c)$ can be neglected. Arguing similarly as in Theorem 1.6, we will see that k is not related to the coefficient of $N_{k+1}\left(r,\frac{1}{f}\right)$. So we can obtain the following corollary.

COROLLARY 1.7. *Let f be a non-constant entire function of finite order and $n \geqslant 5$ be an integer. If $[f^{(k)}(z)]^n$ and $f^n(z+c)$ share $(1,2)$, then $f^{(k)}(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

If the shifts $f(z+c)$ in Theorem 1.5 and 1.6 are replaced by q-difference $f(qz)$, where $q \in \mathbb{C} \setminus \{0\}$, we obtain:

THEOREM 1.8. *Let f be a non-constant meromorphic function of zero order and n be a positive integer. If one of the following conditions is satisfied:*

(I) $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1,2)$, $(\infty,0)$ and $n \geqslant 2k+8$;

(II) $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1,2)$, (∞,∞) and $n \geqslant 2k+7$;

(III) $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1,0)$, $(\infty,0)$ and $n \geqslant 3k+14$;

then $f^{(k)}(z) = tf(qz)$, for a constant t that satisfies $t^n = 1$.

COROLLARY 1.9. *Let f be a non-constant entire function of zero order and $n \geqslant 5$ be an integer. If $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1,2)$, then $f^{(k)}(z) = tf(qz)$, for a constant t that satisfies $t^n = 1$.*

2. Some Lemmas

In this section, we present some lemmas which will be needed later on. We will denote by H the following function:

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right),$$

where F and G are non-constant meromorphic functions. From above, it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) those poles of F and G whose multiplicities are different, (iv) zeros of F' which are not the zeros of $F(F-1)$ and zeros of G' which are not the zeros of $G(G-1)$. And we define the following notations which are used in the proof.

$$N_2\left(r,\frac{1}{f}\right)=\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right),$$

where a simple zero point of f is counted once and a multiple zero point of f is counted twice. Let z_0 be a zero of $f-1$ of multiplicity p and a zero of $g-1$ of multiplicity q. We denote by $N_E^{1)}\left(r,\frac{1}{f-1}\right)$ the counting function of those 1-points of f where $p=q=1$; by $N_E^{(2}\left(r,\frac{1}{f-1}\right)$ the counting function of those 1-points of f where $p=q\geqslant 2$; by $N_L\left(r,\frac{1}{f-1}\right)$ the counting function of the 1-points of f whose multiplicities are greater than 1-points of g; each point in these counting functions is counted only once. We are ignoring g in the notations above only because the reader can interpret from the context with which function g we are comparing the function f.

LEMMA 2.1. ([2], Lemma 2.13, page 13) *Let F, G be two non-constant meromorphic functions. If F, G share $(1,2)$ and (∞,k), where $0\leqslant k\leqslant\infty$, and $H\not\equiv 0$, then*

$$T(r,F)\leqslant N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)$$
$$+\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G),$$

where $\overline{N}_*(r,\infty;F,G)$ denotes the reduced counting function of those poles of F whose multiplicities differ from the multiplicities of the corresponding poles of G.

LEMMA 2.2. ([33], Lemma 2, page 108) *Let f be a non-constant meromorphic function, and let $a_1,a_2,\ldots,a_n$ be finite complex numbers, $a_n\neq 0$. Then*

$$T(r,a_nf^n+\cdots+a_2f^2+a_1f+a_0)=nT(r,f)+S(r,f).$$

LEMMA 2.3. ([9], Theorem 2.1, page 109) *Let f be a meromorphic function of finite order $\rho(f)$, and let c be a nonzero constant. Then*

$$T(r,f(z+c))=T(r,f(z))+O(r^{\rho(f)-1+\varepsilon})+O(\log r),$$

for an arbitrary $\varepsilon>0$.

We mention that Lemma 2.3 holds also for $c=0$ as in the case $T(r,f(z+c)) = T(r,f(z))$.

LEMMA 2.4. ([44], Lemma 2.1, page 4) *Let f be a non-constant meromorphic function, p, k be positive integers, then*

$$N_p\left(r,\frac{1}{f^{(k)}}\right) \leqslant N_{p+k}\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f),$$

where $N_p\left(r,\frac{1}{f^{(k)}}\right)$ denotes the counting function of the zeros of $f^{(k)}$ where a zero of multiplicity m is counted m times if $m \leqslant p$ and p times if $m > p$.

We point out that in Lemma 2.4 one obviously has that $\overline{N}\left(r,\frac{1}{f^{(k)}}\right) = N_1\left(r,\frac{1}{f^{(k)}}\right)$.

LEMMA 2.5. ([13], Theorem 2.1, page 465) *Let f be a non-constant meromorphic function of finite order, and let $c \in C$ and $\delta \in (0,1)$. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right) + m\left(r,\frac{f(z)}{f(z+c)}\right) = o\left(\frac{T(r,f)}{r^{\delta}}\right) = S(r,f).$$

LEMMA 2.6. ([39], Lemma 3.3, page 349) *Suppose that two non-constant meromorphic functions F and G share 1 and ∞ IM. Let H be given as above. If $H \not\equiv 0$, then*

$$\begin{aligned} T(r,F)+T(r,G) \leqslant\ & 3\overline{N}(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_E^{1)}\left(r,\frac{1}{F-1}\right) \\ & + 2N_E^{(2}\left(r,\frac{1}{F-1}\right) + 3N_L\left(r,\frac{1}{F-1}\right) + 3N_L\left(r,\frac{1}{G-1}\right) \\ & + S(r,F) + S(r,G). \end{aligned}$$

LEMMA 2.7. ([40], Theorem 1.1, page 538) *Let f be a zero-order meromorphic function, and $q \in \mathbb{C}\setminus\{0\}$. Then*

$$T(r,f(qz)) = (1+o(1))T(r,f(z))$$

and

$$N(r,f(qz)) = (1+o(1))N(r,f(z))$$

on a set of lower logarithmic density 1.

LEMMA 2.8. ([3], Theorem 1.1, page 457) *Let f be a zero-order meromorphic function, and $q \in C\setminus\{0\}$. Then*

$$m\left(r,\frac{f(qz)}{f(z)}\right) = S(r,f)$$

on a set of logarithmic density 1.

3. Proof of Theorem 1.6

Let

$$F = f^n(z+c), \quad G = [f^{(k)}(z)]^n. \tag{1}$$

(I). Suppose $[f^{(k)}(z)]^n$ and $f^n(z+c)$ share $(1,2)$, $(\infty,0)$ and $n \geqslant 2k+8$. Then it follows directly from the assumptions of the theorem that F and G share $(1,2)$ and $(\infty,0)$. Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.1 that

$$\begin{aligned} T(r,F) \leqslant N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) \\ + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G). \end{aligned} \tag{2}$$

According to Lemma 2.2 and Lemma 2.3, we have

$$T(r,F) = nT(r,f(z+c)) + S(r,f) = nT(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f). \tag{3}$$

It is obvious that

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) &= 2\overline{N}\left(r,\frac{1}{f(z+c)}\right) \leqslant 2T(r,f(z+c)) \\ &= 2T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \end{aligned} \tag{4}$$

$$\begin{aligned} \overline{N}(r,F) &= \overline{N}(r,f(z+c)) \leqslant T(r,f(z+c)) \\ &= T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \end{aligned} \tag{5}$$

$$\begin{aligned} \overline{N}_*(r,\infty;F,G) \leqslant \overline{N}(r,F) &\leqslant T(r,f(z+c)) \\ &= T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f). \end{aligned} \tag{6}$$

Since $\overline{E}(\infty,f^{(k)}) = \overline{E}(\infty,f)$, we have

$$\overline{N}(r,G) = \overline{N}(r,[f^{(k)}(z)]^n) = \overline{N}(r,f^{(k)}(z)) = \overline{N}(r,f) \leqslant T(r,f). \tag{7}$$

Lemma 2.4 gives

$$\begin{aligned} N_2\left(r,\frac{1}{G}\right) &= 2\overline{N}\left(r,\frac{1}{f^{(k)}}\right) \leqslant 2N_{k+1}\left(r,\frac{1}{f}\right) + 2k\overline{N}(r,f) + S(r,f) \\ &\leqslant (2+2k)T(r,f) + S(r,f). \end{aligned} \tag{8}$$

Combining (2)–(8), we deduce

$$(n-2k-7)T(r,f) \leqslant O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \tag{9}$$

which contradicts with $n \geqslant 2k+8$. Therefore $H \equiv 0$, that is

$$\frac{F''}{F'} - \frac{2F'}{F-1} = \frac{G''}{G'} - \frac{2G'}{G-1}.$$

By integrating twice to the both sides of the above, we can get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{10}$$

where $A \neq 0$ and B are constants. From (10) we have

$$G = \frac{(B-A)F + (A-B-1)}{BF - (B+1)}. \tag{11}$$

Suppose that $B \neq 0, -1$. From (11), we have

$$\overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) = \overline{N}(r, G). \tag{12}$$

From the second fundamental theorem and Lemma 2.3, we have

$$\begin{aligned} nT(r,f) &= T(r,F) + S(r,f) \leqslant \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) + S(r,f) \\ &\leqslant \overline{N}(r, f(z+c)) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + \overline{N}(r,f) + S(r,f) \\ &\leqslant 3T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \end{aligned} \tag{13}$$

which contradicts with $n \geqslant 2k+8$.

Suppose that $B = -1$. From (11) we have

$$G = \frac{(A+1)F - A}{F}. \tag{14}$$

If $A \neq -1$, we obtain from (14) that

$$\overline{N}\left(r, \frac{1}{F - \frac{A}{A+1}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{15}$$

From the second fundamental theorem, Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned} nT(r,f) &= T(r,F) + S(r,f) \leqslant \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{A}{A+1}}\right) + S(r,f) \\ &\leqslant \overline{N}(r, f(z+c)) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r,f) \\ &\leqslant \overline{N}(r, f(z+c)) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + N_{k+1}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f) \\ &\leqslant (k+3)T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \end{aligned} \tag{16}$$

which contradicts with $n \geqslant 2k+8$. Hence $A=-1$. From (14), we get $FG=1$, that is

$$f^n(z+c)[f^{(k)}(z)]^n = 1. \tag{17}$$

Since $[f^{(k)}(z)]^n$ and $f^n(z+c)$ share $(\infty,0)$, from (17) we get

$$N(r,f^{(k)}) = 0, \quad T(r,f^{(k)}) = T(r,f(z+c)) + S(r,f), \tag{18}$$

and

$$[f^{(k)}(z)]^{2n} = \frac{[f^{(k)}(z)]^n}{f^n(z+c)} = \frac{\frac{[f^{(k)}(z)]^n}{f^n(z)}}{\frac{f^n(z+c)}{f^n(z)}}. \tag{19}$$

From Lemma 2.5 and the logarithmic derivative lemma, we get

$$\begin{aligned}
2nT(r,f^{(k)}) &= T(r,[f^{(k)}]^{2n}) = m(r,[f^{(k)}]^{2n}) + N(r,[f^{(k)}]^{2n}) = m\left(r,[f^{(k)}(z)]^{2n}\right) \\
&= m\left(r,\frac{\frac{[f^{(k)}(z)]^n}{f^n(z)}}{\frac{f^n(z+c)}{f^n(z)}}\right) \leqslant m\left(r,\frac{[f^{(k)}(z)]^n}{f^n(z)}\right) + m\left(r,\frac{f^n(z+c)}{f^n(z)}\right) \\
&\leqslant nm\left(r,\frac{f^{(k)}(z)}{f(z)}\right) + nm\left(r,\frac{f(z+c)}{f(z)}\right) = S(r,f),
\end{aligned}$$

that is

$$T(r,f^{(k)}) = S(r,f). \tag{20}$$

By (18) and (20), we know that

$$T(r,f(z+c)) = T(r,f^{(k)}) = S(r,f), \tag{21}$$

which is a contradiction with Lemma 2.3.

Suppose that $B=0$. From (11), we have

$$G = AF - (A-1). \tag{22}$$

If $A \neq 1$, from (22) we obtain

$$\overline{N}\left(r,\frac{1}{F-\frac{A-1}{A}}\right) = \overline{N}\left(r,\frac{1}{G}\right). \tag{23}$$

From the second fundamental theorem, Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
nT(r,f) &= T(r,F) + S(r,f) \leqslant \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-\frac{A-1}{A}}\right) + S(r,f) \\
&\leqslant \overline{N}(r,f(z+c)) + \overline{N}\left(r,\frac{1}{f(z+c)}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + S(r,f) \\
&\leqslant \overline{N}(r,f(z+c)) + \overline{N}\left(r,\frac{1}{f(z+c)}\right) + N_{k+1}\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f) \\
&\leqslant (k+3)T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f),
\end{aligned} \tag{24}$$

which contradicts with $n \geqslant 2k+8$. Thus $A=1$. From (22) we have $F=G$, that is $f^n(z+c)=[f^{(k)}(z)]^n$. Hence $f^{(k)}(z)=tf(z+c)$, for a constant t with $t^n=1$. We can get the conclusion of Theorem 1.6.

(II). Suppose $[f^{(k)}(z)]^n$ and $f^n(z+c)$ share $(1,2)$, (∞,∞) and $n \geqslant 2k+7$. Then it follows directly from the assumptions of the theorem that F and G share $(1,2)$ and (∞,∞). Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.1 that

$$\begin{aligned} T(r,F) \leqslant\ & N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G) \\ & +\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G). \end{aligned} \tag{25}$$

According to Lemma 2.2 and Lemma 2.3, we have

$$T(r,F)=nT(r,f(z+c))+S(r,f)=nT(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f). \tag{26}$$

It is obvious that

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) &= 2\overline{N}\left(r,\frac{1}{f(z+c)}\right) \leqslant 2T(r,f(z+c)) \\ &= 2T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f), \end{aligned} \tag{27}$$

$$\begin{aligned} \overline{N}(r,F) &= \overline{N}(r,f(z+c)) \leqslant T(r,f(z+c)) \\ &= T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f), \end{aligned} \tag{28}$$

$$\overline{N}(r,G)=\overline{N}(r,f) \leqslant T(r,f), \tag{29}$$

$$\overline{N}_*(r,\infty;F,G)=0. \tag{30}$$

Lemma 2.4 gives

$$\begin{aligned} N_2\left(r,\frac{1}{G}\right) &= 2\overline{N}\left(r,\frac{1}{f^{(k)}}\right) \leqslant 2N_{k+1}\left(r,\frac{1}{f}\right)+2k\overline{N}(r,f)+S(r,f) \\ &\leqslant (2k+2)T(r,f)+S(r,f). \end{aligned} \tag{31}$$

Combining (25)–(31), we deduce

$$(n-2k-6)T(r,f) \leqslant O(r^{\rho(f)-1+\varepsilon})+S(r,f), \tag{32}$$

which contradicts with $n \geqslant 2k+7$. Therefore $H \equiv 0$. Similar to the proof in (I), we can get the conclusion of Theorem 1.6.

(III). Suppose $[f^{(k)}(z)]^n$ and $f^n(z+c)$ share $(1,0)$, $(\infty,0)$ and $n \geqslant 3k+14$. Then it follows directly from the assumptions of the theorem that F and G share $(1,0)$

and $(\infty,0)$. Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.6 that

$$\begin{aligned} T(r,F)+T(r,G) \leqslant 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right) \\ +2N_E^{(2}\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right) \\ +S(r,F)+S(r,G). \end{aligned} \tag{33}$$

Since

$$\begin{aligned} N_E^{1)}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{G-1}\right) \\ \leqslant N\left(r,\frac{1}{G-1}\right) \leqslant T(r,G)+O(1), \end{aligned} \tag{34}$$

we get from (33) and (34) that

$$\begin{aligned} T(r,F) \leqslant 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2N_L\left(r,\frac{1}{F-1}\right) \\ +N_L\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G). \end{aligned} \tag{35}$$

According to Lemma 2.2 and Lemma 2.3, we have

$$T(r,F)=nT(r,f(z+c))+S(r,f)=nT(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f). \tag{36}$$

It is obvious that

$$\begin{aligned} \overline{N}(r,F)=\overline{N}(r,f(z+c)) \leqslant T(r,f(z+c)) \\ =T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f), \end{aligned} \tag{37}$$

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right)=2\overline{N}\left(r,\frac{1}{f(z+c)}\right) \leqslant 2T(r,f(z+c)) \\ =2T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f), \end{aligned} \tag{38}$$

$$\begin{aligned} N_L\left(r,\frac{1}{F-1}\right) &\leqslant N\left(r,\frac{F}{F'}\right) \leqslant N\left(r,\frac{F'}{F}\right)+S(r,f) \\ &\leqslant \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,f) \\ &=\overline{N}(r,f^n(z+c))+\overline{N}\left(r,\frac{1}{f^n(z+c)}\right)+S(r,f) \\ &=\overline{N}(r,f(z+c))+\overline{N}\left(r,\frac{1}{f(z+c)}\right)+S(r,f) \\ &\leqslant 2T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f). \end{aligned} \tag{39}$$

Lemma 2.4 gives

$$N_2\left(r,\frac{1}{G}\right)=2\overline{N}\left(r,\frac{1}{f^{(k)}}\right)\leqslant 2N_{k+1}\left(r,\frac{1}{f}\right)+2k\overline{N}(r,f)+S(r,f) \\ \leqslant (2k+2)T(r,f)+S(r,f), \tag{40}$$

$$\begin{aligned} N_L\left(r,\frac{1}{G-1}\right) &\leqslant N\left(r,\frac{G}{G'}\right)\leqslant N\left(r,\frac{G'}{G}\right)+S(r,f)\\ &\leqslant \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)\\ &\leqslant \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\ &\leqslant \overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)\\ &\leqslant (k+2)T(r,f)+S(r,f). \end{aligned} \tag{41}$$

Combining (35)–(41), we deduce

$$(n-3k-13)T(r,f)\leqslant O(r^{\rho(f)-1+\varepsilon})+S(r,f), \tag{42}$$

which contradicts with $n\geqslant 3k+14$. Therefore $H\equiv 0$. Similar to the proof of (I), we can get the conclusion of Theorem 1.6.

4. Proof of Theorem 1.8

Let

$$F=f^n(qz),\quad G=[f^{(k)}(z)]^n. \tag{43}$$

(I). Suppose $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1,2)$, $(\infty,0)$ and $n\geqslant 2k+8$. Then it follows directly from the assumptions of the theorem that F and G share $(1,2)$ and $(\infty,0)$. Let H be defined as above. Suppose that $H\not\equiv 0$. It follows from Lemma 2.1 that

$$T(r,F)\leqslant N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G) \\ +\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G). \tag{44}$$

According to Lemma 2.2 and Lemma 2.7, we have

$$T(r,F)=nT(r,f(qz))+S(r,f)=nT(r,f)+S(r,f), \tag{45}$$

$$\overline{N}(r,F)=\overline{N}(r,f(qz))=\overline{N}(r,f(z))+S(r,f)\leqslant T(r,f)+S(r,f), \tag{46}$$

$$N_2\left(r,\frac{1}{F}\right)=2\overline{N}\left(r,\frac{1}{f(qz)}\right)\leqslant 2T(r,f(qz))=2T(r,f)+S(r,f). \tag{47}$$

It is obvious that

$$\overline{N}(r,G)=\overline{N}(r,f)\leqslant T(r,f). \tag{48}$$

$$\overline{N}_*(r,\infty;F,G)\leqslant \overline{N}(r,G)=\overline{N}(r,f)\leqslant T(r,f). \tag{49}$$

Lemma 2.4 gives

$$\begin{aligned}N_2\left(r,\frac{1}{G}\right)&=2\overline{N}\left(r,\frac{1}{f^{(k)}}\right)\leqslant 2N_{k+1}\left(r,\frac{1}{f}\right)+2k\overline{N}(r,f)+S(r,f)\\&\leqslant (2k+2)T(r,f)+S(r,f).\end{aligned} \tag{50}$$

Combining (44)–(50), we deduce

$$(n-2k-7)T(r,f)\leqslant S(r,f), \tag{51}$$

which contradicts with $n\geqslant 2k+8$. Therefore $H\equiv 0$. By integration, we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{52}$$

where $A\neq 0$ and B are constants. From (52) we have

$$G=\frac{(B-A)F+(A-B-1)}{BF-(B+1)}. \tag{53}$$

Suppose that $B\neq 0,-1$. From (53), we have

$$\overline{N}\left(r,\frac{1}{F-\frac{B+1}{B}}\right)=\overline{N}(r,G). \tag{54}$$

From the second fundamental theorem and Lemma 2.7, we have

$$\begin{aligned}nT(r,f)=T(r,F)+S(r,f)&\leqslant \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{B+1}{B}}\right)+S(r,f)\\&\leqslant \overline{N}(r,f(qz))+\overline{N}\left(r,\frac{1}{f(qz)}\right)+\overline{N}(r,f)+S(r,f),\end{aligned} \tag{55}$$

which contradicts with $n\geqslant 2k+8$.

Suppose that $B=-1$. From (53) we have

$$G=\frac{(A+1)F-A}{F}. \tag{56}$$

If $A \neq -1$, we obtain from (56) that

$$\overline{N}\left(r, \frac{1}{F - \frac{A}{A+1}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{57}$$

From the second fundamental theorem and Lemma 2.4, we have

$$\begin{aligned} nT(r,f) = T(r,F) + S(r,f) &\leqslant \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{A}{A+1}}\right) + S(r,f) \\ &\leqslant \overline{N}(r, f(qz)) + \overline{N}\left(r, \frac{1}{f(qz)}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r,f) \\ &\leqslant \overline{N}(r, f(qz)) + \overline{N}\left(r, \frac{1}{f(qz)}\right) + N_{k+1}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f) \end{aligned} \tag{58}$$

which contradicts with $n \geqslant 2k+8$. Hence $A = -1$. From (56), we get $FG = 1$, that is

$$f^n(qz)[f^{(k)}(z)]^n = 1. \tag{59}$$

Since $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(\infty, 0)$, from (59) we get

$$N(r, f^{(k)}) = 0, \quad T(r, f^{(k)}) = T(r, f(qz)) + S(r,f), \tag{60}$$

and

$$[f^{(k)}(z)]^{2n} = \frac{[f^{(k)}(z)]^n}{f^n(qz)} = \frac{\frac{[f^{(k)}(z)]^n}{f^n(z)}}{\frac{f^n(qz)}{f^n(z)}}. \tag{61}$$

From Lemma 2.8 and the logarithmic derivative lemma, we get

$$\begin{aligned} 2nT(r, f^{(k)}) = T(r, [f^{(k)}]^{2n}) = m(r, [f^{(k)}]^{2n}) + N(r, [f^{(k)}]^{2n}) &= m\left(r, [f^{(k)}(z)]^{2n}\right) \\ = m\left(r, \frac{\frac{[f^{(k)}(z)]^n}{f^n(z)}}{\frac{f^n(qz)}{f^n(z)}}\right) \leqslant m\left(r, \frac{[f^{(k)}(z)]^n}{f^n(z)}\right) + m\left(r, \frac{f^n(qz)}{f^n(z)}\right) \\ \leqslant nm\left(r, \frac{f^{(k)}(z)}{f(z)}\right) + nm\left(r, \frac{f(qz)}{f(z)}\right) = S(r,f), \end{aligned}$$

that is

$$T(r, f^{(k)}) = S(r,f). \tag{62}$$

By (60) and (62), we know that

$$T(r, f(qz)) = T(r, f^{(k)}) = S(r,f), \tag{63}$$

which is a contradiction with Lemma 2.7.

Suppose that $B=0$. From (53), we have

$$G=AF-(A-1). \tag{64}$$

If $A\neq 1$, from (64) we obtain

$$\overline{N}\left(r,\frac{1}{F-\frac{A-1}{A}}\right)=\overline{N}\left(r,\frac{1}{G}\right). \tag{65}$$

From the second fundamental theorem and Lemma 2.4, we have

$$\begin{aligned} nT(r,f)=T(r,F)+S(r,f) &\leqslant \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{A-1}{A}}\right)+S(r,f)\\ &\leqslant \overline{N}(r,f(qz))+\overline{N}\left(r,\frac{1}{f(qz)}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\ &\leqslant \overline{N}(r,f(qz))+\overline{N}\left(r,\frac{1}{f(qz)}\right)+N_{k+1}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f), \end{aligned} \tag{66}$$

which contradicts with $n\geqslant 2k+8$. Thus $A=1$. From (64) we have $F=G$, that is $f^n(qz)=[f^{(k)}(z)]^n$. Hence $f^{(k)}(z)=tf(qz)$, for a constant t with $t^n=1$. we can get the conclusion of Theorem 1.8.

(II). Suppose $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1,2)$, (∞,∞) and $n\geqslant 2k+7$. Then it follows directly from the assumptions of the theorem that F and G share $(1,2)$ and (∞,∞). Let H be defined as above. Suppose that $H\not\equiv 0$. It follows from Lemma 2.1 that

$$\begin{aligned} T(r,F)\leqslant N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)\\ +\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G). \end{aligned} \tag{67}$$

According to Lemma 2.2 and Lemma 2.7, we have

$$T(r,F)=nT(r,f(qz))+S(r,f)=nT(r,f)+S(r,f), \tag{68}$$

$$\overline{N}(r,F)=\overline{N}(r,f(qz))=\overline{N}(r,f(z))+S(r,f)\leqslant T(r,f)+S(r,f), \tag{69}$$

$$N_2\left(r,\frac{1}{F}\right)=2\overline{N}\left(r,\frac{1}{f(qz)}\right)\leqslant 2T(r,f(qz))=2T(r,f)+S(r,f). \tag{70}$$

It is obvious that

$$\overline{N}(r,G)=\overline{N}(r,f)\leqslant T(r,f). \tag{71}$$

$$\overline{N}_*(r,\infty;F,G)=0. \tag{72}$$

Lemma 2.4 gives

$$N_2\left(r,\frac{1}{G}\right)=2\overline{N}\left(r,\frac{1}{f^{(k)}}\right)\leqslant 2N_{k+1}\left(r,\frac{1}{f}\right)+2k\overline{N}(r,f)+S(r,f)$$
$$\leqslant (2k+2)T(r,f)+S(r,f). \tag{73}$$

Combining (67)–(73), we deduce

$$(n-2k-6)T(r,f)\leqslant S(r,f), \tag{74}$$

which contradicts with $n\geqslant 2k+7$. Therefore $H\equiv 0$. Similar to the proof of (I), we can get the conclusion of Theorem 1.8.

(III). Suppose $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1,0)$, $(\infty,0)$ and $n\geqslant 3k+14$. Then it follows directly from the assumptions of the theorem that F and G share $(1,0)$ and $(\infty,0)$. Let H be defined as above. Suppose that $H\not\equiv 0$. It follows from Lemma 2.6 that

$$T(r,F)+T(r,G)\leqslant 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)$$
$$+N_E^{1)}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{F-1}\right) \tag{75}$$
$$+3N_L\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G).$$

Since

$$N_E^{1)}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{G-1}\right)$$
$$\leqslant N\left(r,\frac{1}{G-1}\right)\leqslant T(r,G)+O(1), \tag{76}$$

we get from (75) and (76) that

$$T(r,F)\leqslant 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2N_L\left(r,\frac{1}{F-1}\right)$$
$$+N_L\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G). \tag{77}$$

According to Lemma 2.2 and Lemma 2.7, we have

$$T(r,F)=nT(r,f(qz))+S(r,f)=nT(r,f)+S(r,f). \tag{78}$$

It is obvious that

$$\overline{N}(r,F)=\overline{N}(r,f(qz))\leqslant T(r,f(qz))=T(r,f)+S(r,f), \tag{79}$$

$$N_2\left(r,\frac{1}{F}\right)=2\overline{N}\left(r,\frac{1}{f(qz)}\right)\leqslant 2T(r,f(qz))=2T(r,f)+S(r,f), \tag{80}$$

$$
\begin{aligned}
N_L\left(r,\frac{1}{F-1}\right) &\leqslant N\left(r,\frac{F}{F'}\right) \leqslant N\left(r,\frac{F'}{F}\right)+S(r,f)\\
&\leqslant \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)\\
&= \overline{N}(r,f^n(qz))+\overline{N}\left(r,\frac{1}{,f^n(qz)}\right)+S(r,f)\\
&= \overline{N}(r,f(qz))+\overline{N}\left(r,\frac{1}{f(qz)}\right)+S(r,f)\\
&\leqslant 2T(r,f)+S(r,f).
\end{aligned}
\tag{81}
$$

Lemma 2.4 gives

$$
\begin{aligned}
N_2\left(r,\frac{1}{G}\right) &= 2\overline{N}\left(r,\frac{1}{f^{(k)}}\right) \leqslant 2N_{k+1}\left(r,\frac{1}{f}\right)+2k\overline{N}(r,f)+S(r,f)\\
&\leqslant (2k+2)T(r,f)+S(r,f),
\end{aligned}
\tag{82}
$$

$$
\begin{aligned}
N_L\left(r,\frac{1}{G-1}\right) &\leqslant N\left(r,\frac{G}{G'}\right) \leqslant N\left(r,\frac{G'}{G}\right)+S(r,f)\\
&\leqslant \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)\\
&= \overline{N}(r,[f^{(k)}]^n)+\overline{N}\left(r,\frac{1}{[f^{(k)}]^n}\right)+S(r,f)\\
&= \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\
&\leqslant \overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)\\
&\leqslant (k+2)T(r,f)+S(r,f).
\end{aligned}
\tag{83}
$$

Combining (77)–(83), we deduce

$$
(n-3k-13)T(r,f) \leqslant S(r,f), \tag{84}
$$

which contradicts with $n \geqslant 3k+14$. Therefore $H \equiv 0$. Similar to the proof of (I), we can get the conclusion of Theorem 1.8.

REFERENCES

[1] A. AL-KHALADI, *On meromorphic functions that share one value with their derivative*, Analysis, 25 (2005), 131–140.

[2] A. BANERJEE, *Uniqueness of meromorphic functions that share two sets*, Southeast Asian Bull. Math. 31 (2007), 7–17.

[3] D. C. BARNETT, R. G. HALBURD, R. J. KORHONEN AND W. MORGAN, *Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations*, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007), 457–474.

[4] R. BRÜCK, *On entire functions which share one value CM with their first derivative*, Results Math. 30 (1996), 21–24.

[5] J. M. CHANG AND Y. Z. ZHU, *Entire functions that share a small function with their derivatives*, J. Math. Anal. Appl. 351 (2009), 491–496.

[6] Z. X. CHEN, *On the difference counterpart of Brück conjecture*, Acta Math. Sci. 34 (2014), 653–659.

[7] Z. X. CHEN AND K. H. SHON, *On conjecture of R. Brück concerning the entire function sharing one value CM with its derivatives*, Taiwanese J. Math. 8 (2004), 235–244.

[8] Z. X. CHEN AND H. X. YI, *On sharing values of meromorphic functions and their differences*, Results Math. 63 (2013), 557–565.

[9] Y. M. CHIANG AND S. J. FENG, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J. 16 (2008), 105–129.

[10] X. J. DONG AND K. LIU, *Some results on differential-difference analogues of Brück conjecture*, Math. Slovaca 67 (2017), 691–700.

[11] J. GRAHL AND C. MENG, *Entire functions sharing a polynomial with their derivatives and normal families*, Analysis, 28 (2008), 51–61.

[12] G. G. GUNDERSEN AND L. Z. YANG, *Entire functions that share one value with one or two of their derivatives*, J. Math. Anal. Appl. 223 (1998), 88–95.

[13] R. G. HALBURD AND R. J. KORHONEN, *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. 31 (2006), 463–478.

[14] W. K. HAYMAN, *Meromorphic Functions*, Clarendon, Oxford, 1964.

[15] J. HEITTOKANGAS, R. J. KORHONEN, R. LAINE, I. RIEPPO AND J. L. ZHANG, *Value sharing results for shifts of meromorphic functions and sufficient conditions for periodicity*, J. Math. Anal. Appl. 355 (2009), 352–363.

[16] I. LAHIRI, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J. 161 (2001), 193–206.

[17] I. LAHIRI, *Uniqueness of a meromorphic function and its derivative*, J. Inequal. Pure Appl. Math. 5 (1) (2004), Art. 20.

[18] P. LI, *Entire functions that share one value with their linear differential polynomials*, Kodai Math. J. 22 (1999) 446–457.

[19] S. LI AND Z. S. GAO, *Entire functions sharing one or two finite values CM with their shifts or difference operators*, Arch. Math. 97 (2011), 475–483.

[20] X. M. LI, C. Y. KANG, H. X. YI, *Uniqueness theorems for entire functions sharing a nonzero complex number with their difference operators*, Arch. Math. 96 (2011), 577–587.

[21] X. M. LI, H. X. YI AND C. Y. KANG, *Notes on entire functions sharing an entire function of a smaller order with their difference operators*, Arch. Math. 99 (2012), 261–270.

[22] K. LIU AND X. J. DONG, *Some results related to complex differential-difference equations of certain types*, Bull. Korean Math. Soc. 51 (2014), 1453–1467.

[23] L. P. LIU AND Y. X. GU, *Uniqueness of meromorphic functions that share one small function with their derivatives*, Kodai Math. J. 27 (2004), 272–279.

[24] F. LÜ, J. F. XU AND H. X. YI, *Entire functions that share one value with their linear differential polynomials*, J. Math. Anal. Appl. 342 (2008) 615–628.

[25] F. LÜ AND H. X. YI, *The Brück conjecture and entire functions sharing polynomials with their k-th derivatives*, J. Korean Math. Soc. 48 (2011), 499–512.

[26] C. MENG, *On unicity of meromorphic function and its kth order derivative*, J. Math. Inequal. 4 (2010), 151–159.

[27] D. C. PRAMANIK, M. BISWAS AND R. MANDAL, *On the study of Brück conjecture and some non-linear complex differential equations*, Arab J. Math. Sci. 23 (2017), 196–204.

[28] X. G. QI, N. LI AND L. Z. YANG, *Uniqueness of meromorphic functions concerning their differences and solutions of difference painleve equations*, Comput. Methods Funct. Theory. 18 (2018), 567–582.

[29] X. G. QI AND K. LIU, *Uniqueness and value distribution of differences of entire functions*, J. Math. Anal. Appl. 379 (2011), 180–187.

[30] L. A. RUBEL AND C. C. YANG, *Values shared by an entire function and its derivatives, In: Complex Analysis*, Kentucky 1976 (Proc. Conf), Lecture Notes in Mathematics, Vol. 599, Springer-Verlag, Berlin, 1977, 101–103.

[31] J. P. WANG, *Entire functions that share a polynomial with one of their derivatives*, Kodai Math. J. 27 (2004), 144–151.

[32] J. P. WANG AND H. X. YI, *Entire functions that share one value CM with their derivatives*, J. Math. Anal. Appl. 277 (2003) 155–163.
[33] C. C. YANG, *On deficiencies of differential polynomials*, Math. Z. 125 (1972), 107–112.
[34] L. Z. YANG, *Solution of a differential equation and its applications*, Kodai Math. J. 22 (1999), 458–464.
[35] L. Z. YANG, *Further results on entire functions that share one value with their derivatives*, J. Math. Anal. Appl. 212 (1997) 529–536.
[36] L. YANG, *Value distribution theory*, Springer-Verlag, Berlin, 1993.
[37] K. W. YU, *On entire and meromorphic functions that share small functions with their derivatives*, J. Inequal. Pure Appl. Math. 4 (1) (2003), Art. 21.
[38] J. ZHANG AND L. W. LIAO, *Entire functions sharing some values with their difference operators*, Sci. China Math. 57 (2014), 2143–2152.
[39] J. L. ZHANG, *Meromorphic functions sharing a small function with their differential polynomials*, Kyungpook Math. J. 50 (3) (2010), 345–355.
[40] J. L. ZHANG AND R. J. KORHONEN, *On the Nevanlinna characteristic of $f(qz)$ and its applications*, J. Math. Anal. Appl. 369 (2010), 537–544.
[41] J. L. ZHANG AND L. Z. YANG, *A power of a meromorphic function sharing a small function with its derivative*, Ann. Acad. Sci. Fenn. Math. 34 (2009), 249–260.
[42] J. L. ZHANG AND L. Z. YANG, *A power of an entire function sharing one value with its derivative*, Comput. Math. Appl. 60 (2010), 2153–2160.
[43] J. L. ZHANG AND L. Z. YANG, *Some results related to a conjecture of R. Brück*, J. Inequal. Pure Appl. Math. 8 (1) (2007), Art. 18.
[44] Q. C. ZHANG, *Meromorphic function that share one small function with its derivative*, J. Inequal. Pure Appl. Math. 6 (4) (2005), Art. 116.

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